CHAPTER 8
Integration Techniques and Improper Integrals

Section 8.1 Basic Integration Rules **719**

Section 8.2 Integration by Parts **730**

Section 8.3 Trigonometric Integrals **751**

Section 8.4 Trigonometric Substitution **764**

Section 8.5 Partial Fractions **786**

Section 8.6 Numerical Integration **800**

Section 8.7 Integration by Tables and Other Integration Techniques **808**

Section 8.8 Improper Integrals **820**

Review Exercises **833**

Problem Solving **846**

CHAPTER 8
Integration Techniques and Improper Integrals

Section 8.1 Basic Integration Rules

1. Use long division to rewrite the function as the sum of a polynomial and a proper rational function.

2. (a) Write the integral as a sum:

$$\int \frac{2+x}{x^2+9}\,dx = \int \frac{2}{x^2+9}\,dx + \int \frac{x}{x^2+9}\,dx$$

(b) Use a trigonometric identity:

$$\int \cot^2 x\,dx = \int \left(\csc^2 x - 1\right)dx$$

3. (a) $\frac{d}{dx}\left[2\sqrt{x^2+1} + C\right] = 2\left(\frac{1}{2}\right)\left(x^2+1\right)^{-1/2}(2x) = \frac{2x}{\sqrt{x^2+1}}$

(b) $\frac{d}{dx}\left[\sqrt{x^2+1} + C\right] = \frac{1}{2}\left(x^2+1\right)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}$

(c) $\frac{d}{dx}\left[\frac{1}{2}\sqrt{x^2+1} + C\right] = \frac{1}{2}\left(\frac{1}{2}\right)\left(x^2+1\right)^{-1/2}(2x) = \frac{x}{2\sqrt{x^2+1}}$

(d) $\frac{d}{dx}\left[\ln\left(x^2+1\right) + C\right] = \frac{2x}{x^2+1}$

$\int \frac{x}{\sqrt{x^2+1}}\,dx$ matches (b).

4. (a) $\frac{d}{dx}\left[\ln\sqrt{x^2+1} + C\right] = \frac{1}{2}\left(\frac{2x}{x^2+1}\right) = \frac{x}{x^2+1}$

(b) $\frac{d}{dx}\left[\frac{2x}{\left(x^2+1\right)^2} + C\right] = \frac{\left(x^2+1\right)^2(2) - (2x)(2)\left(x^2+1\right)(2x)}{\left(x^2+1\right)^4} = \frac{2\left(1-3x^2\right)}{\left(x^2+1\right)^3}$

(c) $\frac{d}{dx}\left[\arctan x + C\right] = \frac{1}{1+x^2}$

(d) $\frac{d}{dx}\left[\ln\left(x^2+1\right) + C\right] = \frac{2x}{x^2+1}$

$\int \frac{1}{x^2+1}\,dx$ matches (c).

5. $\int (5x-3)^4\,dx$

$u = 5x - 3,\ du = 5\,dx,\ n = 4$

Use $\int u^n\,du$.

6. $\int \frac{2t+1}{t^2+t-4}\,dt$

$u = t^2 + t - 4,\ du = (2t+1)\,dt$

Use $\int \frac{du}{u}$.

7. $\int \frac{1}{\sqrt{x}\left(1-2\sqrt{x}\right)}\,dx$

$u = 1 - 2\sqrt{x},\ du = -\frac{1}{\sqrt{x}}\,dx$

Use $\int \frac{du}{u}$.

8. $\int \frac{2}{(2t-1)^2+4}\,dt$

$u = 2t - 1, du = 2\,dt, a = 2$

Use $\int \frac{du}{u^2+a^2}$.

9. $\int \frac{3}{\sqrt{1-t^2}}\,dt$

$u = t, du = dt, a = 1$

Use $\int \frac{du}{\sqrt{a^2-u^2}}$.

10. $\int \frac{-2x}{\sqrt{x^2-4}}\,dx$

$u = x^2 - 4, du = 2x\,dx, n = -\frac{1}{2}$

Use $\int u^n\,du$.

11. $\int t \sin t^2\,dt$

$u = t^2, du = 2t\,dt$

Use $\int \sin u\,du$.

12. $\int \sec 5x \tan 5x\,dx$

$u = 5x, du = 5\,dx$

Use $\int \sec u \tan u\,du$.

13. $\int (\cos x)e^{\sin x}\,dx$

$u = \sin x, du = \cos x\,dx$

Use $\int e^u\,du$.

14. $\int \frac{1}{x\sqrt{x^2-4}}\,dx$

$u = x, du = dx, a = 2$

Use $\int \frac{du}{u\sqrt{u^2-a^2}}$.

15. Let $u = x - 5, du = dx$.

$$\int 14(x-5)^6\,dx = 14\int (x-5)^6\,dx = 2(x-5)^7 + C$$

16. Let $u = t + 6, du = dt$.

$$\int \frac{5}{(t+6)^3}\,dt = 5\int (t+6)^{-3}\,dt = 5 \cdot \frac{(t+6)^{-2}}{-2} + C = \frac{-5}{2(t+6)^2} + C$$

17. Let $u = z - 10, du = dz$.

$$\int \frac{7}{(z-10)^7}\,dz = 7\int (z-10)^{-7}\,dz = -\frac{7}{6(z-10)^6} + C$$

18. Let $u = t^4 + 1, du = 4t^3\,dt$.

$$\int t^3\sqrt{t^4+1}\,dt = \frac{1}{4}\int (t^4+1)^{1/2}(4t^3)\,dt = \frac{1}{4} \cdot \frac{(t^4+1)^{3/2}}{(3/2)} + C = \frac{1}{6}(t^4+1)^{3/2} + C$$

19. $$\int \left[z^2 + \frac{1}{(1-z)^6}\right]dz = \int \left[z^2 + (1-z)^{-6}\right]dz = \frac{z^3}{3} + \frac{(1-z)^{-5}}{5} + C = \frac{z^3}{3} + \frac{1}{5(1-z)^5} + C = \frac{z^3}{3} - \frac{1}{5(z-1)^5} + C$$

20. $$\int \left[4x - \frac{2}{(2x+3)^2}\right]dx = \int 4x\,dx - \int 2(2x+3)^{-2}\,dx = 2x^2 - \frac{(2x+3)^{-1}}{-1} + C = 2x^2 + \frac{1}{2x+3} + C$$

21. Let $u = -t^3 + 9t + 1$,

$du = (-3t^2 + 9)\,dt = -3(t^2 - 3)\,dt$.

$$\int \frac{t^2-3}{-t^3+9t+1}\,dt = -\frac{1}{3}\int \frac{-3(t^2-3)}{-t^3+9t+1}\,dt = -\frac{1}{3}\ln\left|-t^3+9t+1\right| + C$$

22. Let $u = 3x^2 + 6x$, $du = (6x + 6)\,dx = 6(x + 1)\,dx$.

$$\int \frac{x+1}{\sqrt{3x^2+6x}}\,dx = \frac{1}{6}\int (3x^2+6x)^{-1/2} 6(x+1)\,dx$$
$$= \frac{1}{6}\cdot\frac{(3x^2+6x)^{1/2}}{(1/2)} + C$$
$$= \frac{1}{3}\sqrt{3x^2+6x} + C$$

23. $\int \frac{x^2}{x-1}\,dx = \int (x+1)\,dx + \int \frac{1}{x-1}\,dx$
$$= \frac{1}{2}x^2 + x + \ln|x-1| + C$$

24. $\int \frac{3x}{x+4}\,dx = \int \left(3 - \frac{12}{x+4}\right)dx$
$$= 3x - 12\ln|x+4| + C$$

25. $\int \frac{x+2}{x+1}\,dx = \int \frac{x+1+1}{x+1}\,dx$
$$= \int \left(1 + \frac{1}{x+1}\right)dx$$
$$= x + \ln|x+1| + C$$

26. $\int \left(\frac{1}{9z-5} - \frac{1}{9z+5}\right)dz = \frac{1}{9}\ln|9z-5| - \frac{1}{9}\ln|9z+5| + C$
$$= \frac{1}{9}\ln\left|\frac{9z-5}{9z+5}\right| + C$$

27. $\int (5+4x^2)^2\,dx = \int (25 + 40x^2 + 16x^4)\,dx$
$$= 25x + \frac{40}{3}x^3 + \frac{16}{5}x^5 + C$$
$$= \frac{x}{15}(48x^4 + 200x^2 + 375) + C$$

28. $\int x\left(3 + \frac{2}{x}\right)^2 dx = \int \left(9x + 12 + \frac{4}{x}\right)dx$
$$= \frac{9}{2}x^2 + 12x + 4\ln|x| + C$$

29. Let $u = 2\pi x^2$, $du = 4\pi x\,dx$.

$$\int x(\cos 2\pi x^2)\,dx = \frac{1}{4\pi}\int (\cos 2\pi x^2)(4\pi x)\,dx$$
$$= \frac{1}{4\pi}\sin 2\pi x^2 + C$$

30. Let $u = \pi x$, $du = \pi\,dx$.

$$\int \csc \pi x \cot \pi x\,dx = \frac{1}{\pi}\int (\csc \pi x)(\cot \pi x)\,\pi\,dx$$
$$= -\frac{1}{\pi}\csc \pi x + C$$

31. Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int \frac{\sin x}{\sqrt{\cos x}}\,dx = -\int (\cos x)^{-1/2}(-\sin x)\,dx$$
$$= -2\sqrt{\cos x} + C$$

32. $\int \frac{\csc^2 3t}{\cot 3t}\,dt = -\frac{1}{3}\int \frac{1}{\cot 3t}(-\csc^2 3t)\,dt$
$$= -\frac{1}{3}\ln|\cot 3t| + C$$

33. Let $u = 1 + e^x$, $du = e^x\,dx$.

$$\int \frac{2}{e^{-x}+1}\,dx = 2\int \left(\frac{2}{e^{-x}+1}\right)\left(\frac{e^x}{e^x}\right)dx$$
$$= 2\int \frac{e^x}{1+e^x}\,dx = 2\ln(1+e^x) + C$$

34. $\int \frac{4}{3-e^x}\,dx = 4\int \frac{e^{-x}}{3e^{-x}-1}\,dx$
$$= -\frac{4}{3}\ln|3e^{-x} - 1| + C$$

35. $\int \frac{\ln x^2}{x}\,dx = 2\int (\ln x)\frac{1}{x}\,dx$
$$= 2\frac{(\ln x)^2}{2} + C = (\ln x)^2 + C$$

36. Let $u = \ln(\cos x)$, $du = \frac{-\sin x}{\cos x}\,dx = -\tan x\,dx$.

$$\int (\tan x)(\ln \cos x)\,dx = -\int (\ln \cos x)(-\tan x)\,dx$$
$$= \frac{-[\ln(\cos x)]^2}{2} + C$$

37. $\int \frac{1+\cos\alpha}{\sin\alpha}\,d\alpha = \int \csc\alpha\,d\alpha + \int \cot\alpha\,d\alpha$
$$= -\ln|\csc\alpha + \cot\alpha| + \ln|\sin\alpha| + C$$

38. $\dfrac{1}{\cos\theta - 1} = \dfrac{1}{\cos\theta - 1}\cdot\dfrac{\cos\theta + 1}{\cos\theta + 1} = \dfrac{\cos\theta + 1}{\cos^2\theta - 1}$

$= \dfrac{\cos\theta + 1}{-\sin^2\theta} = -\csc\theta\cdot\cot\theta - \csc^2\theta$

$$\int \frac{1}{\cos\theta - 1}\,d\theta = \int\left(-\csc\theta\cot\theta - \csc^2\theta\right)d\theta$$

$$= \csc\theta + \cot\theta + C$$

$$= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} + C$$

$$= \frac{1 + \cos\theta}{\sin\theta} + C$$

39. Let $u = 4t + 1$, $du = 4\,dt$.

$$\int \frac{-1}{\sqrt{1 - (4t+1)^2}}\,dt = -\frac{1}{4}\int \frac{4}{\sqrt{1 - (4t+1)^2}}\,dt$$

$$= -\frac{1}{4}\arcsin(4t + 1) + C$$

40. Let $u = 2x$, $du = 2dx$, $a = 5$.

$$\int \frac{1}{25 + 4x^2}\,dx = \frac{1}{2}\int \frac{1}{5^2 + (2x)^2}(2)\,dx$$

$$= \frac{1}{10}\arctan\frac{2x}{5} + C$$

41. Let $u = \cos\left(\dfrac{2}{t}\right)$, $du = \dfrac{2\sin(2/t)}{t^2}\,dt$.

$$\int \frac{\tan(2/t)}{t^2}\,dt = \frac{1}{2}\int \frac{1}{\cos(2/t)}\left[\frac{2\sin(2/t)}{t^2}\right]dt$$

$$= \frac{1}{2}\ln\left|\cos\left(\frac{2}{t}\right)\right| + C$$

42. Let $u = -\dfrac{1}{t^3} = -t^{-3}$, $du = 3t^{-4}\,dt$.

$$\int \frac{e^{-1/t^3}}{t^4}\,dt = \frac{1}{3}\int e^{-1/t^3}\left(3t^{-4}\,dt\right)$$

$$= \frac{1}{3}e^{-1/t^3} + C$$

43. $$\int \frac{6}{z\sqrt{9z^2 - 25}}\,dz = \int \frac{6}{3z\sqrt{(3z)^2 - 5^2}}(3dz)$$

$$= \frac{6}{5}\operatorname{arcsec}\left(\frac{|3z|}{5}\right) + C$$

44. $$\int \frac{1}{(x-1)\sqrt{4x^2 - 8x + 3}}\,dx = \int \frac{2}{[2(x-1)]\sqrt{[2(x-1)]^2 - 1}}\,dx = \operatorname{arcsec}|2(x-1)| + C$$

45. $$\int \frac{4}{4x^2 + 4x + 65}\,dx = \int \frac{1}{[x + (1/2)]^2 + 16}\,dx$$

$$= \frac{1}{4}\arctan\left[\frac{x + (1/2)}{4}\right] + C$$

$$= \frac{1}{4}\arctan\left(\frac{2x+1}{8}\right) + C$$

46. $$\int \frac{1}{x^2 - 4x + 9}\,dx = \int \frac{1}{x^2 - 4x + 4 + 5}\,dx$$

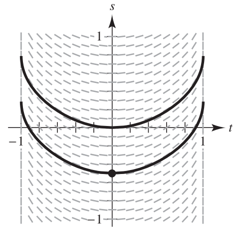
$$= \int \frac{1}{(x-2)^2 + (\sqrt{5})^2}\,dx$$

$$= \frac{1}{\sqrt{5}}\arctan\left(\frac{x-2}{\sqrt{5}}\right) + C$$

$$= \frac{\sqrt{5}}{5}\arctan\left(\frac{\sqrt{5}}{5}(x-2)\right) + C$$

47. $\dfrac{ds}{dt} = \dfrac{t}{\sqrt{1 - t^4}}$, $\left(0, -\dfrac{1}{2}\right)$

(a)



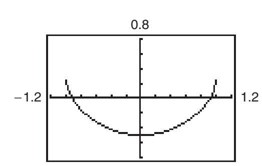
(b) $u = t^2$, $du = 2t\,dt$

$$\int \frac{t}{\sqrt{1 - t^4}}\,dt = \frac{1}{2}\int \frac{2t}{\sqrt{1 - (t^2)^2}}\,dt$$

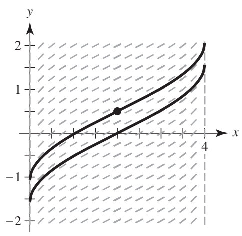
$$= \frac{1}{2}\arcsin t^2 + C$$

$\left(0, -\dfrac{1}{2}\right)$: $-\dfrac{1}{2} = \dfrac{1}{2}\arcsin 0 + C \Rightarrow C = -\dfrac{1}{2}$

$$s = \frac{1}{2}\arcsin t^2 - \frac{1}{2}$$



48. (a) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{4x - x^2}}, \quad \left(2, \dfrac{1}{2}\right)$



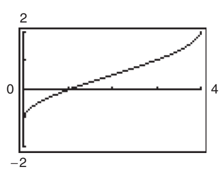
(b) $y = \displaystyle\int \frac{1}{\sqrt{4x - x^2}}\,dx$

$= \displaystyle\int \frac{1}{\sqrt{4 - (x^2 - 4x + 4)}}\,dx$

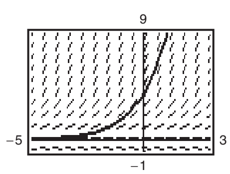
$= \displaystyle\int \frac{1}{\sqrt{4 - (x - 2)^2}}\,dx = \arcsin\left(\frac{x - 2}{2}\right) + C$

$\left(2, \dfrac{1}{2}\right)$: $\dfrac{1}{2} = \arcsin(0) + C \Rightarrow C = \dfrac{1}{2}$

$y = \arcsin\left(\dfrac{x - 2}{2}\right) + \dfrac{1}{2}$

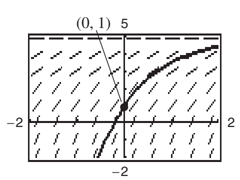


49.



$y = 4e^{0.8x}$

50.



$y = 5 - 4e^{-x}$

51. $\dfrac{dy}{dx} = \left(e^x + 5\right)^2 = e^{2x} + 10e^x + 25$

$y = \displaystyle\int \left(e^{2x} + 10e^x + 25\right) dx$

$= \dfrac{1}{2}e^{2x} + 10e^x + 25x + C$

52. $\dfrac{dy}{dx} = \left(4 - e^{2x}\right)^2 = 16 - 8e^{2x} + e^{4x}$

$y = \displaystyle\int \left(16 - 8e^{2x} + e^{4x}\right) dx$

$= 16x - 4e^{2x} + \dfrac{1}{4}e^{4x} + C$

53. $\dfrac{dr}{dt} = \dfrac{10e^t}{\sqrt{1 - e^{2t}}}$

$r = \displaystyle\int \frac{10e^t}{\sqrt{1 - \left(e^t\right)^2}}\,dt$

$= 10 \arcsin\left(e^t\right) + C$

54. $\dfrac{dr}{dt} = \dfrac{\left(1 + e^t\right)^2}{e^{3t}} = \dfrac{1 + 2e^t + e^{2t}}{e^{3t}} = e^{-3t} + 2e^{-2t} + e^{-t}$

$r = \displaystyle\int \left(e^{-3t} + 2e^{-2t} + e^{-t}\right) dt$

$= -\dfrac{1}{3}e^{-3t} - e^{-2t} - e^{-t} + C$

55. $\dfrac{dy}{dx} = \dfrac{\sec^2 x}{4 + \tan^2 x}$

Let $u = \tan x$, $du = \sec^2 x\,dx$.

$y = \displaystyle\int \frac{\sec^2 x}{4 + \tan^2 x}\,dx = \frac{1}{2}\arctan\left(\frac{\tan x}{2}\right) + C$

56. $y' = \dfrac{1}{x\sqrt{4x^2 - 9}}$

Let $u = 2x$, $du = 2dx$, $a = 3$.

$y = \displaystyle\int \frac{1}{x\sqrt{4x^2 - 9}}\,dx = \int \frac{1}{(2x)\sqrt{(2x)^2 - 3^2}}(2)\,dx$

$= \dfrac{1}{3}\operatorname{arcsec}\dfrac{|2x|}{3} + C$

57. $\displaystyle\int_{2/3}^{1} (2 - 3t)^4\,dt = \int_{2/3}^{1} (3t - 2)^4\,dt$

$= \left[\dfrac{1}{3}\dfrac{(3t - 2)^5}{5}\right]_{2/3}^{1}$

$= \dfrac{1}{15}(1 - 0) = \dfrac{1}{15}$

58. $\displaystyle\int_{-1}^{0} \frac{5}{(t + 2)^{11}}\,dt = \int_{-1}^{0} 5(t + 2)^{-11}\,dt$

$= \left[5\dfrac{(t + 2)^{-10}}{(-10)}\right]_{-1}^{0} = \left[\dfrac{-1}{2(t + 2)^{10}}\right]_{-1}^{0}$

$= -\dfrac{1}{2}\left[\dfrac{1}{2^{10}} - \dfrac{1}{1}\right] = \dfrac{1023}{2048} \approx 0.5$

59. Let $u = 2x$, $du = 2\,dx$.

$$\int_0^{\pi/4} \cos 2x\,dx = \frac{1}{2}\int_0^{\pi/4} \cos 2x(2)\,dx = \left[\frac{1}{2}\sin 2x\right]_0^{\pi/4} = \frac{1}{2}$$

60. Let $u = \sin t$, $du = \cos t\,dt$.

$$\int_0^{\pi} \sin^2 t \cos t\,dt = \left[\frac{1}{3}\sin^3 t\right]_0^{\pi} = 0$$

61. Let $u = -x^2$, $du = -2x\,dx$.

$$\int_0^1 xe^{-x^2}\,dx = -\frac{1}{2}\int_0^1 e^{-x^2}(-2x)\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^1 = \frac{1}{2}\left(1 - e^{-1}\right) \approx 0.316$$

62. Let $u = 1 - \ln x$, $du = \dfrac{-1}{x}\,dx$.

$$\int_1^e \frac{1 - \ln x}{x}\,dx = -\int_1^e (1 - \ln x)\left(\frac{-1}{x}\right)dx = \left[-\frac{1}{2}(1 - \ln x)^2\right]_1^e = \frac{1}{2}$$

63. $$\int_2^3 \frac{\ln(x+1)^3}{x+1}\,dx = 3\int_2^3 \ln(x+1)\frac{1}{x+1}\,dx = \left[3\frac{\left[\ln(x+1)\right]^2}{2}\right]_2^3 = \frac{3}{2}\left[\left[\ln(x+1)\right]^2\right]_2^3 = \frac{3}{2}\left[(\ln 4)^2 - (\ln 3)^2\right] \approx 1.072$$

64. $$\int_{-3}^1 \frac{e^x}{e^{2x} + 2e^x + 1}\,dx = \int_{-3}^1 \frac{e^x}{\left(e^x + 1\right)^2}\,dx = \int_{-3}^1 \left(e^x + 1\right)^{-2} e^x\,dx = \left[-\left(e^x + 1\right)^{-1}\right]_{-3}^1 = \left[\frac{-1}{e^x + 1}\right]_{-3}^1 = \frac{-1}{e+1} + \frac{1}{e^{-3} + 1} \approx 0.684$$

65. Let $u = x^2 + 36$, $du = 2x\,dx$.

$$\int_0^8 \frac{2x}{\sqrt{x^2 + 36}}\,dx = \int_0^8 \left(x^2 + 36\right)^{-1/2}(2x)\,dx = 2\left[\left(x^2 + 36\right)^{1/2}\right]_0^8 = 8$$

66. $$\int_1^3 \frac{2x^2 + 3x - 2}{x}\,dx = \int_1^3 \left(2x + 3 - \frac{2}{x}\right)dx = \left[x^2 + 3x - 2\ln|x|\right]_1^3 = (9 + 9 - 2\ln 3) - (1 + 3 - 0) = 14 - 2\ln 3$$

67. $$\int_3^5 \frac{2t}{t^2 - 4t + 4}\,dt = \int_3^5 \frac{2t - 4}{t^2 - 4t + 4}\,dt + \int_3^5 \frac{4}{(t-2)^2}\,dt = \left[\ln\left(t^2 - 4t + 4\right) - \frac{4}{t-2}\right]_3^5 = \left(\ln 9 - \frac{4}{3}\right) - (\ln 1 - 4) = \ln 9 + \frac{8}{3} \approx 4.864$$

68. $$\int_2^4 \frac{4x^3}{x^4 - 6x^2 + 9} = \int_2^4 \frac{4x^3 - 12x}{x^4 - 6x^2 + 9}\,dx + \int_2^4 \left(x^2 - 3\right)^{-2}(12x)\,dx = \left[\ln\left(x^4 - 6x^2 + 9\right) - \frac{6}{x^2 - 3}\right]_2^4 = \left(\ln 169 - \frac{6}{13}\right) - (\ln 1 - 6) = \ln 169 + \frac{72}{13} = 2\ln 13 + \frac{72}{13} \approx 10.688$$

69. Let $u = 3x$, $du = 3\,dx$.

$$\int_0^{2/\sqrt{3}} \frac{1}{4 + 9x^2}\,dx = \frac{1}{3}\int_0^{2/\sqrt{3}} \frac{3}{4 + (3x)^2}\,dx$$
$$= \left[\frac{1}{6}\arctan\left(\frac{3x}{2}\right)\right]_0^{2/\sqrt{3}}$$
$$= \frac{\pi}{18} \approx 0.175$$

70. $\displaystyle\int_0^7 \frac{1}{\sqrt{100 - x^2}}\,dx = \left[\arcsin\left(\frac{x}{10}\right)\right]_0^7 = \arcsin\left(\frac{7}{10}\right)$

71. $$\int_{-4}^0 3^{1-x}\,dx = \left[\frac{-1}{\ln 3}3^{1-x}\right]_{-4}^0$$
$$= -\frac{1}{\ln 3}\left(3 - 3^5\right)$$
$$= \frac{240}{\ln 3} \approx 218.457$$

72. $$\int_0^1 7^{x^2+2x}(x + 1)\,dx = \frac{1}{2}\int_0^1 7^{x^2+2x}(2x + 2)\,dx$$
$$= \left[\frac{1}{2\ln 7}\cdot 7^{x^2+2x}\right]_0^1$$
$$= \frac{1}{2\ln 7}\left[7^3 - 1\right] = \frac{171}{\ln 7} \approx 87.877$$

73. $$A = \int_0^{3/2} (-4x + 6)^{3/2}\,dx$$
$$= -\tfrac{1}{4}\int_0^{3/2} (6 - 4x)^{3/2}(-4)\,dx$$
$$= -\tfrac{1}{4}\left[\tfrac{2}{5}(6 - 4x)^{5/2}\right]_0^{3/2}$$
$$= -\tfrac{1}{10}\left(0 - 6^{5/2}\right)$$
$$= \tfrac{18}{5}\sqrt{6} \approx 8.8182$$

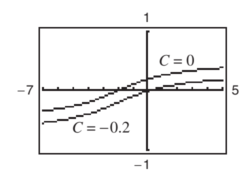
74. $$A = \int_0^5 \frac{3x + 2}{x^2 + 9}\,dx$$
$$= \int_0^5 \frac{3x}{x^2 + 9}\,dx + \int_0^5 \frac{2}{x^2 + 9}\,dx$$
$$= \left[\frac{3}{2}\ln\left|x^2 + 9\right| + \frac{2}{3}\arctan\left(\frac{x}{3}\right)\right]_0^5$$
$$= \frac{3}{2}\ln(34) + \frac{2}{3}\arctan\left(\frac{5}{3}\right) - \frac{3}{2}\ln 9$$
$$= \frac{3}{2}\ln\left(\frac{34}{9}\right) + \frac{2}{3}\arctan\left(\frac{5}{3}\right)$$
$$\approx 2.6806$$

75. $$y^2 = x^2\left(1 - x^2\right)$$
$$y = \pm\sqrt{x^2\left(1 - x^2\right)}$$
$$A = 4\int_0^1 x\sqrt{1 - x^2}\,dx$$
$$= -2\int_0^1 \left(1 - x^2\right)^{1/2}(-2x)\,dx$$
$$= -\frac{4}{3}\left[\left(1 - x^2\right)^{3/2}\right]_0^1$$
$$= -\frac{4}{3}(0 - 1) = \frac{4}{3}$$

76. $A = \displaystyle\int_0^{\pi/2} \sin 2x\,dx = -\tfrac{1}{2}\left[\cos 2x\right]_0^{\pi/2} = -\tfrac{1}{2}(-1 - 1) = 1$

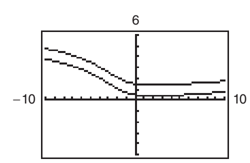
77. $\displaystyle\int \frac{1}{x^2 + 4x + 13}\,dx = \frac{1}{3}\arctan\left(\frac{x + 2}{3}\right) + C$

The antiderivatives are vertical translations of each other.



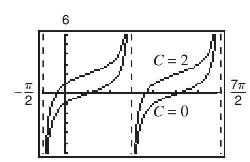
78. $\displaystyle\int \frac{x - 2}{x^2 + 4x + 13}\,dx = \frac{1}{2}\ln\left(x^2 + 4x + 13\right) - \frac{4}{3}\arctan\left(\frac{x + 2}{3}\right) + C$

The antiderivatives are vertical translations of each other.



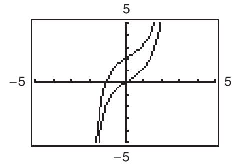
79. $\displaystyle\int \frac{1}{1 + \sin\theta}\,d\theta = \tan\theta - \sec\theta + C \quad \left(\text{or } \frac{-2}{1 + \tan(\theta/2)}\right)$

The antiderivatives are vertical translations of each other.



80. $\int \left(\frac{e^x + e^{-x}}{2}\right)^3 dx = \frac{1}{24}\left[e^{3x} + 9e^x - 9e^{-x} - e^{-3x}\right] + C$

The antiderivatives are vertical translations of each other.



81. No. When $u = x^2$, it does not follow that $x = \sqrt{u}$ because x is negative on $[-1, 0)$.

82. $$\frac{\sin x}{\cos x} + \frac{\cos x}{1 + \sin x} = \frac{\sin x(1 + \sin x) + \cos^2 x}{\cos x(1 + \sin x)}$$
$$= \frac{\sin x + \sin^2 x + \cos^2 x}{\cos x(1 + \sin x)}$$
$$= \frac{\sin x + 1}{\cos x(1 + \sin x)}$$
$$= \frac{1}{\cos x} = \sec x$$

So,

$$\int \sec x\, dx = \int \left[\frac{\sin x}{\cos x} + \frac{\cos x}{1 + \sin x}\right] dx$$
$$= -\ln|\cos x| + \ln|1 + \sin x| + C$$
$$= \ln\left|\frac{1 + \sin x}{\cos x}\right| + C$$
$$= \ln|\sec x + \tan x| + C$$

83. $\sin x + \cos x = a \sin(x + b)$

$\sin x + \cos x = a \sin x \cos b + a \cos x \sin b$

$\sin x + \cos x = (a \cos b) \sin x + (a \sin b) \cos x$

Equate coefficients of like terms to obtain the following.

$1 = a \cos b$ and $1 = a \sin b$

So, $a = 1/\cos b$. Now, substitute for a in $1 = a \sin b$.

$$1 = \left(\frac{1}{\cos b}\right) \sin b$$

$$1 = \tan b \Rightarrow b = \frac{\pi}{4}$$

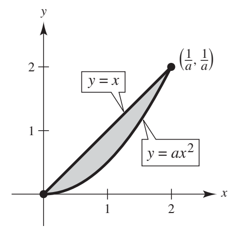
Because $b = \frac{\pi}{4}$, $a = \frac{1}{\cos(\pi/4)} = \sqrt{2}$. So,

$$\sin x + \cos x = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right).$$

$$\int \frac{dx}{\sin x + \cos x} = \int \frac{dx}{\sqrt{2} \sin(x + (\pi/4))}$$
$$= \frac{1}{\sqrt{2}} \int \csc\left(x + \frac{\pi}{4}\right) dx$$
$$= -\frac{1}{\sqrt{2}} \ln\left|\csc\left(x + \frac{\pi}{4}\right) + \cot\left(x + \frac{\pi}{4}\right)\right| + C$$

84. $\int_0^{1/a} (x - ax^2)\, dx = \left[\frac{1}{2}x^2 - \frac{a}{3}x^3\right]_0^{1/a} = \frac{1}{6a^2}$

Let $\frac{1}{6a^2} = \frac{2}{3}$, $12a^2 = 3$, $a = \frac{1}{2}$.



85. (a) They are equivalent because

$e^{x + C_1} = e^x \cdot e^{C_1} = Ce^x$, $C = e^{C_1}$.

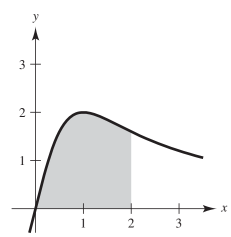
(b) They differ by a constant.

$\sec^2 x + C_1 = (\tan^2 x + 1) + C_1 = \tan^2 x + C$

86. $\int_0^5 f(x)\, dx < 0$ because there is more area below the x-axis than above.

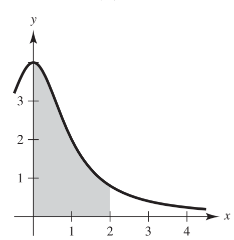
87. $\int_0^2 \frac{4x}{x^2 + 1} dx \approx 3$

Matches (a).

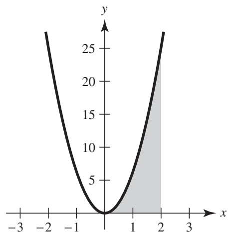


88. $\int_0^2 \frac{4}{x^2 + 1} dx \approx 4$

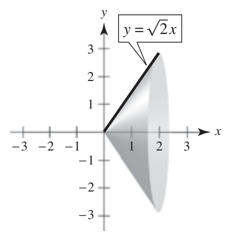
Matches (d).



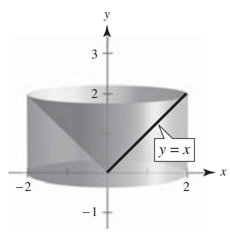
89. (a) $y = 2\pi x^2, \quad 0 \le x \le 2$



(b) $y = \sqrt{2}x, \quad 0 \le x \le 2$

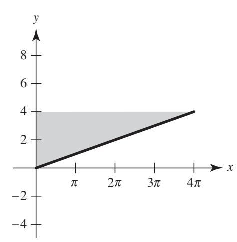


(c) $y = x, \quad 0 \le x \le 2$



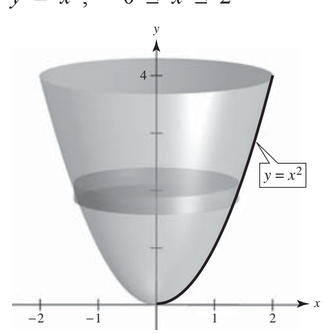
90. (a) $x = \pi y, \quad 0 \le y \le 4$

$y = \dfrac{1}{\pi}x, \quad 0 \le x \le 4\pi$



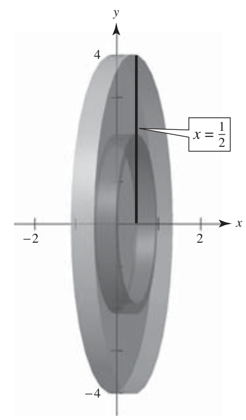
(b) $x = \sqrt{y}, \quad 0 \le y \le 4$

$y = x^2, \quad 0 \le x \le 2$



(c) $x = \dfrac{1}{2}, \quad 0 \le y \le 4$

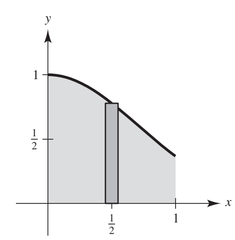
$$2\pi\int_0^4 y\left(\frac{1}{2}\right)dy$$



91. (a) **Shell Method:**

Let $u = -x^2$, $du = -2x\,dx$.

$$\begin{aligned}V &= 2\pi\int_0^1 xe^{-x^2}\,dx \\ &= -\pi\int_0^1 e^{-x^2}(-2x)\,dx \\ &= \left[-\pi e^{-x^2}\right]_0^1 \\ &= \pi\left(1 - e^{-1}\right) \approx 1.986\end{aligned}$$



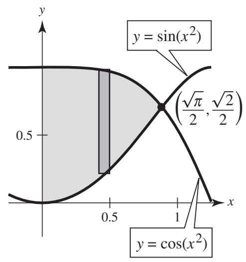
(b) **Shell Method:**

$$\begin{aligned}V &= 2\pi\int_0^b xe^{-x^2}\,dx \\ &= \left[-\pi e^{-x^2}\right]_0^b \\ &= \pi\left(1 - e^{-b^2}\right) = \frac{4}{3}\end{aligned}$$

$$e^{-b^2} = \frac{3\pi - 4}{3\pi}$$

$$b = \sqrt{\ln\left(\frac{3\pi}{3\pi - 4}\right)} \approx 0.743$$

92.



Shell Method:

$$V = 2\pi\int_0^{\sqrt{\pi}/2} x\left(\cos\left(x^2\right) - \sin\left(x^2\right)\right) dx$$

$$= \pi\left[\sin\left(x^2\right) + \cos\left(x^2\right)\right]_0^{\frac{\sqrt{\pi}}{2}}$$

$$= \pi\left[\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1)\right]$$

$$= \pi\left(\sqrt{2} - 1\right)$$

93. $y = f(x) = \ln(\sin x)$

$$f'(x) = \frac{\cos x}{\sin x}$$

$$s = \int_{\pi/4}^{\pi/2}\sqrt{1 + \frac{\cos^2 x}{\sin^2 x}}\, dx = \int_{\pi/4}^{\pi/2}\sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}}\, dx$$

$$= \int_{\pi/4}^{\pi/2}\frac{1}{\sin x}\, dx = \int_{\pi/4}^{\pi/2}\csc x\, dx$$

$$= \Big[-\ln\left|\csc x + \cot x\right|\Big]_{\pi/4}^{\pi/2}$$

$$= -\ln(1) + \ln\left(\sqrt{2} + 1\right)$$

$$= \ln\left(\sqrt{2} + 1\right) \approx 0.8814$$

94. $y = \ln(\cos x), \quad 0 \le x \le \pi/3$

$$y' = \frac{-\sin x}{\cos x} = -\tan x$$

$$\left[1 + (y')^2\right] = 1 + \tan^2 x = \sec^2 x$$

$$s = \int_0^{\pi/3}\sqrt{1 + (y')^2}\, dx = \int_0^{\pi/3}\sec x\, dx$$

$$= \Big[\ln\left|\sec x + \tan x\right|\Big]_0^{\pi/3}$$

$$= \ln\left(2 + \sqrt{3}\right) - \ln(1) = \ln\left(2 + \sqrt{3}\right) \approx 1.317$$

95. $y = 2\sqrt{x}$

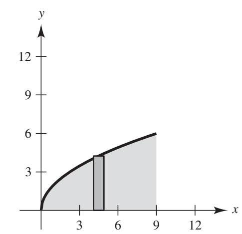
$$y' = \frac{1}{\sqrt{x}}$$

$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x + 1}{x}$$

$$S = 2\pi\int_0^9 2\sqrt{x}\sqrt{\frac{x + 1}{x}}\, dx$$

$$= 2\pi\int_0^9 2\sqrt{x + 1}\, dx$$

$$= \left[4\pi\left(\frac{2}{3}\right)(x + 1)^{3/2}\right]_0^9 = \frac{8\pi}{3}\left(10\sqrt{10} - 1\right) \approx 256.545$$



96. $A = \int_0^2 \frac{1}{2x + 1}\, dx = \left[\frac{1}{2}\ln(2x + 1)\right]_0^2 = \frac{1}{2}\ln 5 \approx 0.805$

$$\bar{x} = \frac{1}{A}\int_0^2 x\left(\frac{1}{2x + 1}\right) dx$$

$$= \frac{1}{A}\int_0^2 \frac{1}{2}\cdot\frac{2x + 1 - 1}{2x + 1}\, dx$$

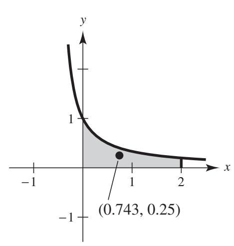
$$= \frac{1}{2A}\int_0^2\left(1 - \frac{1}{2x + 1}\right) dx$$

$$= \frac{1}{2A}\left[x - \frac{1}{2}\ln(2x + 1)\right]_0^2$$

$$= \frac{1}{2A}\left[2 - \frac{1}{2}\ln 5\right] = \frac{4 - \ln 5}{2\ln 5} \approx 0.743$$

$$\bar{y} = \frac{1}{A}\int_0^2 \frac{1}{2}\cdot\frac{1}{2x + 1}\, dx = \frac{1}{2A}\int_0^2 \frac{1}{2x + 1}\, dx$$

$$= \frac{1}{2A}\left[\frac{1}{2}\ln(2x + 1)\right]_0^2 = \frac{\ln 5}{4\ln 5} = 0.25$$



97.
$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$= \frac{1}{3-(-3)}\int_{-3}^{3} \frac{1}{1+x^2}\,dx$$
$$= \frac{1}{6}\Big[\arctan(x)\Big]_{-3}^{3}$$
$$= \frac{1}{6}\Big[\arctan(3) - \arctan(-3)\Big]$$
$$= \frac{1}{3}\arctan(3) \approx 0.4163$$

98.
$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$= \frac{1}{(\pi/n) - 0}\int_0^{\pi/n} \sin(nx)\,dx$$
$$= \frac{n}{\pi}\left[\frac{-1}{n}\cos(nx)\right]_0^{\pi/n}$$
$$= -\frac{1}{\pi}\Big[\cos(\pi) - \cos(0)\Big] = \frac{2}{\pi}$$

99.
$$y = \tan(\pi x)$$
$$y' = \pi\sec^2(\pi x)$$
$$1 + (y')^2 = 1 + \pi^2\sec^4(\pi x)$$
$$s = \int_0^{1/4}\sqrt{1 + \pi^2\sec^4(\pi x)}\,dx \approx 1.0320$$

100.
$$y = x^{2/3}$$
$$y' = \frac{2}{3x^{1/3}}$$
$$1 + (y')^2 = 1 + \frac{4}{9x^{2/3}}$$
$$s = \int_1^8 \sqrt{1 + \frac{4}{9x^{2/3}}}\,dx \approx 7.6337$$

101. (a) $\int \cos^3 x\,dx = \int (1 - \sin^2 x)\cos x\,dx = \sin x - \dfrac{\sin^3 x}{3} + C = \dfrac{1}{3}\sin x(\cos^2 x + 2) + C$

(b) $\int \cos^5 x\,dx = \int (1 - \sin^2 x)^2 \cos x\,dx = \int (1 - 2\sin^2 x + \sin^4 x)\cos x\,dx$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{\sin^5 x}{5} + C = \frac{1}{15}\sin x(3\cos^4 x + 4\cos^2 x + 8) + C$$

(c) $\int \cos^7 x\,dx = \int (1 - \sin^2 x)^3 \cos x\,dx$

$$= \int (1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x)\cos x\,dx$$
$$= \sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$$
$$= \frac{1}{35}\sin x(5\cos^6 x + 6\cos^4 x + 8\cos^2 x + 16) + C$$

(d) $\int \cos^{15} x\,dx = \int (1 - \sin^2 x)^7 \cos x\,dx$

You would expand $(1 - \sin^2 x)^7$.

102. (a) $\int \tan^3 x\,dx = \int (\sec^2 x - 1)\tan x\,dx$

$$= \int \sec^2 x\tan x\,dx - \int \tan x\,dx$$
$$= \frac{\tan^2 x}{2} - \int \tan x\,dx$$
$$\int \tan^3 x\,dx = \frac{\tan^2 x}{2} + \ln|\cos x| + C$$

(b) $\int \tan^5 x\,dx = \int (\sec^2 x - 1)\tan^3 x\,dx$

$$= \frac{\tan^4 x}{4} - \int \tan^3 x\,dx$$

(c) $\int \tan^{2k+1} x\,dx = \int (\sec^2 x - 1)\tan^{2k-1} x\,dx$

$$= \frac{\tan^{2k} x}{2k} - \int \tan^{2k-1} x\,dx$$

(d) You would use these formulas recursively.

103. Let $f(x) = \frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left|x + \sqrt{x^2+1}\right|\right) + C.$

$$f'(x) = \frac{1}{2}\left(x\frac{1}{2}\left(x^2+1\right)^{-1/2}(2x) + \sqrt{x^2+1} + \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{1}{2}\left(x^2+1\right)^{-1/2}(2x)\right)\right)$$

$$= \frac{1}{2}\left(\frac{x^2}{\sqrt{x^2+1}} + \sqrt{x^2+1} + \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)\right)$$

$$= \frac{1}{2}\left(\frac{x^2+\left(x^2+1\right)}{\sqrt{x^2+1}} + \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right)\right)$$

$$= \frac{1}{2}\left(\frac{2x^2+1}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}}\right) = \frac{1}{2}\left(\frac{2\left(x^2+1\right)}{\sqrt{x^2+1}}\right) = \sqrt{x^2+1}$$

So, $\int\sqrt{x^2+1}\,dx = \frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left|x+\sqrt{x^2+1}\right|\right) + C.$

Let $g(x) = \frac{1}{2}\left(x\sqrt{x^2+1} + \operatorname{arcsinh}(x)\right).$

$$g'(x) = \frac{1}{2}\left(x\frac{1}{2}\left(x^2+1\right)^{-1/2}(2x) + \sqrt{x^2+1} + \frac{1}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{2}\left(\frac{x^2}{\sqrt{x^2+1}} + \sqrt{x^2+1} + \frac{1}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{2}\left(\frac{x^2+\left(x^2+1\right)+1}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{2}\left(\frac{2\left(x^2+1\right)}{\sqrt{x^2+1}}\right) = \sqrt{x^2+1}$$

So, $\int\sqrt{x^2+1}\,dx = \frac{1}{2}\left(x\sqrt{x^2+1} + \operatorname{arcsinh}(x)\right) + C.$

104. Let $I = \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}\,dx.$

I is defined and continuous on $[2, 4]$. Note the symmetry: as x goes from 2 to 4, $9 - x$ goes from 7 to 5 and $x + 3$ goes from 5 to 7. So, let $y = 6 - x, dy = -dx.$

$$I = \int_4^2 \frac{\sqrt{\ln(3+y)}}{\sqrt{\ln(3+y)} + \sqrt{\ln(9-y)}}(-dy) = \int_2^4 \frac{\sqrt{\ln(3+y)}}{\sqrt{\ln(3+y)} + \sqrt{\ln(9-y)}}\,dy$$

Adding:

$$2I = \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}\,dx + \int_2^4 \frac{\sqrt{\ln(3+x)}}{\sqrt{\ln(3+x)} + \sqrt{\ln(9-x)}}\,dx = \int_2^4 dx = 2 \Rightarrow I = 1$$

You can easily check this result numerically.

Section 8.2 Integration by Parts

1. Integration by parts is based on the formula for the derivative of a product.

2. You should try letting u be a portion of the integrand whose derivative is a function simpler than u. You should try letting dv be a function whose integral is easy to calculate.

3. Let u be that single term and $dv = dx$.

4. The tabular method is useful for integrals of the form $\int x^n \sin ax\,dx$, $\int x^n \cos ax\,dx$, and $\int x^n e^{ax}\,dx$.

5. $\int xe^{9x}\,dx$

$u = x,\ dv = e^{9x}\,dx$

6. $\int x^2e^{2x}\,dx$

$u = x^2,\ dv = e^{2x}\,dx$

7. $\int (\ln x)^2\,dx$

$u = (\ln x)^2,\ dv = dx$

8. $\int \ln 4x\,dx$

$u = \ln 4x,\ dv = dx$

9. $\int x\sec^2 x\,dx$

$u = x,\ dv = \sec^2 x\,dx$

10. $\int x^2 \cos x\,dx$

$u = x^2,\ dv = \cos x\,dx$

11. $dv = x^3\,dx \Rightarrow v = \int x^3\,dx = \dfrac{x^4}{4}$

$u = \ln x \quad \Rightarrow du = \dfrac{1}{x}\,dx$

$$\begin{aligned}\int x^3 \ln x\,dx &= uv - \int v\,du\\ &= (\ln x)\frac{x^4}{4} - \int\left(\frac{x^4}{4}\right)\frac{1}{x}\,dx\\ &= \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3\,dx\\ &= \frac{x^4}{4}\ln x - \frac{1}{16}x^4 + C\\ &= \frac{1}{16}x^4(4\ln x - 1) + C\end{aligned}$$

12. $dv = e^{x/2}\,dx \quad \Rightarrow \quad v = \int e^{x/2}\,dx = 2e^{x/2}$

$u = 7 - x \quad \Rightarrow \quad du = -dx$

$$\begin{aligned}\int (7 - x)e^{x/2}\,dx &= uv - \int v\,du\\ &= (7 - x)\left(2e^{x/2}\right) + \int 2e^{x/2}\,dx\\ &= 2(7 - x)e^{x/2} + 4e^{x/2} + C\\ &= (18 - 2x)e^{x/2} + C\end{aligned}$$

13. $dv = \sin 4x\,dx \quad \Rightarrow \quad v = \int \sin 4x\,dx = -\dfrac{1}{4}\cos 4x$

$u = 2x + 1 \quad \Rightarrow \quad du = 2\,dx$

$$\begin{aligned}\int (2x + 1)\sin 4x\,dx &= uv - \int v\,du\\ &= (2x + 1)\left(-\frac{1}{4}\cos 4x\right) - \int\left(-\frac{1}{4}\cos 4x\right)(2\,dx)\\ &= -\frac{1}{4}(2x + 1)\cos 4x + \frac{1}{8}\sin 4x + C\end{aligned}$$

14. $dv = \cos 4x\,dx \Rightarrow v = \int \cos 4x\,dx = \dfrac{1}{4}\sin 4x$

$u = x \quad \Rightarrow du = dx$

$$\begin{aligned}\int x\cos 4x\,dx &= uv - \int v\,du\\ &= x\left(\frac{1}{4}\sin 4x\right) - \int \frac{1}{4}\sin 4x\,dx\\ &= \frac{x}{4}\sin 4x + \frac{1}{16}\cos 4x + C\end{aligned}$$

15. $dv = e^{4x}\,dx \quad \Rightarrow \quad v = \int e^{4x}\,dx = \dfrac{1}{4}e^{4x}\,dx$

$u = x \quad \Rightarrow \quad du = dx$

$$\begin{aligned}\int xe^{4x}\,dx &= x\left(\frac{1}{4}e^{4x}\right) - \int\left(\frac{1}{4}e^{4x}\right)dx\\ &= \frac{x}{4}e^{4x} - \frac{1}{16}e^{4x} + C\\ &= \frac{e^{4x}}{16}(4x - 1) + C\end{aligned}$$

16. $dv = e^{-2x}\,dx \Rightarrow v = \int e^{-2x}\,dx = -\frac{1}{2}e^{-2x}$

$u = 5x \Rightarrow du = 5dx$

$$\int \frac{5x}{e^{2x}}\,dx = \int 5xe^{-2x}\,dx$$
$$= (5x)\left(-\frac{1}{2}e^{-2x}\right) - \int\left(-\frac{1}{2}e^{-2x}\right)5dx$$
$$= -\frac{5}{2}xe^{-2x} + \frac{5}{2}\int e^{-2x}\,dx$$
$$= -\frac{5}{2}xe^{-2x} - \frac{5}{4}e^{-2x} + C$$
$$= -\frac{5}{4}e^{-2x}(2x + 1) + C$$

17. Use integration by parts three times.

(1) $dv = e^x\,dx \Rightarrow v = \int e^x\,dx = e^x$

$u = x^3 \Rightarrow du = 3x^2\,dx$

(2) $dv = e^x\,dx \Rightarrow v = \int e^x\,dx = e^x$

$u = x^2 \Rightarrow du = 2x\,dx$

(3) $dv = e^x\,dx \Rightarrow v = \int e^x\,dx = e^x$

$u = x \Rightarrow du = dx$

$$\int x^3e^x\,dx = x^3e^x - 3\int x^2e^x\,dx = x^3e^x - 3x^2e^x + 6\int xe^x\,dx$$
$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C = e^x\left(x^3 - 3x^2 + 6x - 6\right) + C$$

18. $\int \frac{e^{1/t}}{t^2}\,dt = -\int e^{1/t}\left(\frac{-1}{t^2}\right)dt = -e^{1/t} + C$

19. $dv = t\,dt \Rightarrow v = \int t\,dt = \frac{t^2}{2}$

$u = \ln(t + 1) \Rightarrow du = \frac{1}{t + 1}\,dt$

$$\int t\ln(t + 1)\,dt = \frac{t^2}{2}\ln(t + 1) - \frac{1}{2}\int \frac{t^2}{t + 1}\,dt$$
$$= \frac{t^2}{2}\ln(t + 1) - \frac{1}{2}\int\left(t - 1 + \frac{1}{t + 1}\right)dt$$
$$= \frac{t^2}{2}\ln(t + 1) - \frac{1}{2}\left[\frac{t^2}{2} - t + \ln(t + 1)\right] + C$$
$$= \frac{1}{4}\left[2\left(t^2 - 1\right)\ln|t + 1| - t^2 + 2t\right] + C$$

20. $dv = x^5\,dx \Rightarrow v = \int x^5\,dx = \frac{1}{6}x^6$

$u = \ln 3x \Rightarrow du = \frac{1}{x}\,dx$

$$\int x^5\ln 3x\,dx = \frac{x^6}{6}\ln 3x - \int \frac{x^6}{6}\left(\frac{1}{x}\right)dx$$
$$= \frac{x^6}{6}\ln 3x - \frac{x^6}{36} + C$$

21. Let $u = \ln x$, $du = \frac{1}{x}\,dx$.

$$\int \frac{(\ln x)^2}{x}\,dx = \int(\ln x)^2\left(\frac{1}{x}\right)dx = \frac{(\ln x)^3}{3} + C$$

22. $dv = x^{-3}\,dx \Rightarrow v = \int x^{-3}\,dx = -\frac{1}{2}x^{-2}$

$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$$\int \frac{\ln x}{x^3}\,dx = -\frac{1}{2}x^{-2}\ln x - \int\left(-\frac{1}{2}x^{-2}\right)\frac{1}{x}\,dx$$
$$= -\frac{1}{2x^2}\ln x + \frac{1}{2}\int x^{-3}\,dx$$
$$= -\frac{1}{2x^2}\ln x + \left(\frac{1}{2}\right)\frac{x^{-2}}{-2} + C$$
$$= -\frac{1}{2x^2}\ln x - \frac{1}{4x^2} + C$$

23. $dv = \dfrac{1}{(2x+1)^2}\,dx \quad\Rightarrow\quad v = \int (2x+1)^{-2}\,dx$

$$= -\frac{1}{2(2x+1)}$$

$u = xe^{2x} \quad\Rightarrow\quad du = \left(2xe^{2x} + e^{2x}\right)dx$

$$= e^{2x}(2x+1)\,dx$$

$$\int \frac{xe^{2x}}{(2x+1)^2}\,dx = -\frac{xe^{2x}}{2(2x+1)} + \int \frac{e^{2x}}{2}\,dx$$

$$= \frac{-xe^{2x}}{2(2x+1)} + \frac{e^{2x}}{4} + C = \frac{e^{2x}}{4(2x+1)} + C$$

24. $dv = \dfrac{x}{(x^2+1)^2}\,dx \quad\Rightarrow\quad v = \int (x^2+1)^{-2}\,x\,dx = -\dfrac{1}{2(x^2+1)}$

$u = x^2e^{x^2} \quad\Rightarrow\quad du = \left(2x^3e^{x^2} + 2xe^{x^2}\right)dx = 2xe^{x^2}(x^2+1)\,dx$

$$\int \frac{x^3e^{x^2}}{(x^2+1)^2}\,dx = -\frac{x^2e^{x^2}}{2(x^2+1)} + \int xe^{x^2}\,dx = -\frac{x^2e^{x^2}}{2(x^2+1)} + \frac{e^{x^2}}{2} + C = \frac{e^{x^2}}{2(x^2+1)} + C$$

25. $dv = \sqrt{x-5}\,dx \quad\Rightarrow\quad v = \int (x-5)^{1/2}\,dx = \frac{2}{3}(x-5)^{3/2}$

$u = x \quad\Rightarrow\quad du = dx$

$$\int x\sqrt{x-5}\,dx = x\tfrac{2}{3}(x-5)^{3/2} - \int \tfrac{2}{3}(x-5)^{3/2}\,dx$$

$$= \tfrac{2}{3}x(x-5)^{3/2} - \tfrac{4}{15}(x-5)^{5/2} + C$$

$$= \tfrac{2}{15}(x-5)^{3/2}\left(5x - 2(x-5)\right) + C$$

$$= \tfrac{2}{15}(x-5)^{3/2}(3x+10) + C$$

26. $dv = (1-6x)^{-1/2}\,dx \quad\Rightarrow\quad v = \int (1-6x)^{-1/2}\,dx = -\dfrac{1}{3}(1-6x)^{1/2}$

$u = 2x \quad\Rightarrow\quad du = 2dx$

$$\int \frac{2x}{\sqrt{1-6x}}\,dx = \int (1-6x)^{-1/2}\,2x\,dx$$

$$= uv - \int v\,du$$

$$= (2x)\left[-\frac{1}{3}(1-6x)^{1/2}\right] - \int -\frac{1}{3}(1-6x)^{1/2}\,(2\,dx)$$

$$= -\frac{2x}{3}(1-6x)^{1/2} - \frac{2}{27}(1-6x)^{3/2} + C$$

$$= \sqrt{1-6x}\left[-\frac{2x}{3} - \frac{2}{27}(1-6x)\right] + C$$

$$= \sqrt{1-6x}\left[\frac{-6x-2}{27}\right] + C$$

27. $dv = \csc^2 x\, dx \quad \Rightarrow \quad v = \int \csc^2 x\, dx = -\cot x$

$u = x \quad \Rightarrow \quad du = dx$

$$\int x \csc^2 x\, dx = uv - \int v\, du$$
$$= x(-\cot x) - \int(-\cot x)dx$$
$$= -x \cot x + \ln|\sin x| + C$$

28. $u = t, du = dt, dv = \csc t \cot t\, dt, v = -\csc t$

$$\int t \csc t \cot t\, dt = -t \csc t + \int \csc t\, dt = -t \csc t - \ln|\csc t + \cot t| + C$$

29. Use integration by parts three times.

(1) $u = x^3, du = 3x^2\, dx, dv = \sin x\, dx, v = -\cos x$

$$\int x^3 \sin dx = -x^3 \cos x + 3\int x^2 \cos x\, dx$$

(2) $u = x^2, du = 2x\, dx, dv = \cos x\, dx, v = \sin x$

$$\int x^3 \sin x\, dx = -x^3 \cos x + 3\left(x^2 \sin x - 2\int x \sin x\, dx\right) = -x^3 \cos x + 3x^2 \sin x - 6\int x \sin x\, dx$$

(3) $u = x, du = dx, dv = \sin x\, dx, v = -\cos x$

$$\int x^3 \sin x\, dx = -x^3 \cos x + 3x^2 \sin x - 6\left(-x \cos x + \int \cos x\, dx\right)$$
$$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$$
$$= \left(6x - x^3\right) \cos x + \left(3x^2 - 6\right) \sin x + C$$

30. Use integration by parts twice.

(1) $u = x^2, du = 2x\, dx, dv = \cos x\, dx, v = \sin x$

$$\int x^2 \cos x\, dx = x^2 \sin x - 2\int x \sin x\, dx$$

(2) $u = x, du = dx, dv = \sin x\, dx, v = -\cos x$

$$\int x^2 \cos x\, dx = x^2 \sin x - 2\left(-x \cos x + \int \cos x\, dx\right) = x^2 \sin x + 2x \cos x - 2 \sin x + C$$

31. $dv = dx \quad \Rightarrow \quad v = \int dx = x$

$u = \arctan x \quad \Rightarrow \quad du = \dfrac{1}{1 + x^2}\, dx$

$$\int \arctan x\, dx = x \arctan x - \int \frac{x}{1 + x^2}\, dx$$
$$= x \arctan x - \frac{1}{2} \ln\left(1 + x^2\right) + C$$

32. $dv = dx \quad \Rightarrow \quad v = \int dx = x$

$u = \arccos x \quad \Rightarrow \quad du = -\dfrac{1}{\sqrt{1 - x^2}}\, dx$

$$4\int \arccos x\, dx = 4\left(x \arccos x + \int \frac{x}{\sqrt{1 - x^2}}\, dx\right)$$
$$= 4\left(x \arccos x - \sqrt{1 - x^2}\right) + C$$

33. Use integration by parts twice.

(1) $dv = e^{-3x}\,dx \Rightarrow v = \int e^{-3x}\,dx = -\frac{1}{3}e^{-3x}$

$u = \sin 5x \Rightarrow du = 5\cos 5x\,dx$

$$\int e^{-3x}\sin 5x\,dx = \sin 5x\left(-\tfrac{1}{3}e^{-3x}\right) - \int\left(-\tfrac{1}{3}e^{-3x}\right)5\cos x\,dx = -\tfrac{1}{3}e^{-3x}\sin 5x + \tfrac{5}{3}\int e^{-3x}\cos 5x\,dx$$

(2) $dv = e^{-3x}\,dx \Rightarrow v = \int e^{-3x}\,dx = -\frac{1}{3}e^{-3x}$

$u = \cos 5x \Rightarrow du = -5\sin 5x\,dx$

$$\int e^{-3x}\sin 5x\,dx = -\tfrac{1}{3}e^{-3x}\sin 5x + \tfrac{5}{3}\left[\left(-\tfrac{1}{3}e^{-3x}\cos 5x - \int\left(-\tfrac{1}{3}e^{-3x}\right)(-5\sin 5x)dx\right]$$

$$= -\tfrac{1}{3}e^{-3x}\sin 5x - \tfrac{5}{9}e^{-3x}\cos 5x - \tfrac{25}{9}\int e^{-3x}\sin 5x\,dx$$

$$\left(1 + \tfrac{25}{9}\right)\int e^{-3x}\sin 5x\,dx = -\tfrac{1}{3}e^{-3x}\sin 5x - \tfrac{5}{9}e^{-3x}\cos 5x$$

$$\int e^{-3x}\sin 5x\,dx = \tfrac{9}{34}\left(-\tfrac{1}{3}e^{-3x}\sin 5x - \tfrac{5}{9}e^{-3x}\cos 5x\right) + C = -\tfrac{3}{34}e^{-3x}\sin 5x - \tfrac{5}{34}e^{-3x}\cos 5x + C$$

34. Use integration by parts twice.

(1) $dv = e^{4x}\,dx \Rightarrow v = \int e^{4x}\,dx = \frac{1}{4}e^{4x}$

$u = \cos 2x \Rightarrow du = -2\sin 2x\,dx$

$$\int e^{4x}\cos 2x\,dx = \tfrac{1}{4}e^{4x}\cos 2x - \int\tfrac{1}{4}e^{4x}(-2\sin 2x)\,dx$$

$$= \tfrac{1}{4}e^{4x}\cos 2x + \tfrac{1}{2}\int e^{4x}\sin 2x\,dx$$

(2) $dv = e^{4x}\,dx \Rightarrow v = \int e^{4x}\,dx = \frac{1}{4}e^{4x}$

$u = \sin 2x \Rightarrow du = 2\cos 2x\,dx$

$$\int e^{4x}\cos 2x\,dx = \tfrac{1}{4}e^{4x}\cos 2x + \tfrac{1}{2}\left[\tfrac{1}{4}e^{4x}\sin 2x - \int\tfrac{1}{4}e^{4x}(2\cos 2x)\,dx\right]$$

$$= \tfrac{1}{4}e^{4x}\cos 2x + \tfrac{1}{8}e^{4x}\sin 2x - \tfrac{1}{4}\int e^{4x}\cos 2x\,dx + C$$

$$\left(1 + \tfrac{1}{4}\right)\int e^{4x}\cos 2x\,dx = \tfrac{1}{4}e^{4x}\cos 2x + \tfrac{1}{8}e^{4x}\sin 2x + C$$

$$\int e^{4x}\cos 2x\,dx = \tfrac{1}{5}e^{4x}\cos 2x + \tfrac{1}{10}e^{4x}\sin 2x + C$$

35. $dv = dx \Rightarrow v = x$

$u = \ln x \Rightarrow du = \dfrac{1}{x}\,dx$

$y' = \ln x$

$$y = \int \ln x\,dx = x\ln x - \int x\left(\frac{1}{x}\right)dx = x\ln x - x + C = x(-1 + \ln x) + C$$

36. $dv = dx \Rightarrow v = \int dx = x$

$$u = \arctan\frac{x}{2} \Rightarrow du = \frac{1}{1 + (x/2)^2}\left(\frac{1}{2}\right)dx = \frac{2}{4 + x^2}\,dx$$

$$y = \int \arctan\frac{x}{2}\,dx = x\arctan\frac{x}{2} - \int\frac{2x}{4 + x^2}\,dx = x\arctan\frac{x}{2} - \ln\left(4 + x^2\right) + C$$

37. Use integration by parts twice.

(1) $dv = \dfrac{1}{\sqrt{3+5t}}\,dt \;\Rightarrow\; v = \displaystyle\int (3+5t)^{-1/2}\,dt = \frac{2}{5}(3+5t)^{1/2}$

$u = t^2 \;\Rightarrow\; du = 2t\,dt$

$$\int \frac{t^2}{\sqrt{3+5t}}\,dt = \frac{2}{5}t^2(3+5t)^{1/2} - \int \frac{2}{5}(3+5t)^{1/2}2t\,dt$$
$$= \frac{2}{5}t^2(3+5t)^{1/2} - \frac{4}{5}\int t(3+5t)^{1/2}\,dt$$

(2) $dv = (3+5t)^{1/2}\,dt \;\Rightarrow\; v = \displaystyle\int (3+5t)^{1/2}\,dt = \frac{2}{15}(3+5t)^{3/2}$

$u = t \;\Rightarrow\; du = dt$

$$\int \frac{t^2}{\sqrt{3+5t}}\,dt = \frac{2}{5}t^2(3+5t)^{1/2} - \frac{4}{5}\left[\frac{2}{15}t(3+5t)^{3/2} - \int \frac{2}{15}(3+5t)^{3/2}\,dt\right]$$
$$= \frac{2}{5}t^2(3+5t)^{1/2} - \frac{8}{75}t(3+5t)^{3/2} + \frac{8}{75}\int (3+5t)^{3/2}\,dt$$
$$= \frac{2}{5}t^2(3+5t)^{1/2} - \frac{8}{75}t(3+5t)^{3/2} + \frac{16}{1875}(3+5t)^{5/2} + C$$
$$= \frac{2}{1875}\sqrt{3+5t}\left(375t^2 - 100t(3+5t) + 8(3+5t)^2\right) + C$$
$$= \frac{2}{625}\sqrt{3+5t}\left(25t^2 - 20t + 24\right) + C$$

38. Use integration by parts twice.

(1) $dv = \sqrt{x-3}\,dx \;\Rightarrow\; v = \displaystyle\int (x-3)^{1/2}dx = \tfrac{2}{3}(x-3)^{3/2}$

$u = x^2 \;\Rightarrow\; du = 2x\,dx$

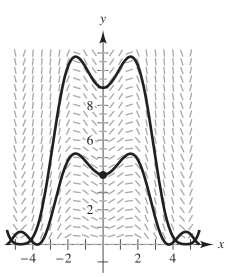
$$\int x^2\sqrt{x-3}\,dx = \tfrac{2}{3}x^2(x-3)^{3/2} - \int \tfrac{2}{3}(x-3)^{3/2}2x\,dx$$
$$= \tfrac{2}{3}x^2(x-3)^{3/2} - \tfrac{4}{3}\int (x-3)^{3/2}\,x\,dx$$

(2) $dv = (x-3)^{3/2}dx \;\Rightarrow\; v = \displaystyle\int (x-3)^{3/2}\,dx = \tfrac{2}{5}(x-3)^{5/2}$

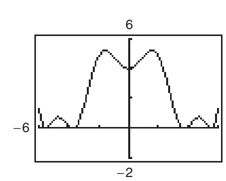
$u = x \;\Rightarrow\; du = dx$

$$\int x^2\sqrt{x-3}\,dx = \tfrac{2}{3}x^2(x-3)^{3/2} - \tfrac{4}{3}\left[\tfrac{2}{5}x(x-3)^{5/2} - \int \tfrac{2}{5}(x-3)^{5/2}\,dx\right]$$
$$= \tfrac{2}{3}x^2(x-3)^{3/2} - \tfrac{8}{15}x(x-3)^{5/2} + \tfrac{8}{15}\left[\tfrac{2}{7}(x-3)^{7/2}\right] + C$$
$$= \tfrac{2}{35}(x-3)^{3/2}\left(5x^2 + 12x + 24\right) + C$$

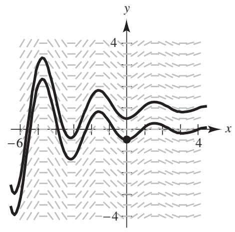
39. (a)



(b) $$\frac{dy}{dx} = x\sqrt{y}\cos x, \quad (0, 4)$$
$$\int \frac{dy}{\sqrt{y}} = \int x\cos x\,dx$$
$$\int y^{-1/2}\,dy = \int x\cos x\,dx \qquad (u = x, du = dx, dv = \cos x\,dx, v = \sin x)$$
$$2y^{1/2} = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$
$$(0, 4):\ 2(4)^{1/2} = 0 + 1 + C \Rightarrow C = 3$$
$$2\sqrt{y} = x\sin x + \cos x + 3$$



40. (a)



(b) $\dfrac{dy}{dx} = e^{-x/3}\sin 2x, \quad \left(0, -\dfrac{18}{37}\right)$

$y = \int e^{-x/3}\sin 2x\,dx$

Use integration by parts twice.

(1) $u = \sin 2x,\ du = 2\cos 2x$

$dv = e^{-x/3}\,dx,\ v = -3e^{-x/3}$

$\int e^{-x/3}\sin 2x\,dx = -3e^{-x/3}\sin 2x + \int 6e^{-x/3}\cos 2x\,dx$

(2) $u = \cos 2x,\ du = -2\sin 2x$

$dv = e^{-x/3}\,dx,\ v = -3e^{-x/3}$

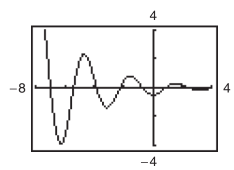
$$\int e^{-x/3}\sin 2x\,dx = -3e^{-x/3}\sin 2x + 6\left(-3e^{-x/3}\cos 2x - \int 6e^{-x/3}\sin 2x\,dx\right) + C$$

$$37\int e^{-x/3}\sin 2x\,dx = -3e^{-x/3}\sin 2x - 18e^{-x/3}\cos 2x + C$$

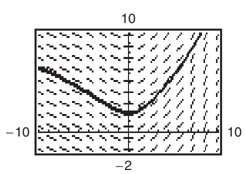
$$y = \int e^{-x/3}\sin 2x\,dx = \frac{1}{37}\left(-3e^{-x/3}\sin 2x - 18e^{-x/3}\cos 2x\right) + C$$

$$\left(0, \frac{-18}{37}\right): \frac{-18}{37} = \frac{1}{37}[0 - 18] + C \Rightarrow C = 0$$

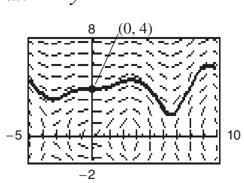
$$y = \frac{-1}{37}\left(3e^{-x/3}\sin 2x + 18e^{-x/3}\cos 2x\right)$$



41. $\dfrac{dy}{dx} = \dfrac{x}{y}e^{x/8},\ y(0) = 2$



42. $\dfrac{dy}{dx} = \dfrac{x}{y}\sin x,\ y(0) = 4$



43. $u = x,\ du = dx,\ dv = e^{x/2}\,dx,\ v = 2e^{x/2}$

$$\int xe^{x/2}\,dx = 2xe^{x/2} - \int 2e^{x/2}\,dx$$
$$= 2xe^{x/2} - 4e^{x/2} + C$$

So,

$$\int_0^3 xe^{x/2}\,dx = \left[2xe^{x/2} - 4e^{x/2}\right]_0^3$$
$$= \left(6e^{3/2} - 4e^{3/2}\right) - (-4)$$
$$= 4 + 2e^{3/2} \approx 12.963$$

44. Use integration by parts twice.

(1) $u = x^2, du = 2x\,dx, dv = e^{-2x}\,dx,$

$v = -\frac{1}{2}e^{-2x}$

$$\int x^2e^{-2x}\,dx = -\frac{1}{2}x^2e^{-2x} - \int\left(-\frac{1}{2}e^{-2x}\right)2x\,dx$$
$$= -\frac{1}{2}x^2e^{-2x} + \int xe^{-2x}\,dx$$

(2) $u = x, du = dx, dv = e^{-2x}\,dx, v = -\frac{1}{2}e^{-2x}$

$$\int x^2e^{-2x}\,dx = -\frac{1}{2}x^2e^{-2x} + \left(-\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\,dx\right)$$
$$= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$$
$$= e^{-2x}\left(-\frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{4}\right)$$

So,

$$\int_0^2 x^2e^{-2x}\,dx = \left[e^{-2x}\left(-\frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{4}\right)\right]_0^2$$
$$= e^{-4}\left(-2 - 1 - \frac{1}{4}\right) - \left(-\frac{1}{4}\right)$$
$$= \frac{-13}{4e^4} + \frac{1}{4} \approx 0.190$$

45. $u = x, du = dx, dv = \cos 2x\,dx, v = \frac{1}{2}\sin 2x$

$$\int x\cos 2x\,dx = \frac{1}{2}x\sin 2x - \int\frac{1}{2}\sin 2x\,dx$$
$$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$$

So,

$$\int_0^{\pi/4} x\cos 2x\,dx = \left[\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right]_0^{\pi/4}$$
$$= \left(\frac{\pi}{8}(1) + 0\right) - \left(0 + \frac{1}{4}\right)$$
$$= \frac{\pi}{8} - \frac{1}{4} \approx 0.143$$

46. $dv = \sin 2x\,dx \Rightarrow v = \int\sin 2x\,dx = -\frac{1}{2}\cos 2x$

$u = x \Rightarrow du = dx$

$$\int x\sin 2x\,dx = \frac{-1}{2}x\cos 2x + \frac{1}{2}\int\cos 2x\,dx$$
$$= \frac{-1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$$
$$= \frac{1}{4}(\sin 2x - 2x\cos 2x) + C$$

So,

$$\int_0^{\pi} x\sin 2x\,dx = \left[\frac{1}{4}(\sin 2x - 2x\cos 2x)\right]_0^{\pi} = -\frac{\pi}{2}.$$

47. $u = \arccos x, du = -\frac{1}{\sqrt{1 - x^2}}\,dx, dv = dx, v = x$

$$\int\arccos x\,dx = x\arccos x + \int\frac{x}{\sqrt{1 - x^2}}\,dx$$
$$= x\arccos x - \sqrt{1 - x^2} + C$$

So,

$$\int_0^{1/2}\arccos x = \left[x\arccos x - \sqrt{1 - x^2}\right]_0^{1/2}$$
$$= \frac{1}{2}\arccos\left(\frac{1}{2}\right) - \sqrt{\frac{3}{4}} + 1$$
$$= \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1 \approx 0.658.$$

48. $dv = x\,dx \Rightarrow v = \int x\,dx = \frac{x^2}{2}$

$u = \arcsin x^2 \Rightarrow du = \frac{2x}{\sqrt{1 - x^4}}\,dx$

$$\int x\arcsin x^2\,dx = \frac{x^2}{2}\arcsin x^2 - \int\frac{x^3}{\sqrt{1 - x^4}}\,dx$$
$$= \frac{x^2}{2}\arcsin x^2 + \frac{1}{4}(2)\left(1 - x^4\right)^{1/2} + C$$
$$= \frac{1}{2}\left(x^2\arcsin x^2 + \sqrt{1 - x^4}\right) + C$$

So, $\int_0^1 x\arcsin x^2\,dx = \frac{1}{2}\left[x^2\arcsin x^2 + \sqrt{1 - x^4}\right]_0^1$

$$= \frac{1}{4}(\pi - 2).$$

49. Use integration by parts twice.

(1) $dv = e^x\,dx \Rightarrow v = \int e^x\,dx = e^x$ (2) $dv = e^x\,dx \Rightarrow v = \int e^x\,dx = e^x$

$u = \sin x \Rightarrow du = \cos x\,dx$ $u = \cos x \Rightarrow du = -\sin x\,dx$

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$$

$$2\int e^x \sin x\,dx = e^x(\sin x - \cos x)$$

$$\int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

So, $\int_0^1 e^x \sin x\,dx = \left[\frac{e^x}{2}(\sin x - \cos x)\right]_0^1 = \frac{e}{2}(\sin 1 - \cos 1) + \frac{1}{2} = \frac{e(\sin 1 - \cos 1) + 1}{2} \approx 0.909.$

50. $u = \ln(4 + x^2),\ du = \frac{2x}{4 + x^2}\,dx,\ dv = dx,\ v = x$

$$\begin{aligned}\int \ln(4 + x^2)\,dx &= x\ln(4 + x^2) - \int \frac{2x^2}{4 + x^2}\,dx\\ &= x\ln(4 + x^2) - 2\int\left(1 - \frac{4}{4 + x^2}\right)dx\\ &= x\ln(4 + x^2) - 2\left(x - \frac{4}{2}\arctan\frac{x}{2}\right) + C\\ &= x\ln(4 + x^2) - 2x + 4\arctan\frac{x}{2} + C\end{aligned}$$

So, $\int_0^1 \ln(4 + x^2)\,dx = \left[x\ln(4 + x^2) - 2x + 4\arctan\frac{x}{2}\right]_0^1 = \left(\ln 5 - 2 + 4\arctan\left(\frac{1}{2}\right)\right) \approx 1.464.$

51. $dv = x\,dx,\ v = \frac{x^2}{2},\ u = \operatorname{arcsec} x,\ du = \frac{1}{x\sqrt{x^2 - 1}}\,dx$

$$\int x \operatorname{arcsec} x\,dx = \frac{x^2}{2}\operatorname{arcsec} x - \int \frac{x^2/2}{x\sqrt{x^2 - 1}}\,dx = \frac{x^2}{2}\operatorname{arcsec} x - \frac{1}{4}\int \frac{2x}{\sqrt{x^2 - 1}}\,dx = \frac{x^2}{2}\operatorname{arcsec} x - \frac{1}{2}\sqrt{x^2 - 1} + C$$

So,

$$\int_2^4 x \operatorname{arcsec} x\,dx = \left[\frac{x^2}{2}\operatorname{arcsec} x - \frac{1}{2}\sqrt{x^2 - 1}\right]_2^4 = \left(8\operatorname{arcsec} 4 - \frac{\sqrt{15}}{2}\right) - \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) = 8\operatorname{arcsec} 4 - \frac{\sqrt{15}}{2} + \frac{\sqrt{3}}{2} - \frac{2\pi}{3} \approx 7.380.$$

52. $u = x,\ du = dx,\ dv = \sec^2 2x\,dx,\ v = \frac{1}{2}\tan 2x$

$$\int x\sec^2 2x\,dx = \frac{1}{2}x\tan 2x - \int \frac{1}{2}\tan 2x\,dx = \frac{1}{2}x\tan 2x + \frac{1}{4}\ln|\cos 2x| + C$$

So,

$$\int_0^{\pi/8} x\sec^2 2x\,dx = \left[\frac{1}{2}x\tan 2x + \frac{1}{4}\ln|\cos 2x|\right]_0^{\pi/8} = \frac{\pi}{16}(1) + \frac{1}{4}\ln\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{16} - \frac{1}{8}\ln(2) \approx 0.1097.$$

53. $$\begin{aligned}\int x^2 e^{2x}\,dx &= x^2\left(\tfrac{1}{2}e^{2x}\right) - (2x)\left(\tfrac{1}{4}e^{2x}\right) + 2\left(\tfrac{1}{8}e^{2x}\right) + C\\ &= \tfrac{1}{2}x^2e^{2x} - \tfrac{1}{2}xe^{2x} + \tfrac{1}{4}e^{2x} + C\\ &= \tfrac{1}{4}e^{2x}(2x^2 - 2x + 1) + C\end{aligned}$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^2	e^{2x}
−	$2x$	$\frac{1}{2}e^{2x}$
+	2	$\frac{1}{4}e^{2x}$
−	0	$\frac{1}{8}e^{2x}$

54. $\int (1 - x)(e^{-x} + 1)\,dx = (1 - x)(-e^{-x} + x) - (-1)\left(e^{-x} + \frac{x^2}{2}\right) + C$

$$= \left(-e^{-x} + x + xe^{-x} - x^2\right) + \left(e^{-x} + \frac{x^2}{2}\right) + C$$

$$= -\frac{x^2}{2} + x + xe^{-x} + C$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	$(1 - x)$	$e^{-x} + 1$
–	-1	$-e^{-x} + x$
+	0	$e^{-x} + \frac{x^2}{2}$

55. $\int (x + 2)^2 \sin x\,dx = (x + 2)^2(-\cos x) - 2(x + 2)(-\sin x) + 2(\cos x) + C$

$$= -\cos x\,(x + 2)^2 + 2\sin x\,(x + 2) + 2\cos x + C$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	$(x + 2)^2$	$\sin x$
–	$2(x + 2)$	$-\cos x$
+	2	$-\sin x$
–	0	$\cos x$

56. $\int x^3 \cos 2x\,dx = x^3\left(\frac{1}{2}\sin 2x\right) - 3x^2\left(-\frac{1}{4}\cos 2x\right) + 6x\left(-\frac{1}{8}\sin 2x\right) - 6\left(\frac{1}{16}\cos 2x\right) + C$

$$= \frac{1}{2}x^3\sin 2x + \frac{3}{4}x^2\cos 2x - \frac{3}{4}x\sin 2x - \frac{3}{8}\cos 2x + C$$

$$= \frac{1}{8}\left(4x^3\sin 2x + 6x^2\cos 2x - 6x\sin 2x - 3\cos 2x\right) + C$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^3	$\cos 2x$
–	$3x^2$	$\frac{1}{2}\sin 2x$
+	$6x$	$-\frac{1}{4}\cos 2x$
–	6	$-\frac{1}{8}\sin 2x$
+	0	$\frac{1}{16}\cos 2x$

57. $\int (6 + x)\sqrt{4x + 9}\,dx = (6 + x)\left(\frac{1}{6}\right)(4x + 9)^{3/2} - \frac{1}{60}(4x + 9)^{5/2} + C$

$$= \frac{1}{60}(4x + 9)^{3/2}[60 + 10x - 4x - 9] + C$$

$$= \frac{1}{60}(4x + 9)^{3/2}(51 + 6x) + C$$

$$= \frac{1}{20}(4x + 9)^{3/2}(17 + 2x) + C$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	$6 + x$	$(4x + 9)^{1/2}$
–	1	$\frac{1}{6}(4x + 9)^{3/2}$
+	0	$\frac{1}{60}(4x + 9)^{5/2}$

58. $\int x^2(x-2)^{3/2}\,dx = \frac{2}{5}x^2(x-2)^{5/2} - \frac{8}{35}x(x-2)^{7/2} + \frac{16}{315}(x-2)^{9/2} + C = \frac{2}{315}(x-2)^{5/2}(35x^2 + 40x + 32) + C$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^2	$(x-2)^{3/2}$
−	$2x$	$\frac{2}{5}(x-2)^{5/2}$
+	2	$\frac{4}{35}(x-2)^{7/2}$
−	0	$\frac{8}{315}(x-2)^{9/2}$

59. Answers will vary. *Sample answer*: $\int x^3 \sin x\,dx$.

It takes three applications of integration by parts for the term x^3 to become a constant.

Other possible answers: $\int x^3 \cos x\,dx$, $\int x^3 e^x\,dx$

60. In order for the integration by parts technique to be efficient, you want dv to be the most complicated portion of the integrand and you want u to be the portion of the integrand whose derivative is a function simpler than u. Suppose you let $u = \sin x$ and $dv = x\,dx$. Then $du = \cos x\,dx$ and $v = x^2/2$. So

$$\int x \sin x\,dx = uv - \int v\,du = \frac{x^2}{2}\sin x - \int \frac{x^2}{2}\cos x\,dx,$$

which is a more complicated integral than the original one.

61. (a) No

Substitution

(b) Yes

$u = \ln x$, $dv = x\,dx$

(c) Yes

$u = x^2$, $dv = e^{-3x}\,dx$

(d) No

Substitution

(e) Yes. Let $u = x$ and

$dv = \dfrac{1}{\sqrt{x+1}}\,dx$.

(Substitution also works. Let $u = \sqrt{x+1}$.)

(f) No

Substitution

62. (a) The slope of f at $x = 2$ is approximately 1.4 because $f'(2) \approx 1.4$.

(b) $f' < 0$ on $(0, 1) \Rightarrow f$ is decreasing on $(0, 1)$.
$f' > 0$ on $(1, \infty) \Rightarrow f$ is increasing on $(1, \infty)$.

63. $u = \sqrt{x} \Rightarrow u^2 = x \Rightarrow 2u\,du = dx$

$\int \sin\sqrt{x}\,dx = \int \sin u(2u\,du) = 2\int u \sin u\,du$

Integration by parts:

$w = u$, $dw = du$, $dv = \sin u\,du$, $v = -\cos u$

$$\begin{aligned}2\int u \sin u\,du &= 2\left(-u\cos u + \int \cos u\,du\right)\\ &= 2(-u\cos u + \sin u) + C\\ &= 2\left(-\sqrt{x}\cos\sqrt{x} + \sin\sqrt{x}\right) + C\end{aligned}$$

64. $u = x^2$, $du = 2x\,dx$

$\int 2x^3\cos(x^2)\,dx = \int x^2\cos(x^2)(2x)\,dx = \int u\cos u\,du$

Integration by parts:

$w = u$, $dw = du$, $dv = \cos u\,du$, $v = \sin u$

$$\begin{aligned}\int u\cos u\,du &= u\sin u - \int \sin u\,du\\ &= u\sin u + \cos u + C\\ &= x^2\sin(x^2) + \cos(x^2) + C\end{aligned}$$

65. $u = x^2$, $du = 2x\,dx$

$\int x^5 e^{x^2}\,dx = \frac{1}{2}\int e^{x^2}x^4\,2x\,dx = \frac{1}{2}\int e^u u^2\,du$

Integration by parts twice.

(1) $w = u^2$, $dw = 2u\,du$, $dv = e^u\,du$, $v = e^u$

$$\begin{aligned}\frac{1}{2}\int e^u u^2\,du &= \frac{1}{2}\left[u^2e^u - \int 2ue^u\,du\right]\\ &= \frac{1}{2}u^2e^u - \int ue^u\,du\end{aligned}$$

(2) $w = u$, $dw = du$, $dv = e^u\,du$, $v = e^u$

$$\begin{aligned}\frac{1}{2}\int e^u u^2\,du &= \frac{1}{2}u^2e^u - \left(ue^u - \int e^u\,du\right)\\ &= \frac{1}{2}u^2e^u - ue^u + e^u + C\\ &= \frac{1}{2}x^4e^{x^2} - x^2e^{x^2} + e^{x^2} + C\\ &= \frac{e^{x^2}}{2}(x^4 - 2x^2 + 2) + C\end{aligned}$$

66. Let $u = \sqrt{2x}, u^2 = 2x, 2u\,du = 2dx.$

$$\int e^{\sqrt{2x}}\,dx = \int e^u(u\,du)$$

Now use integration by parts.

$$dv = e^u\,du \Rightarrow v = \int e^u\,du = e^u$$

$$w = u \Rightarrow dw = du$$

$$\begin{aligned}\int e^{\sqrt{2x}}\,dx &= ue^u - \int e^u\,du \\ &= ue^u - e^u + C \\ &= \sqrt{2x}\,e^{\sqrt{2x}} - e^{\sqrt{2x}} + C\end{aligned}$$

67. (a) $dv = \dfrac{x}{\sqrt{4+x^2}}\,dx \Rightarrow v = \int\left(4+x^2\right)^{-1/2}x\,dx = \sqrt{4+x^2}$

$u = x^2 \Rightarrow du = 2x\,dx$

$$\begin{aligned}\int \frac{x^3}{\sqrt{4+x^2}}\,dx &= x^2\sqrt{4+x^2} - 2\int x\sqrt{4+x^2}\,dx \\ &= x^2\sqrt{4+x^2} - \frac{2}{3}\left(4+x^2\right)^{3/2} + C = \frac{1}{3}\sqrt{4+x^2}\left(x^2-8\right) + C\end{aligned}$$

(b) $u = 4 + x^2 \Rightarrow x^2 = u - 4$ and $2x\,dx = du \Rightarrow x\,dx = \dfrac{1}{2}du$

$$\begin{aligned}\int \frac{x^3}{\sqrt{4+x^2}}\,dx &= \int \frac{x^2}{\sqrt{4+x^2}}x\,dx = \int\left(\frac{u-4}{\sqrt{u}}\right)\frac{1}{2}\,du \\ &= \frac{1}{2}\int\left(u^{1/2} - 4u^{-1/2}\right)du = \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 8u^{1/2}\right) + C \\ &= \frac{1}{3}u^{1/2}(u-12) + C \\ &= \frac{1}{3}\sqrt{4+x^2}\left[\left(4+x^2\right) - 12\right] + C = \frac{1}{3}\sqrt{4+x^2}\left(x^2-8\right) + C\end{aligned}$$

68. (a) $dv = \sqrt{4-x}\,dx \Rightarrow v = \int(4-x)^{1/2}\,dx = -\frac{2}{3}(4-x)^{3/2}$

$u = x \Rightarrow du = dx$

$$\begin{aligned}\int x\sqrt{4-x}\,dx &= -\tfrac{2}{3}x(4-x)^{3/2} + \tfrac{2}{3}\int(4-x)^{3/2}\,dx \\ &= -\tfrac{2}{3}x(4-x)^{3/2} - \tfrac{4}{15}(4-x)^{5/2} + C \\ &= -\tfrac{2}{15}(4-x)^{3/2}\left[5x + 2(4-x)\right] + C = -\tfrac{2}{15}(4-x)^{3/2}(3x+8) + C\end{aligned}$$

(b) $u = 4 - x \Rightarrow x = 4 - u$ and $dx = -du$

$$\begin{aligned}\int x\sqrt{4-x}\,dx &= -\int(4-u)\sqrt{u}\,du \\ &= -\int\left(4u^{1/2} - u^{3/2}\right)du \\ &= -\tfrac{8}{3}u^{3/2} + \tfrac{2}{5}u^{5/2} + C \\ &= -\tfrac{2}{15}u^{3/2}(20-3u) + C \\ &= -\tfrac{2}{15}(4-x)^{3/2}\left[20 - 3(4-x)\right] + C \\ &= -\tfrac{2}{15}(4-x)^{3/2}(3x+8) + C\end{aligned}$$

69. $n = 0: \int \ln x\,dx = x(\ln x - 1) + C$

$n = 1: \int x \ln x\,dx = \frac{x^2}{4}(2 \ln x - 1) + C$

$n = 2: \int x^2 \ln x\,dx = \frac{x^3}{9}(3 \ln x - 1) + C$

$n = 3: \int x^3 \ln x\,dx = \frac{x^4}{16}(4 \ln x - 1) + C$

$n = 4: \int x^4 \ln x\,dx = \frac{x^5}{25}(5 \ln x - 1) + C$

In general, $\int x^n \ln x\,dx = \frac{x^{n+1}}{(n+1)^2}\left[(n+1)\ln x - 1\right] + C.$

70. $n = 0: \int e^x\,dx = e^x + C$

$n = 1: \int xe^x\,dx = xe^x - e^x + C = xe^x - \int e^x\,dx$

$n = 2: \int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x + C = x^2e^x - 2\int xe^x\,dx$

$n = 3: \int x^3e^x\,dx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C = x^3e^x - 3\int x^2e^x\,dx$

$n = 4: \int x^4e^x\,dx = x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + C = x^4e^x - 4\int x^3e^x\,dx$

In general, $\int x^ne^x\,dx = x^ne^x - n\int x^{n-1}e^x\,dx.$

71. $dv = \sin x\,dx \Rightarrow v = -\cos x$

$u = x^n \Rightarrow du = nx^{n-1}\,dx$

$\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1} \cos x\,dx$

72. $dv = \cos x\,dx \Rightarrow v = \sin x$

$u = x^n \Rightarrow du = nx^{n-1}\,dx$

$\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1} \sin x\,dx$

73. $dv = x^n\,dx \Rightarrow v = \frac{x^{n+1}}{n+1}$

$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$$\int x^n \ln x\,dx = \frac{x^{n+1}}{n+1}\ln x - \int \frac{x^n}{n+1}\,dx$$
$$= \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C$$
$$= \frac{x^{n+1}}{(n+1)^2}\left[(n+1)\ln x - 1\right] + C$$

74. $dv = e^{ax}\,dx \Rightarrow v = \frac{1}{a}e^{ax}$

$u = x^n \Rightarrow du = nx^{n-1}dx$

$\int x^n e^{ax}\,dx = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\,dx$

75. Use integration by parts twice.

(1) $dv = e^{ax}\,dx \Rightarrow v = \frac{1}{a}e^{ax}$ (2) $dv = e^{ax}\,dx \Rightarrow v = \frac{1}{a}e^{ax}$

$u = \sin bx \Rightarrow du = b\cos bx\,dx$ $u = \cos bx \Rightarrow du = -b\sin bx\,dx$

$$\int e^{ax} \sin bx\,dx = \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\int e^{ax}\cos bx\,dx$$
$$= \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\left(\frac{e^{ax}\cos bx}{a} + \frac{b}{a}\int e^{ax}\sin bx\,dx\right) = \frac{e^{ax}\sin bx}{a} - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx\,dx$$

Therefore, $\left(1 + \frac{b^2}{a^2}\right)\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2}$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2} + C.$$

76. Use integration by parts twice.

(1) $\left[\begin{array}{l} dv = e^{ax}\,dx \Rightarrow v = \frac{1}{a}e^{ax} \\ u = \cos bx \Rightarrow du = -b\sin bx \end{array}\right.$ (2) $\left[\begin{array}{l} dv = e^{ax}\,dx \Rightarrow v = \frac{1}{a}e^{ax} \\ u = \sin bx \Rightarrow du = b\cos bx \end{array}\right.$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}\cos bx}{a} + \frac{b}{a}\int e^{ax}\sin bx\,dx = \frac{e^{ax}\cos bx}{a} + \frac{b}{a}\left(\frac{e^{ax}\sin bx}{a} - \frac{b}{a}\int e^{ax}\cos bx\,dx\right)$$

$$= \frac{e^{ax}\cos bx}{a} + \frac{be^{ax}\sin bx}{a^2} - \frac{b^2}{a^2}\int e^{ax}\cos bx\,dx$$

Therefore, $\left(1 + \frac{b^2}{a^2}\right)\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2}$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2} + C.$$

77. $n = 2$ (Use formula in Exercise 67.)

$$\int x^2\sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx$$

$$= -x^2\cos x + 2\left[x\sin x - \int\sin x\,dx\right] \text{(Use formula in Exercise 68; } (n = 1).\text{)}$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C$$

78. $n = 2$ (Use formula in Exercise 68.)

$$\int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx, \quad \text{(Use formula in Exercise 67.)} \quad (n = 1)$$

$$= x^2\sin x - 2\left(-x\cos x + \int\cos x\,dx\right) = x^2\sin x + 2x\cos x - 2\sin x + C$$

79. $n = 5$ (Use formula in Exercise 69.)

$$\int x^5\ln x\,dx = \frac{x^6}{6^2}(-1 + 6\ln x) + C = \frac{x^6}{36}(-1 + 6\ln x) + C$$

80. $n = 3, a = 2$ (Use formula in Exercise 70 three times.)

$$\int x^3e^{2x}\,dx = \frac{x^3e^{2x}}{2} - \frac{3}{2}\int x^2e^{2x}\,dx, \quad (n = 3, a = 2)$$

$$= \frac{x^3e^{2x}}{2} - \frac{3}{2}\left[\frac{x^2e^{2x}}{2} - \int xe^{2x}\,dx\right], \quad (n = 2, a = 2)$$

$$= \frac{x^3e^{2x}}{2} - \frac{3x^2e^{2x}}{4} + \frac{3}{2}\left[\frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x}\,dx\right]$$

$$= \frac{x^3e^{2x}}{2} - \frac{3x^2e^{2x}}{4} + \frac{3xe^{2x}}{4} - \frac{3e^{2x}}{8} + C, \quad (n = 1, a = 2)$$

$$= \frac{e^{2x}}{8}(4x^3 - 6x^2 + 6x - 3) + C$$

81. $a = -3, b = 4$ (Use formula in Exercise 71.)

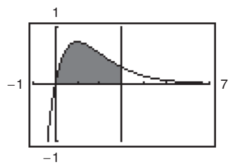
$$\int e^{-3x}\sin 4x\,dx = \frac{e^{-3x}(-3\sin 4x - 4\cos 4x)}{(-3)^2 + (4)^2} + C$$

$$= \frac{-e^{-3x}(3\sin 4x + 4\cos 4x)}{25} + C$$

82. $a = 2, b = 3$ (Use formula in Exercise 72.)

$$\int e^{2x} \cos 3x \, dx = \frac{e^{2x}(2 \cos 3x + 3 \sin 3x)}{13} + C$$

83.



$$dv = e^{-x}\,dx \Rightarrow v = \int e^{-x}\,dx = -e^{-x}$$

$$u = 2x \Rightarrow du = 2\,dx$$

$$\int 2xe^{-x}dx = 2x(-e^{-x}) - \int -2e^{-x}\,dx$$

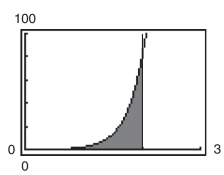
$$= -2xe^{-x} - 2e^{-x} + C$$

$$A = \int_0^3 2xe^{-x}\,dx = \left[-2xe^{-x} - 2e^{-x}\right]_0^3$$

$$= (-6e^{-3} - 2e^{-3}) - (-2)$$

$$= 2 - 8e^{-3} \approx 1.602$$

84.



$$A = \int_0^2 \frac{1}{10}xe^{3x}\,dx = \frac{1}{10}\int_0^2 xe^{3x}\,dx$$

$$dv = e^{3x}\,dx \Rightarrow v = \int e^{3x}\,dx = \frac{1}{3}e^{3x}$$

$$u = x \Rightarrow du = dx$$

$$\frac{1}{10}\int xe^{3x}\,dx = \frac{1}{10}\left[\frac{x}{3}e^{3x} - \int \frac{1}{3}e^{3x}\,dx\right]$$

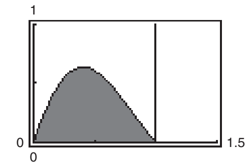
$$= \frac{x}{30}e^{3x} - \frac{1}{90}e^{3x} + C$$

$$A = \left[\frac{x}{30}e^{3x} - \frac{1}{90}e^{3x}\right]_0^2$$

$$= \left(\frac{1}{15}e^6 - \frac{1}{90}e^6\right) + \frac{1}{90}$$

$$= \frac{1}{90}(5e^6 + 1) \approx 22.424$$

85.



$$A = \int_0^1 e^{-x} \sin(\pi x)\,dx$$

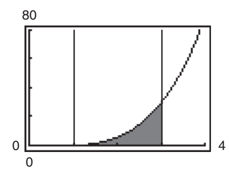
$$= \left[\frac{e^{-x}(-\sin \pi x - \pi \cos \pi x)}{1 + \pi^2}\right]_0^1$$

$$= \frac{1}{1 + \pi^2}\left(\frac{\pi}{e} + \pi\right)$$

$$= \frac{\pi}{1 + \pi^2}\left(\frac{1}{e} + 1\right)$$

≈ 0.395 (See Exercise 71.)

86.



$$dv = x^3\,dx \Rightarrow v = \int x^3\,dx = \frac{x^4}{4}$$

$$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$$

$$\int x^3 \ln x\,dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\left(\frac{1}{x}\,dx\right)$$

$$= \frac{x^4}{4}\ln x - \int \frac{x^3}{4}\,dx$$

$$= \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$$

$$A = \int_1^3 x^3 \ln x\,dx = \left[\frac{x^4}{4}\ln x - \frac{x^4}{16}\right]_1^3$$

$$= \left(\frac{81}{4}\ln 3 - \frac{81}{16}\right) + \frac{1}{16}$$

$$= \frac{81}{4}\ln 3 - 5 \approx 17.247$$

87. (a) $dv = dx \quad \Rightarrow \quad v = x$

$u = \ln x \quad \Rightarrow \quad du = \frac{1}{x}\,dx$

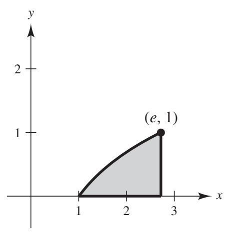
$A = \int_1^e \ln x\,dx = \left[x \ln x - x\right]_1^e = 1$ (Use integration by parts once.)

(b) $R(x) = \ln x, r(x) = 0$

$$V = \pi\int_1^e (\ln x)^2\,dx$$

$$= \pi\left[x(\ln x)^2 - 2x\ln x + 2x\right]_1^e$$ (Use integration by parts twice, see Exercise 3.)

$$= \pi(e - 2) \approx 2.257$$



(c) $p(x) = x, h(x) = \ln x$

$$V = 2\pi\int_1^e x\ln x\,dx = 2\pi\left[\frac{x^2}{4}(-1 + 2\ln x)\right]_1^e$$

$$= \frac{(e^2 + 1)\pi}{2} \approx 13.177$$ (See Exercise 73.)

(d) $$\bar{x} = \frac{\int_1^e x\ln x\,dx}{1} = \frac{e^2 + 1}{4} \approx 2.097$$

$$\bar{y} = \frac{\frac{1}{2}\int_1^e (\ln x)^2\,dx}{1} = \frac{e - 2}{2} \approx 0.359$$

$$(\bar{x}, \bar{y}) = \left(\frac{e^2 + 1}{4}, \frac{e - 2}{2}\right) \approx (2.097, 0.359)$$

88. $y = x\sin x, \quad 0 \le x \le \pi$

(a) $A = \int_0^\pi x\sin x\,dx$

$= -x\cos x + \int \cos x\,dx$ (Exercise 71)

$= -x\cos x + \sin x\Big]_0^\pi$

$= -\pi(-1) = \pi$

(b) $V = \int_0^{\pi} \pi(x \sin x)^2\, dx = \pi \int_0^{\pi} x^2 \sin^2 x\, dx$

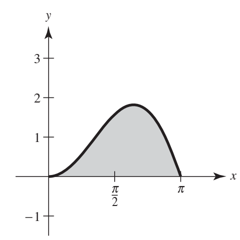
Let $u = x^2$, $du = 2x\, dx$, $dv = \sin^2 x\, dx = \dfrac{1 - \cos 2x}{2}\, dx$, $v = \dfrac{1}{2}x - \dfrac{\sin 2x}{4}$.

$$\begin{aligned}\int x^2 \sin^2 x\, dx &= x^2\left(\frac{1}{2}x - \frac{\sin 2x}{4}\right) - \int\left(\frac{1}{2}x - \frac{\sin 2x}{4}\right)(2x\, dx)\\ &= \frac{1}{2}x^3 - \frac{x^2 \sin 2x}{4} - \int\left(x^2 - \frac{x \sin 2x}{2}\right) dx\\ &= \frac{1}{2}x^3 - \frac{x^2 \sin 2x}{4} - \frac{x^3}{3} + \int \frac{x \sin 2x}{2}\, dx\\ &= \frac{1}{6}x^3 - \frac{1}{4}x^2 \sin 2x + \frac{1}{8}(\sin 2x - 2x \cos 2x) + C \quad \text{(Integration by Parts)}\end{aligned}$$

$$V = \pi \int_0^{\pi} x^2 \sin^2 x\, dx = \pi\left[\frac{1}{6}x^3 - \frac{1}{4}x^2 \sin 2x + \frac{1}{8}(\sin 2x - 2x \cos 2x)\right]_0^{\pi} = \frac{1}{6}\pi^4 - \frac{1}{4}\pi^2$$

(c) $V = \int_0^{\pi} 2\pi x(x \sin x) dx = 2\pi\left[2 \cos x + 2x \sin x - x^2 \cos x\right]_0^{\pi} = 2\pi(\pi^2 - 4) = 2\pi^3 - 8\pi$

(d) $m = \int_0^{\pi} x \sin(x)\, dx = \left[\sin x - x \cos x\right]_0^{\pi} = \pi$



$$\begin{aligned}M_x &= \int_0^{\pi} \frac{1}{2}(x \sin x)^2\, dx\\ &= \frac{1}{2}\left(\frac{1}{6}\pi^3 - \frac{1}{4}\pi\right) \quad \text{(See part (a).)}\\ &= \frac{1}{12}\pi^3 - \frac{1}{8}\pi\end{aligned}$$

$M_y = \int_0^{\pi} x(x \sin x)\, dx = \pi^2 - 4$ (See part (b).)

$\bar{x} = \dfrac{M_y}{m} = \dfrac{\pi^2 - 4}{\pi} \approx 1.8684$, $\bar{y} = \dfrac{M_x}{m} = \dfrac{(1/12)\pi^3 - (1/8)\pi}{\pi} = \dfrac{1}{2}\pi^2 - \dfrac{1}{8} \approx 0.6975$

89. In Example 6, you showed that the centroid of an equivalent region was $(1, \pi/8)$. By symmetry, the centroid of this region is $(\pi/8, 1)$. You can also solve this problem directly.

$$\begin{aligned}A = \int_0^1 \left(\frac{\pi}{2} - \arcsin x\right) dx &= \left[\frac{\pi}{2}x - x \arcsin x - \sqrt{1 - x^2}\right]_0^1 \quad \text{(Example 3)}\\ &= \left(\frac{\pi}{2} - \frac{\pi}{2} - 0\right) - (-1) = 1\end{aligned}$$

$\bar{x} = \dfrac{M_y}{A} = \int_0^1 x\left(\dfrac{\pi}{2} - \arcsin x\right) dx = \dfrac{\pi}{8}$, $\bar{y} = \dfrac{M_x}{A} = \int_0^1 \dfrac{(\pi/2) + \arcsin x}{2}\left(\dfrac{\pi}{2} - \arcsin x\right) dx = 1$

90. $f(x) = x^2, g(x) = 2^x$

$f(2) = g(2) = 4, f(4) = g(4) = 16$

$$m = \int_2^4 \left(x^2 - 2^x\right) dx = \left[\frac{x^3}{3} - \frac{1}{\ln 2}2^x\right]_2^4 = \left(\frac{64}{3} - \frac{16}{\ln 2}\right) - \left(\frac{8}{3} - \frac{4}{\ln 2}\right) = \frac{56}{3} - \frac{12}{\ln 2} \approx 1.3543$$

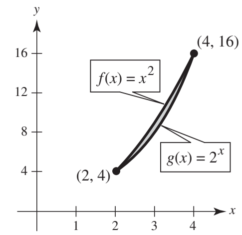
$$M_x = \int_2^4 \frac{1}{2}\left(x^2 + 2^x\right)\left(x^2 - 2^x\right) dx$$

$$= \frac{1}{2}\int_2^4 \left(x^4 - 2^{2x}\right) dx$$

$$= \frac{1}{2}\left[\frac{x^5}{5} - \frac{2^{2x}}{2\ln 2}\right]_2^4$$

$$= \frac{1}{2}\left[\left(\frac{1024}{5} - \frac{128}{\ln 2}\right) - \left(\frac{32}{5} - \frac{8}{\ln 2}\right)\right]$$

$$= \frac{496}{5} - \frac{60}{\ln 2} \approx 12.6383$$



$$M_y = \int_2^4 x\left[x^2 - 2^x\right] dx = -\frac{56}{\ln 2} + \frac{12}{(\ln 2)^2} \approx 4.1855$$

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) \approx (3.0905, 9.3318)$$

91. $$\text{Average value} = \frac{1}{\pi}\int_0^\pi e^{-4t}(\cos 2t + 5\sin 2t)\,dt$$

$$= \frac{1}{\pi}\left[e^{-4t}\left(\frac{-4\cos 2t + 2\sin 2t}{20}\right) + 5e^{-4t}\left(\frac{-4\sin 2t - 2\cos 2t}{20}\right)\right]_0^\pi \quad \text{(From Exercises 71 and 72)}$$

$$= \frac{7}{10\pi}\left(1 - e^{-4\pi}\right) \approx 0.223$$

92. (a) $\text{Average} = \int_1^2 (1.6t\ln t + 1)\,dt = \left[0.8t^2\ln t - 0.4t^2 + t\right]_1^2 = 3.2(\ln 2) - 0.2 \approx 2.018$

(b) $\text{Average} = \int_3^4 (1.6t\ln t + 1)\,dt = \left[0.8t^2\ln t - 0.4t^2 + t\right]_3^4 = 12.8(\ln 4) - 7.2(\ln 3) - 1.8 \approx 8.035$

93. $c(t) = 100{,}000 + 4000t, r = 5\% = 0.05, t_1 = 10$

$$P = \int_0^{10} (100{,}000 + 4000t)e^{-0.05t}\,dt = 4000\int_0^{10} (25 + t)e^{-0.05t}\,dt$$

Let $u = 25 + t, dv = e^{-0.05t}dt, du = dt, v = -\frac{100}{5}e^{-0.05t}$.

$$P = 4000\left\{\left[(25 + t)\left(-\frac{100}{5}e^{-0.05t}\right)\right]_0^{10} + \frac{100}{5}\int_0^{10} e^{-0.05t}dt\right\} = 4000\left\{\left[(25 + t)\left(-\frac{100}{5}e^{-0.05t}\right)\right]_0^{10} - \left[\frac{10{,}000}{25}e^{-0.05t}\right]_0^{10}\right\} \approx \$931{,}265$$

94. $c(t) = 1000 + 120t,\ r = 2\% = 0.02,\ t_1 = 30$

$$P = \int_0^{30} (1000 + 120t)e^{-0.02t}\,dt$$

Let $u = 1000 + 120t,\ dv = e^{-0.02t}\,dt,\ du = 120\,dt,\ v = -50e^{-0.02t}$

$$\begin{aligned} P &= \Big[(1000 + 120t)\big(-50e^{-0.02t}\big)\Big]_0^{30} - \int_0^{30} -50e^{-0.02t}(120\,dt) \\ &= \big[-230{,}000e^{-0.6} + 50{,}000\big] - \Big[300{,}000e^{-0.02t}\Big]_0^{30} \\ &= -230{,}000e^{-0.6} + 50{,}000 - 300{,}000e^{-0.6} + 300{,}000 \\ &= 350{,}000 - 530{,}000e^{-0.6} \\ &\approx \$59{,}130 \end{aligned}$$

95. $$\int_{-\pi}^{\pi} x \sin nx\,dx = \left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi} = -\frac{\pi}{n}\cos \pi n - \frac{\pi}{n}\cos(-\pi n) = -\frac{2\pi}{n}\cos \pi n = \begin{cases} -(2\pi/n), & \text{if } n \text{ is even} \\ (2\pi/n), & \text{if } n \text{ is odd} \end{cases}$$

96. $$\begin{aligned} \int_{-\pi}^{\pi} x^2 \cos nx\,dx &= \left[\frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx\right]_{-\pi}^{\pi} \\ &= \frac{2\pi}{n^2}\cos n\pi + \frac{2\pi}{n^2}\cos(-n\pi) \\ &= \frac{4\pi}{n^2}\cos n\pi \\ &= \begin{cases} (4\pi/n^2), & \text{if } n \text{ is even} \\ -(4\pi/n^2), & \text{if } n \text{ is odd} \end{cases} \\ &= \frac{(-1)^n 4\pi}{n^2} \end{aligned}$$

97. Let $u = x,\ dv = \sin\left(\frac{n\pi}{2}x\right)dx,\ du = dx,\ v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right).$

$$\begin{aligned} I_1 = \int_0^1 x \sin\left(\frac{n\pi}{2}x\right)dx &= \left[\frac{-2x}{n\pi}\cos\left(\frac{n\pi}{2}x\right)\right]_0^1 + \frac{2}{n\pi}\int_0^1 \cos\left(\frac{n\pi}{2}x\right)dx \\ &= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}x\right)\right]_0^1 \\ &= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

Let $u = (-x + 2),\ dv = \sin\left(\frac{n\pi}{2}x\right)dx,\ du = -dx,\ v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right).$

$$\begin{aligned} I_2 = \int_1^2 (-x + 2)\sin\left(\frac{n\pi}{2}x\right)dx &= \left[\frac{-2(-x + 2)}{n\pi}\cos\left(\frac{n\pi}{2}x\right)\right]_1^2 - \frac{2}{n\pi}\int_1^2 \cos\left(\frac{n\pi}{2}x\right)dx \\ &= \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}x\right)\right]_1^2 \\ &= \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

$$h(I_1 + I_2) = b_n = h\left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right)\right] = \frac{8h}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

98. (a) $A = \int_0^{\pi} x \sin x \, dx = \left[\sin x - x \cos x\right]_0^{\pi} = \pi$

(b) $\int_{\pi}^{2\pi} x \sin x \, dx = \left[\sin x - x \cos x\right]_{\pi}^{2\pi} = -2\pi - \pi = -3\pi$

$A = 3\pi$

(c) $\int_{2\pi}^{3\pi} x \sin x \, dx = \left[\sin x - x \cos x\right]_{2\pi}^{3\pi} = 3\pi + 2\pi = 5\pi$

$A = 5\pi$

The area between $y = x \sin x$ and $y = 0$ on $\left[n\pi, (n+1)\pi\right]$ is $(2n+1)\pi$:

$\int_{n\pi}^{(n+1)\pi} x \sin x \, dx = \left[\sin x - x \cos x\right]_{n\pi}^{(n+1)\pi} = \pm(n+1)\pi \pm n\pi = \pm(2n+1)\pi$

$A = \left|\pm(2n+1)\pi\right| = (2n+1)\pi$

99. For any integrable function, $\int f(x)dx = C + \int f(x)dx$, but this cannot be used to imply that $C = 0$.

100. Use the fact that $\sin x \geq \dfrac{2x}{\pi}$ on the interval $\left[0, \dfrac{\pi}{2}\right]$. To see this, note that if $g(x) = \sin x - \dfrac{2x}{\pi}$, then $g(0) = g\left(\dfrac{\pi}{2}\right) = 0$.
Furthermore, the sine function is concave downward on this interval.

Let $f(r) = \int_0^{\pi/2} x^r \sin x \, dx$. Then you have $f(r) < \int_0^{\pi/2} x^r \, dx = \left[\dfrac{x^{r+1}}{r+1}\right]_0^{\pi/2} = \dfrac{\left(\pi/2\right)^{r+1}}{r+1}$.

Furthermore, $f(r) = \int_0^{\pi/2} x^r \sin x \, dx > \int_0^{\pi/2} x^r\left(\dfrac{2x}{\pi}\right)dx = \left[\dfrac{2}{\pi}\dfrac{x^{r+2}}{r+2}\right]_0^{\pi/2} = \dfrac{2}{\pi}\dfrac{\left(\pi/2\right)^{r+2}}{r+2} = \dfrac{\left(\pi/2\right)^{r+1}}{r+2}$.

So,

$$\frac{\left(\pi/2\right)^{r+1}}{r+2} < f(r) < \frac{\left(\pi/2\right)^{r+1}}{r+1}$$

$$r\left(\frac{2}{\pi}\right)^{r+1}\frac{\left(\pi/2\right)^{r+1}}{r+2} < r\left(\frac{2}{\pi}\right)^{r+1} f(r) < r\left(\frac{2}{\pi}\right)^{r+1}\frac{\left(\pi/2\right)^{r+1}}{r+1}$$

$$\frac{r}{r+2} < r\left(\frac{2}{\pi}\right)^{r+1} f(r) < \frac{r}{r+1}$$

Taking limits by the Squeeze Theorem, $\lim_{r\to\infty}\left[r\left(\dfrac{2}{\pi}\right)^{r+1} f(r)\right] = 1$.

Now analyze the following limit: $\lim_{r\to\infty}\left[\dfrac{f(r)}{f(r+1)}\right] = \lim_{r\to\infty}\left[\dfrac{r\left(2/\pi\right)^{r+1} f(r)}{(r+1)r\left(2/\pi\right)^{r+2} f(r+1)}\right]\left[\dfrac{2(r+1)}{\pi r}\right] = \dfrac{2}{\pi}$

Using integration by parts, let $u = \cos x$, $du = -\sin x \, dx$, $dv = x^r \, dx$, $v = \dfrac{x^{r+1}}{r+1}$.

So, you have $\int_0^{\pi/2} x^r \cos x \, dx = \left[\dfrac{x^{r+1}}{r+1}\cos x\right]_0^{\pi/2} - \int_0^{\pi/2} \dfrac{-x^{r+1}}{r+1}\sin x \, dx = \dfrac{1}{r+1} f(r+1)$.

Now, consider the limit with $c = -1$:

$$\lim_{r\to\infty}\frac{r^{-1}\int_0^{\pi/2} x^r \sin x \, dx}{\int_0^{\pi/2} x^r \cos x \, dx} = \lim_{r\to\infty}\frac{f(r)}{r}\frac{(r+1)}{f(r+1)} = \frac{2}{\pi}$$

Therefore, $c = -1$ and $L = \dfrac{2}{\pi}$.

Section 8.3 Trigonometric Integrals

1. The integral $\int \sin^8 x\,dx$ requires more steps. The other integral can be found by u-substitution $(u = \sin x, du = \cos x\,dx)$.

2. The power of the tangent is odd. So, save a secant-tangent factor, and convert the remaining tangents to secants.

3. Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int \cos^5 x \sin x\,dx = -\int \cos^5 x\,(-\sin x)\,dx = -\frac{\cos^6 x}{6} + C$$

4. Let $u = \sin 2x$, $du = 2\cos 2x\,dx$.

$$\int \sin^7 2x \cos 2x\,dx = \frac{1}{2}\int \sin^7 2x(2\cos 2x)\,dx = \frac{1}{2}\left(\frac{\sin^8 2x}{8}\right) + C = \frac{1}{16}\sin^8 2x + C$$

5. $$\int \cos^3 x \sin^4 x\,dx = \int \cos x\left(1 - \sin^2 x\right)\sin^4 x\,dx = \int\left(\sin^4 x - \sin^6 x\right)\cos x\,dx = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$$

6. $$\int \sin^3 3x\,dx = \int \sin^2 3x \sin 3x\,dx = \int\left(1 - \cos^2 3x\right)\sin 3x\,dx = \int \sin 3x\,dx - \int \cos^2 3x(\sin 3x\,dx) = -\frac{1}{3}\cos 3x + \frac{\cos^3 3x}{9} + C$$

7. $$\int \sin^3 x \cos^2 x\,dx = \int\left(1 - \cos^2 x\right)\cos^2 x \sin x\,dx = \int\left(\cos^2 x - \cos^4 x\right)\sin x\,dx = -\int\left(\cos^2 x - \cos^4 x\right)(-\sin x)\,dx = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$$

8. Let $u = \sin\frac{x}{3}$, $du = \frac{1}{3}\cos\frac{x}{3}\,dx$.

$$\int \cos^3\frac{x}{3}\,dx = \int\left(\cos\frac{x}{3}\right)\left(1 - \sin^2\frac{x}{3}\right)dx = 3\int\left(1 - \sin^2\frac{x}{3}\right)\left(\frac{1}{3}\cos\frac{x}{3}\right)dx = 3\left(\sin\frac{x}{3} - \frac{1}{3}\sin^3\frac{x}{3}\right) + C = 3\sin\frac{x}{3} - \sin^3\frac{x}{3} + C$$

9. $$\begin{aligned}\int \sin^3 2\theta\sqrt{\cos 2\theta}\,d\theta &= \int\left(1 - \cos^2 2\theta\right)\sqrt{\cos 2\theta}\sin 2\theta\,d\theta \\ &= \int\left[(\cos 2\theta)^{1/2} - (\cos 2\theta)^{5/2}\right]\sin 2\theta\,d\theta \\ &= -\tfrac{1}{2}\int\left[(\cos 2\theta)^{1/2} - (\cos 2\theta)^{5/2}\right](-2\sin 2\theta)\,d\theta \\ &= -\tfrac{1}{2}\left[\tfrac{2}{3}(\cos 2\theta)^{3/2} - \tfrac{2}{7}(\cos 2\theta)^{7/2}\right] + C \\ &= -\tfrac{1}{3}(\cos 2\theta)^{3/2} + \tfrac{1}{7}(\cos 2\theta)^{7/2} + C\end{aligned}$$

10. $$\begin{aligned}\int \frac{\cos^5 t}{\sqrt{\sin t}}\,dt &= \int \cos t\left(1 - \sin^2 t\right)^2(\sin t)^{-1/2}\,dt \\ &= \int\left(1 - 2\sin^2 t + \sin^4 t\right)(\sin t)^{-1/2}\cos t\,dt \\ &= \int\left[(\sin t)^{-1/2} - 2(\sin t)^{3/2} + (\sin t)^{7/2}\right]\cos t\,dt \\ &= 2\sqrt{\sin t} - \frac{4}{5}(\sin t)^{5/2} + \frac{2}{9}(\sin t)^{9/2} + C\end{aligned}$$

11. $$\int \cos^2 3x\,dx = \int \frac{1 + \cos 6x}{2}\,dx = \frac{1}{2}\left(x + \frac{1}{6}\sin 6x\right) + C = \frac{1}{12}(6x + \sin 6x) + C$$

12. $$\int \sin^4 6\theta\, d\theta = \int\left(\frac{1-\cos 12\theta}{2}\right)\left(\frac{1-\cos 12\theta}{2}\right) d\theta$$
$$= \frac{1}{4}\int\left(1 - 2\cos 12\theta + \cos^2 12\theta\right) d\theta$$
$$= \frac{1}{4}\int\left(1 - 2\cos 12\theta + \frac{1+\cos 24\theta}{2}\right) d\theta$$
$$= \frac{1}{4}\int\left(\frac{3}{2} - 2\cos 12\theta + \frac{1}{2}\cos 24\theta\right) d\theta$$
$$= \frac{1}{4}\left(\frac{3}{2}\theta - \frac{1}{6}\sin 12\theta + \frac{1}{48}\sin 24\theta\right) + C = \frac{3}{8}\theta - \frac{1}{24}\sin 12\theta + \frac{1}{192}\sin 24\theta + C$$

13. $dv = \cos^2 x\, dx = \frac{1+\cos 2x}{2}\, dx \Rightarrow v = \frac{1}{2}x + \frac{\sin 2x}{4}$

$u = 8x \Rightarrow du = 8\, dx$

$$\int 8x\cos^2 x\, dx = uv - \int v\, du$$
$$= 8x\left(\frac{1}{2}x + \frac{\sin 2x}{4}\right) - \int\left(\frac{1}{2}x + \frac{\sin 2x}{4}\right)(8\, dx)$$
$$= 4x^2 + 2x\sin 2x - 2x^2 + \cos 2x + C$$
$$= 2x^2 + 2x\sin 2x + \cos 2x + C$$

14. Use integration by parts twice.

$dv = \sin^2 x\, dx = \frac{1-\cos 2x}{2} \Rightarrow v = \frac{x}{2} - \frac{\sin 2x}{4} = \frac{1}{4}(2x - \sin 2x)$

$u = x^2 \Rightarrow du = 2x\, dx$

$dv = \sin 2x\, dx \quad\Rightarrow\quad v = -\frac{1}{2}\cos 2x$

$u = x \quad\Rightarrow\quad du = dx$

$$\int x^2\sin^2 x\, dx = \frac{1}{4}x^2(2x - \sin 2x) - \frac{1}{2}\int\left(2x^2 - x\sin 2x\right) dx$$
$$= \frac{1}{2}x^3 - \frac{1}{4}x^2\sin 2x - \frac{1}{3}x^3 + \frac{1}{2}\int x\sin 2x\, dx$$
$$= \frac{1}{6}x^3 - \frac{1}{4}x^2\sin 2x + \frac{1}{2}\left(-\frac{1}{2}x\cos 2x + \frac{1}{2}\int\cos 2x\, dx\right)$$
$$= \frac{1}{6}x^3 - \frac{1}{4}x^2\sin 2x - \frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C$$
$$= \frac{1}{24}\left(4x^3 - 6x^2\sin 2x - 6x\cos 2x + 3\sin 2x\right) + C$$

15. $\int_0^{\pi/2}\cos^3 x\, dx = \frac{2}{3} \quad (n = 3, \text{odd})$

16. $\int_0^{\pi/2}\cos^6 x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{\pi}{2}\right) = \frac{5}{32}\pi \quad (n = 6, \text{even})$

17. $\int_0^{\pi/2}\sin^2 x\, dx = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4} \quad (n = 2, \text{even})$

18. $\int_0^{\pi/2}\sin^9 x\, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\left(\frac{8}{9}\right) = \frac{128}{315} \quad (n = 9, \text{odd})$

19. $\int_0^{\pi/2} \sin^{10} x\,dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\left(\frac{9}{10}\right)\left(\frac{\pi}{2}\right) = \frac{63}{512}\pi \qquad (n = 10, \text{even})$

20. $\int_0^{\pi/2} \cos^{11} x\,dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\left(\frac{8}{9}\right)\left(\frac{10}{11}\right) = \frac{256}{693} \qquad (n = 11, \text{odd})$

21.
$$\int \sec 4x\,dx = \frac{1}{4}\int \sec 4x(4\,dx)$$
$$= \frac{1}{4}\ln\left|\sec 4x + \tan 4x\right| + C$$

22.
$$\int \sec^4 x\,dx = \int \left(1 + \tan^2 x\right)\sec^2 x\,dx$$
$$= \tan x + \frac{1}{3}\tan^3 x + C$$

23. $dv = \sec^2 \pi x\,dx \implies v = \frac{1}{\pi}\tan \pi x$

$u = \sec \pi x \implies du = \pi \sec \pi x \tan \pi x\,dx$

$$\int \sec^3 \pi x\,dx = \frac{1}{\pi}\sec \pi x \tan \pi x - \int \sec \pi x \tan^2 \pi x\,dx = \frac{1}{\pi}\sec \pi x \tan \pi x - \int \sec \pi x\left(\sec^2 \pi x - 1\right)dx$$
$$2\int \sec^3 \pi x\,dx = \frac{1}{\pi}\left(\sec \pi x \tan \pi x + \ln\left|\sec \pi x + \tan \pi x\right|\right) + C_1$$
$$\int \sec^3 \pi x\,dx = \frac{1}{2\pi}\left(\sec \pi x \tan \pi x + \ln\left|\sec \pi x + \tan \pi x\right|\right) + C$$

24.
$$\int \tan^6 3x\,dx = \int \left(\sec^2 3x - 1\right)\tan^4 3x\,dx$$
$$= \int \tan^4 3x \sec^2 3x\,dx - \int \tan^4 3x\,dx$$
$$= \int \tan^4 3x \sec^2 3x\,dx - \int \tan^2 3x\left(\sec^2 3x - 1\right)dx$$
$$= \int \tan^4 3x \sec^2 3x\,dx - \int \tan^2 3x \sec^2 3x\,dx + \int\left(\sec^2 3x + 1\right)dx$$
$$= \frac{\tan^5 3x}{15} - \frac{\tan^3 3x}{9} + \frac{\tan 3x}{3} + x + C$$

25.
$$\int \tan^5 \frac{x}{2}\,dx = \int\left(\sec^2 \frac{x}{2} - 1\right)\tan^3 \frac{x}{2}\,dx$$
$$= \int \tan^3 \frac{x}{2}\sec^2 \frac{x}{2}\,dx - \int \tan^3 \frac{x}{2}\,dx$$
$$= \frac{\tan^4 \frac{x}{2}}{2} - \int\left(\sec^2 \frac{x}{2} - 1\right)\tan \frac{x}{2}\,dx$$
$$= \frac{1}{2}\tan^4 \frac{x}{2} - \tan^2 \frac{x}{2} - 2\ln\left|\cos \frac{x}{2}\right| + C$$

26. $\int \tan^3 \frac{\pi x}{2}\sec^2 \frac{\pi x}{2}\,dx = \frac{1}{2\pi}\tan^4 \frac{\pi x}{2} + C$

27. Let $u = \sec 2t$, $du = 2\sec 2t \tan 2t\,dt$.

$$\int \tan^3 2t \cdot \sec^3 2t\,dt = \int\left(\sec^2 2t - 1\right)\sec^3 2t \cdot \tan 2t\,dt$$
$$= \int\left(\sec^4 2t - \sec^2 2t\right)\left(\sec 2t \tan 2t\right)dt = \frac{\sec^5 2t}{10} - \frac{\sec^3 2t}{6} + C$$

28. $\int \tan^5 x \sec^4 x\,dx = \int \tan^5 x(\tan^2 x + 1)\sec^2 x\,dx$

$= \int (\tan^7 x + \tan^5 x)\sec^2 x\,dx$

$= \frac{1}{8}\tan^8 x + \frac{1}{6}\tan^6 x + C$

29. $\int \sec^6 4x \tan 4x\,dx = \frac{1}{4}\int \sec^5 4x(4\sec 4x \tan 4x)\,dx$

$= \frac{\sec^6 4x}{24} + C$

30. $\int \sec^2 \frac{x}{2}\tan\frac{x}{2}\,dx = 2\int \sec\frac{x}{2}\left(\frac{1}{2}\sec\frac{x}{2}\tan\frac{x}{2}\right)dx$

$= \sec^2\frac{x}{2} + C$ or

$\int \sec^2 \frac{x}{2}\tan\frac{x}{2}\,dx = 2\int \tan\frac{x}{2}\left(\frac{1}{2}\sec^2\frac{x}{2}\right)dx = \tan^2\frac{x}{2} + C$

31. $\int \sec^5 x \tan^3 x\,dx = \int \sec^4 x \tan^2 x(\sec x \tan x)\,dx$

$= \int \sec^4 x(\sec^2 x - 1)(\sec x \tan x)\,dx$

$= \int (\sec^6 x - \sec^4 x)(\sec x \tan x)\,dx$

$= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C$

32. $\int \tan^3 3x\,dx = \int (\sec^2 3x - 1)\tan 3x\,dx$

$= \frac{1}{3}\int \tan 3x(3\sec^2 3x)\,dx + \frac{1}{3}\int \frac{-3\sin 3x}{\cos 3x}\,dx$

$= \frac{1}{6}\tan^2 3x + \frac{1}{3}\ln|\cos 3x| + C$

33. $\int \frac{\tan^2 x}{\sec x}\,dx = \int \frac{(\sec^2 x - 1)}{\sec x}\,dx$

$= \int (\sec x - \cos x)\,dx$

$= \ln|\sec x + \tan x| - \sin x + C$

34. $\int \frac{\tan^2 x}{\sec^5 x} = \int \frac{\sin^2 x}{\cos^2 x}\cdot \cos^5 x\,dx$

$= \int \sin^2 x \cdot \cos^3 x\,dx$

$= \int \sin^2 x(1 - \sin^2 x)\cos x\,dx$

$= \int (\sin^2 x - \sin^4 x)\cos x\,dx$

$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

35. $r = \int \sin^4(\pi\theta)\,d\theta = \frac{1}{4}\int \left[1 - \cos(2\pi\theta)\right]^2 d\theta$

$= \frac{1}{4}\int \left[1 - 2\cos(2\pi\theta) + \cos^2(2\pi\theta)\right]d\theta$

$= \frac{1}{4}\int \left[1 - 2\cos(2\pi\theta) + \frac{1 + \cos(4\pi\theta)}{2}\right]d\theta$

$= \frac{1}{4}\left[\theta - \frac{1}{\pi}\sin(2\pi\theta) + \frac{\theta}{2} + \frac{1}{8\pi}\sin(4\pi\theta)\right] + C$

$= \frac{1}{32\pi}\left[12\pi\theta - 8\sin(2\pi\theta) + \sin(4\pi\theta)\right] + C$

36. $s = \int \sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\,d\alpha$

$= \int \left(\frac{1 - \cos\alpha}{2}\right)\left(\frac{1 + \cos\alpha}{2}\right)d\alpha = \int \frac{1 - \cos^2\alpha}{4}\,d\alpha$

$= \frac{1}{4}\int \sin^2\alpha\,d\alpha = \frac{1}{8}\int (1 - \cos 2\alpha)\,d\alpha$

$= \frac{1}{8}\left(\theta - \frac{\sin 2\alpha}{2}\right) + C$

$= \frac{1}{16}(2\alpha - \sin 2\alpha) + C$

37. $y = \int \tan^3 3x \sec 3x\,dx$

$= \int (\sec^2 3x - 1)\sec 3x \tan 3x\,dx$

$= \frac{1}{3}\int \sec^2 3x(3\sec 3x \tan 3x)\,dx - \frac{1}{3}\int 3\sec 3x \tan 3x\,dx$

$= \frac{1}{9}\sec^3 3x - \frac{1}{3}\sec 3x + C$

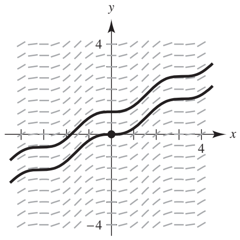
38. $y = \int \sqrt{\tan x}\sec^4 x\,dx$

$= \int \tan^{1/2} x(\tan^2 x + 1)\sec^2 x\,dx$

$= \int (\tan^{5/2} x + \tan^{1/2} x)\sec^2 x\,dx$

$= \frac{2}{7}\tan^{7/2} x + \frac{2}{3}\tan^{3/2} x + C$

39. (a)

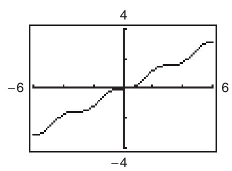


(b) $\dfrac{dy}{dx} = \sin^2 x, \quad (0, 0)$

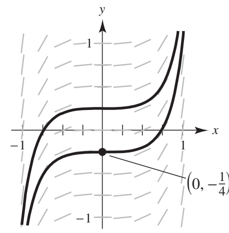
$$y = \int \sin^2 x\,dx = \int \frac{1 - \cos 2x}{2}\,dx$$

$$= \frac{1}{2}x - \frac{\sin 2x}{4} + C$$

$$(0, 0):\ 0 = C,\ y = \frac{1}{2}x - \frac{\sin 2x}{4}$$



40. (a)

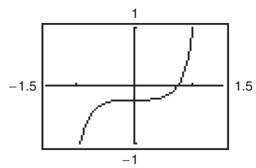


(b) $\dfrac{dy}{dx} = \sec^2 x \tan^2 x, \quad \left(0, -\dfrac{1}{4}\right)$

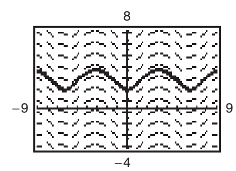
$$y = \int \sec^2 x \tan^2 x\,dx \qquad u = \tan x,\ du = \sec^2 x\,dx$$

$$y = \frac{\tan^3 x}{3} + C$$

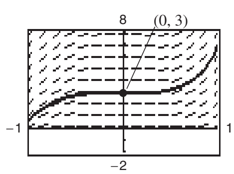
$$\left(0, -\frac{1}{4}\right):\ -\frac{1}{4} = C \Rightarrow y = \frac{1}{3}\tan^3 x - \frac{1}{4}$$



41. $\dfrac{dy}{dx} = \dfrac{3 \sin x}{y},\ y(0) = 2$



42. $\dfrac{dy}{dx} = 3\sqrt{y}\tan^2 x,\ y(0) = 3$



43. $\displaystyle\int \cos 2x \cos 6x\,dx = \frac{1}{2}\int \left[\cos((2 - 6)x) + \cos((2 + 6)x)\right] dx$

$$= \frac{1}{2}\int \left[\cos(-4x) + \cos 8x\right] dx$$

$$= \frac{1}{2}\int (\cos 4x + \cos 8x)\,dx$$

$$= \frac{1}{2}\left[\frac{\sin 4x}{4} + \frac{\sin 8x}{8}\right] + C$$

$$= \frac{\sin 4x}{8} + \frac{\sin 8x}{16} + C$$

$$= \frac{1}{16}(2 \sin 4x + \sin 8x) + C$$

44. $$\int \cos(5\theta)\cos(3\theta)\,d\theta = \frac{1}{2}\int\left[\cos(5-3)\theta + \cos(5+3)\theta\right]d\theta$$
$$= \frac{1}{2}\int(\cos 2\theta + \cos 8\theta)\,d\theta$$
$$= \frac{1}{2}\left[\frac{\sin 2\theta}{2} + \frac{\sin 8\theta}{8}\right] + C$$
$$= \frac{\sin 2\theta}{4} + \frac{\sin 8\theta}{16} + C$$

45. $$\int \sin 2t \cos 9t\,dt = \frac{1}{2}\int\left[\sin\left[(2-9)t\right] + \sin\left[(2+9)t\right]\right]dt$$
$$= \frac{1}{2}\int\left[\sin(-7t) + \sin(11t)\right]dt$$
$$= -\frac{1}{2}\int(\sin 7t + \sin 11t)\,dt$$
$$= \frac{1}{14}\cos 7t - \frac{1}{22}\cos 11t + C$$

46. $$\int \sin 8x \cos 7x\,dx = \frac{1}{2}\int\left[\sin\left[(8-7)x\right] + \sin\left[(8+7)x\right]\right]dx$$
$$= \frac{1}{2}\int(\sin x + \sin 15x)\,dx$$
$$= -\frac{1}{2}\cos x - \frac{1}{30}\cos 15x + C$$

47. $$\int \sin\theta \sin 3\theta\,d\theta = \tfrac{1}{2}\int(\cos 2\theta - \cos 4\theta)\,d\theta$$
$$= \tfrac{1}{2}\left(\tfrac{1}{2}\sin 2\theta - \tfrac{1}{4}\sin 4\theta\right) + C$$
$$= \tfrac{1}{8}(2\sin 2\theta - \sin 4\theta) + C$$

48. $$\int \sin 5x \sin 4x\,dx = \frac{1}{2}\int(\cos x - \cos 9x)\,dx$$
$$= \frac{1}{2}\left(\sin x - \frac{\sin 9x}{9}\right) + C$$
$$= \frac{\sin x}{2} - \frac{\sin 9x}{18} + C$$
$$= \frac{1}{18}(9\sin x - \sin 9x) + C$$

49. $$\int \cot^3 2x\,dx = \int\left(\csc^2 2x - 1\right)\cot 2x\,dx$$
$$= -\frac{1}{2}\int \cot 2x\left(-2\csc^2 2x\right)dx - \frac{1}{2}\int\frac{2\cos 2x}{\sin 2x}\,dx$$
$$= -\frac{1}{4}\cot^2 2x - \frac{1}{2}\ln\left|\sin 2x\right| + C$$
$$= \frac{1}{4}\left(\ln\left|\csc^2 2x\right| - \cot^2 2x\right) + C$$

50. $\int \tan^5 \frac{x}{4} \sec^4 \frac{x}{4}\,dx$

$= \int \tan^5 \frac{x}{4}\left(\tan^2 \frac{x}{4} + 1\right)\sec^2 \frac{x}{4}\,dx$

$= \int \left(\tan^7 \frac{x}{4} + \tan^5 \frac{x}{4}\right)\sec^2 \frac{x}{4}\,dx$

$= \frac{\tan^8 \frac{x}{4}}{2} + \frac{2\tan^6 \frac{x}{4}}{3} + C$

$= \frac{1}{2}\tan^8 \frac{x}{4} + \frac{2}{3}\tan^6 \frac{x}{4} + C$

51. $\int \csc^4 3x\,dx = \int \csc^2 3x\left(1 + \cot^2 3x\right)dx$

$= \int \csc^2 3x\,dx + \int \cot^2 3x \csc^2 3x\,dx$

$= -\frac{1}{3}\cot 3x - \frac{1}{9}\cot^3 3x + C$

52. $\int \cot^3 \frac{x}{2}\csc^4 \frac{x}{2}\,dx = \int \cot^2 \frac{x}{2}\csc^3 \frac{x}{2}\left(\csc \frac{x}{2}\cot \frac{x}{2}\right)dx$

$= \int \left(\csc^2 \frac{x}{2} - 1\right)\csc^3 \frac{x}{2}\left(\csc \frac{x}{2}\cot \frac{x}{2}\right)dx$

$= \int \left(\csc^5 \frac{x}{2} - \csc^3 \frac{x}{2}\right)\left(\csc \frac{x}{2}\cot \frac{x}{2}\right)dx$

$= -\frac{1}{3}\csc^6 \frac{x}{2} + \frac{1}{2}\csc^4 \frac{x}{2} + C$

53. $\int \frac{\cot^2 t}{\csc t}\,dt = \int \frac{\csc^2 t - 1}{\csc t}\,dt$

$= \int (\csc t - \sin t)\,dt$

$= \ln\left|\csc t - \cot t\right| + \cos t + C$

54. $\int \frac{\cot^3 t}{\csc t}\,dt = \int \frac{\cos^3 t}{\sin^2 t}\,dt = \int \frac{\left(1 - \sin^2 t\right)\cos t}{\sin^2 t}\,dt$

$= \int \frac{\cos t}{\sin^2 t}\,dt - \int \cos t\,dt$

$= \frac{-1}{\sin t} - \sin t + C = -\csc t - \sin t + C$

55. $\int \frac{1}{\sec x \tan x}\,dx = \int \frac{\cos^2 x}{\sin x}\,dx = \int \frac{1 - \sin^2 x}{\sin x}\,dx$

$= \int (\csc x - \sin x)\,dx$

$= \ln\left|\csc x - \cot x\right| + \cos x + C$

56. $\int \frac{\sin^2 x - \cos^2 x}{\cos x}\,dx = \int \frac{1 - 2\cos^2 x}{\cos x}\,dx$

$= \int (\sec x - 2\cos x)\,dx$

$= \ln\left|\sec x + \tan x\right| - 2\sin x + C$

57. $\int \left(\tan^4 t - \sec^4 t\right)dt = \int \left(\tan^2 t + \sec^2 t\right)\left(\tan^2 t - \sec^2 t\right)dt, \qquad \left(\tan^2 t - \sec^2 t = -1\right)$

$= -\int \left(\tan^2 t + \sec^2 t\right)dt = -\int \left(2\sec^2 t - 1\right)dt = -2\tan t + t + C$

58. $\int \frac{1 - \sec t}{\cos t - 1}\,dt = \int \frac{\cos t - 1}{(\cos t - 1)\cos t}\,dt = \int \sec t\,dt = \ln\left|\sec t + \tan t\right| + C$

59. $\int_{-\pi}^{\pi} \sin^2 x\,dx = 2\int_0^{\pi} \frac{1 - \cos 2x}{2}\,dx$

$= \left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \pi$

60. $\int_0^{\pi/3} \tan^2 x\,dx = \int_0^{\pi/3} \left(\sec^2 x - 1\right)dx$

$= \left[\tan x - x\right]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3}$

61. $\int_0^{\pi/4} 6\tan^3 x\,dx = 6\int_0^{\pi/4} \left(\sec^2 x - 1\right)\tan x\,dx$

$= 6\int_0^{\pi/4} \left[\tan x \sec^2 x - \tan x\right]dx$

$= 6\left[\frac{\tan^2 x}{2} + \ln\left|\cos x\right|\right]_0^{\pi/4}$

$= 6\left[\frac{1}{2} + \ln\left(\frac{\sqrt{2}}{2}\right)\right] = 6\left(\frac{1}{2} - \ln\sqrt{2}\right)$

$= 3(1 - \ln 2)$

62. $$\int_0^{\pi/3} \sec^{3/2} x \tan x\, dx = \int_0^{\pi/3} \sec^{1/2} x (\sec x \tan x)\, dx = \left[\frac{2}{3}\sec^{3/2} x\right]_0^{\pi/3} = \frac{2}{3}\left(2\sqrt{2} - 1\right)$$

63. Let $u = 1 + \sin t$, $du = \cos t\, dt$.

$$\int_0^{\pi/2} \frac{\cos t}{1 + \sin t}\, dt = \left[\ln\left|1 + \sin t\right|\right]_0^{\pi/2} = \ln 2$$

64. $$\int_{\pi/6}^{\pi/3} \sin 6x \cos 4x\, dx = \frac{1}{2}\int_{\pi/6}^{\pi/3} (\sin 2x + \sin 10x)\, dx = \left[-\frac{\cos 2x}{4} - \frac{\cos 10x}{20}\right]_{\pi/6}^{\pi/3} = \left(\frac{1}{8} + \frac{1}{40}\right) - \left(-\frac{1}{8} - \frac{1}{40}\right) = \frac{3}{10}$$

65. $$\int_{-\pi/2}^{\pi/2} 3\cos^3 x\, dx = 3\int_{-\pi/2}^{\pi/2} \left(1 - \sin^2 x\right)\cos x\, dx = 3\left[\sin x - \frac{\sin^3 x}{3}\right]_{-\pi/2}^{\pi/2} = 3\left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)\right] = 4$$

66. $$\begin{aligned}\int_0^{\pi} \sin^5 x\, dx &= \int_0^{\pi} \sin^4 x \sin x\, dx \\ &= \int_0^{\pi} \left(1 - \cos^2 x\right)\left(1 - \cos^2 x\right)\sin x\, dx \\ &= \int_0^{\pi} \left[1 - 2\cos^2 x + \cos^4 x\right]\sin x\, dx \\ &= \left[-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x\right]_0^{\pi} \\ &= \left[-(-1) + \frac{2}{3}(-1) - \frac{1}{5}(-1)\right] - \left[-1 + \frac{2}{3} - \frac{1}{5}\right] \\ &= 2 - \frac{4}{3} + \frac{2}{5} = \frac{16}{15}\end{aligned}$$

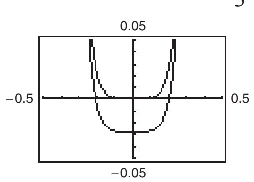
67. (a) Let $u = \tan 3x$, $du = 3\sec^2 3x\, dx$.

$$\begin{aligned}\int \sec^4 3x \tan^3 3x\, dx &= \int \sec^2 3x \tan^3 3x \sec^2 3x\, dx = \frac{1}{3}\int \left(\tan^2 3x + 1\right)\tan^3 3x\left(3\sec^2 3x\right) dx \\ &= \frac{1}{3}\int \left(\tan^5 3x + \tan^3 3x\right)\left(3\sec^2 3x\right) dx = \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + C_1\end{aligned}$$

Or let $u = \sec 3x$, $du = 3\sec 3x \tan 3x\, dx$.

$$\begin{aligned}\int \sec^4 3x \tan^3 3x\, dx &= \int \sec^3 3x \tan^2 3x \sec 3x \tan 3x\, dx \\ &= \frac{1}{3}\int \sec^3 3x\left(\sec^2 3x - 1\right)\left(3\sec 3x \tan 3x\right) dx = \frac{\sec^6 3x}{18} - \frac{\sec^4 3x}{12} + C\end{aligned}$$

(b)



(c) $$\begin{aligned}\frac{\sec^6 3x}{18} - \frac{\sec^4 3x}{12} + C &= \frac{\left(1 + \tan^2 3x\right)^3}{18} - \frac{\left(1 + \tan^2 3x\right)^2}{12} + C \\ &= \frac{1}{18}\tan^6 3x + \frac{1}{6}\tan^4 3x + \frac{1}{6}\tan^2 3x + \frac{1}{18} - \frac{1}{12}\tan^4 3x - \frac{1}{6}\tan^2 3x - \frac{1}{12} + C \\ &= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + \left(\frac{1}{18} - \frac{1}{12}\right) + C \\ &= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + C_2\end{aligned}$$

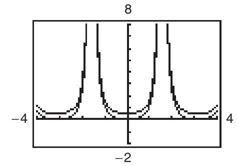
68. (a) Let $u = \tan x$, $du = \sec^2 x\, dx$.

$$\int \sec^2 x \tan x\, dx = \tfrac{1}{2}\tan^2 x + C_1$$

Or let $u = \sec x$, $du = \sec x \tan x\, dx$.

$$\int \sec x(\sec x \tan x)\, dx = \tfrac{1}{2}\sec^2 x + C$$

(b)



(c) $\tfrac{1}{2}\sec^2 x + C = \tfrac{1}{2}(\tan^2 x + 1) + C$

$$= \tfrac{1}{2}\tan^2 x + \left(\tfrac{1}{2} + C\right)$$

$$= \tfrac{1}{2}\tan^2 x + C_2$$

69. (a) $\displaystyle\int \sin x \cos x\, dx = \frac{\sin^2 x}{2} + C$

(b) $\displaystyle -\int \cos x\,(-\sin x)\, dx = -\frac{\cos^2 x}{2} + C$

(c) $dv = \cos x\, dx \Rightarrow\ v = \sin x$

$u = \sin x \qquad \Rightarrow du = \cos x\, dx$

$$\int \sin x \cos x\, dx = \sin^2 x - \int \sin x \cos x\, dx$$

$$2\int \sin x \cos x\, dx = \sin^2 x$$

$$\int \sin x \cos x\, dx = \frac{\sin^2 x}{2} + C$$

(Answers will vary.)

(d) $\displaystyle\int \sin x \cos x\, dx = \int \frac{1}{2}\sin 2x\, dx = -\frac{1}{4}\cos 2x + C$

The answers all differ by a constant.

70. (a) f has a maximum at the points where f' changes from positive to negative: $x = -\pi, \pi$.

(b) f has a minimum at the points where f' changes from negative to positive: $x = 0$.

71. $\displaystyle A = \int_0^{\pi/2} (\sin x - \sin^3 x)\, dx$

$$= \int_0^{\pi/2} \sin x\, dx - \int_0^{\pi/2} \sin^3 x\, dx$$

$$= \left[-\cos x\right]_0^{\pi/2} - \tfrac{2}{3} \qquad \text{(Wallis's Formula)}$$

$$= 1 - \tfrac{2}{3} = \tfrac{1}{3}$$

72. $\displaystyle A = \int_0^1 \sin^2(\pi x)\, dx$

$$= \int_0^1 \frac{1 - \cos(2\pi x)}{2}\, dx$$

$$= \left[\frac{1}{2}x - \frac{\sin 2\pi x}{4\pi}\right]_0^1 = \frac{1}{2}$$

73. $\displaystyle A = \int_{-\pi/4}^{\pi/4} \left[\cos^2 x - \sin^2 x\right] dx$

$$= \int_{-\pi/4}^{\pi/4} \cos 2x\, dx$$

$$= \left[\frac{\sin 2x}{2}\right]_{-\pi/4}^{\pi/4} = \frac{1}{2} + \frac{1}{2} = 1$$

74. $\displaystyle A = \int_{-\pi/2}^{\pi/4} \left[\cos^2 x - \sin x \cos x\right] dx$

$$= \int_{-\pi/2}^{\pi/4} \left[\frac{1 + \cos 2x}{2} - \sin x \cos x\right] dx$$

$$= \left[\frac{1}{2}x + \frac{\sin 2x}{4} - \frac{\sin^2 x}{2}\right]_{-\pi/2}^{\pi/4}$$

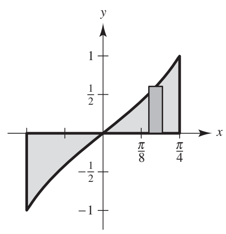
$$= \left(\frac{\pi}{8} + \frac{1}{4} - \frac{1}{4}\right) - \left(-\frac{\pi}{4} - \frac{1}{2}\right)$$

$$= \frac{3\pi}{8} + \frac{1}{2}$$

75. Disks

$R(x) = \tan x,\ r(x) = 0$

$$V = 2\pi\int_0^{\pi/4} \tan^2 x\, dx = 2\pi\int_0^{\pi/4} (\sec^2 x - 1)\, dx = 2\pi\Big[\tan x - x\Big]_0^{\pi/4} = 2\pi\left(1 - \frac{\pi}{4}\right) \approx 1.348$$



76. $$V = \pi\int_0^{\pi/2} \left[\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)\right] dx = \pi\int_0^{\pi/2} \cos x\, dx = \pi\Big[\sin x\Big]_0^{\pi/2} = \pi$$

77. (a) $V = \pi\int_0^{\pi} \sin^2 x\, dx = \frac{\pi}{2}\int_0^{\pi} (1 - \cos 2x)\, dx = \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi^2}{2}$

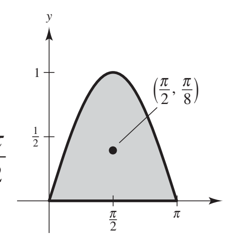
(b) $A = \int_0^{\pi} \sin x\, dx = \Big[-\cos x\Big]_0^{\pi} = 1 + 1 = 2$

Let $u = x,\ dv = \sin x\, dx,\ du = dx,\ v = -\cos x.$

$\bar{x} = \frac{1}{A}\int_0^{\pi} x \sin x\, dx = \frac{1}{2}\left[\Big[-x\cos x\Big]_0^{\pi} + \int_0^{\pi} \cos x\, dx\right] = \frac{1}{2}\Big[-x\cos x + \sin x\Big]_0^{\pi} = \frac{\pi}{2}$

$\bar{y} = \frac{1}{2A}\int_0^{\pi} \sin^2 x\, dx = \frac{1}{8}\int_0^{\pi} (1 - \cos 2x)\, dx = \frac{1}{8}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi}{8}$

$(\bar{x}, \bar{y}) = \left(\frac{\pi}{2}, \frac{\pi}{8}\right)$



78. (a) $V = \pi\int_0^{\pi/2} \cos^2 x\, dx = \frac{\pi}{2}\int_0^{\pi/2} (1 + \cos 2x)\, dx = \frac{\pi}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2} = \frac{\pi^2}{4}$

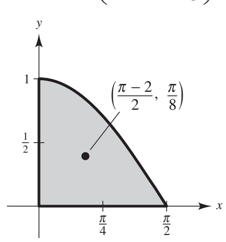
(b) $A = \int_0^{\pi/2} \cos x\, dx = \Big[\sin x\Big]_0^{\pi/2} = 1$

Let $u = x,\ dv = \cos x\, dx,\ du = dx,\ v = \sin x.$

$\bar{x} = \int_0^{\pi/2} x\cos x\, dx = \Big[x\sin x\Big]_0^{\pi/2} - \int_0^{\pi/2} \sin x\, dx = \Big[x\sin x + \cos x\Big]_0^{\pi/2} = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$

$\bar{y} = \frac{1}{2}\int_0^{\pi/2} \cos^2 x\, dx = \frac{1}{4}\int_0^{\pi/2} (1 + \cos 2x)\, dx = \frac{1}{4}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2} = \frac{\pi}{8}$

$(\bar{x}, \bar{y}) = \left(\frac{\pi - 2}{2}, \frac{\pi}{8}\right)$



79. $dv = \sin x\, dx \Rightarrow \quad v = -\cos x$

$u = \sin^{n-1} x \quad \Rightarrow du = (n-1)\sin^{n-2} x \cos x\, dx$

$$\int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1 - \sin^2 x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$$

Therefore, $n\int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx$

$$\int \sin^n x\, dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx.$$

80. $dv = \cos x\, dx \Rightarrow \quad v = \sin x$

$u = \cos^{n-1} x \quad \Rightarrow du = -(n-1)\cos^{n-2} x \sin x\, dx$

$$\int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1 - \cos^2 x)\, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x\, dx$$

Therefore, $n\int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx$

$$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx.$$

81. Let $u = \sin^{n-1} x, du = (n-1)\sin^{n-2} x \cos x\, dx, dv = \cos^m x \sin x\, dx, v = \dfrac{-\cos^{m+1} x}{m+1}$.

$$\int \cos^m x \sin^n x\, dx = \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{n-2} x \cos^{m+2} x\, dx$$

$$= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{n-2} x \cos^m x(1 - \sin^2 x)\, dx$$

$$= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{n-2} x \cos^m x\, dx - \frac{n-1}{m+1}\int \sin^n x \cos^m x\, dx$$

$$\frac{m+n}{m+1}\int \cos^m x \sin^n x\, dx = \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{n-2} x \cos^m x\, dx$$

$$\int \cos^m x \sin^n x\, dx = \frac{-\cos^{m+1} x \sin^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \cos^m x \sin^{n-2} x\, dx$$

82. Let $u = \sec^{n-2} x, du = (n-2)\sec^{n-2} x \tan x\, dx, dv = \sec^2 x\, dx, v = \tan x$.

$$\int \sec^n x\, dx = \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x \tan^2 x\, dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)\, dx$$

$$= \sec^{n-2} x \tan x - (n-2)\left[\int \sec^n x\, dx - \int \sec^{n-2} x\, dx\right]$$

$$(n-1)\int \sec^n x\, dx = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x\, dx$$

$$\int \sec^n x\, dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx$$

83. $\displaystyle\int \sin^5 x\, dx = -\frac{\sin^4 x \cos x}{5} + \frac{4}{5}\int \sin^3 x\, dx$

$$= -\frac{\sin^4 x \cos x}{5} + \frac{4}{5}\left(-\frac{\sin^2 x \cos x}{3} + \frac{2}{3}\int \sin x\, dx\right)$$

$$= -\frac{1}{5}\sin^4 x \cos x - \frac{4}{15}\sin^2 x \cos x - \frac{8}{15}\cos x + C$$

$$= -\frac{\cos x}{15}(3\sin^4 x + 4\sin^2 x + 8) + C$$

84. $$\int \cos^4 x\,dx = \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\int \cos^2 x\,dx$$
$$= \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\left(\frac{\cos x \sin x}{2} + \frac{1}{2}\int dx\right)$$
$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + C$$
$$= \frac{1}{8}\left(2\cos^3 x \sin x + 3\cos x \sin x + 3x\right) + C$$

85. $$\int \sin^4 x \cos^2 x\,dx = -\frac{\cos^3 x \sin^3 x}{6} + \frac{1}{2}\int \cos^2 x \sin^2 x\,dx$$
$$= -\frac{\cos^3 x \sin^3 x}{6} + \frac{1}{2}\left(-\frac{\cos^3 x \sin x}{4} + \frac{1}{4}\int \cos^2 x\,dx\right)$$
$$= -\frac{1}{6}\cos^3 x \sin^3 x - \frac{1}{8}\cos^3 x \sin x + \frac{1}{8}\left(\frac{\cos x \sin x}{2} + \frac{x}{2}\right) + C$$
$$= -\frac{1}{48}\left(8\cos^3 x \sin^3 x + 6\cos^3 x \sin x - 3\cos x \sin x - 3x\right) + C$$

86. $$\int \sec^4 \frac{2\pi x}{5}\,dx = \frac{5}{2\pi}\int \sec^4\left(\frac{2\pi x}{5}\right)\frac{2\pi}{5}\,dx$$
$$= \frac{5}{2\pi}\left[\frac{1}{3}\sec^2\left(\frac{2\pi x}{5}\right)\tan\left(\frac{2\pi x}{5}\right) + \frac{2}{3}\int \sec^2\left(\frac{2\pi x}{5}\right)\frac{2\pi}{5}\,dx\right]$$
$$= \frac{5}{6\pi}\left[\sec^2\left(\frac{2\pi x}{5}\right)\tan\left(\frac{2\pi x}{5}\right) + 2\tan\left(\frac{2\pi x}{5}\right)\right] + C$$
$$= \frac{5}{6\pi}\tan\left(\frac{2\pi x}{5}\right)\left[\sec^2\left(\frac{2\pi x}{5}\right) + 2\right] + C$$

87. (a) n is odd and $n \geq 3$.

$$\int_0^{\pi/2} \cos^n x\,dx = \left[\frac{\cos^{n-1} x \sin x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx$$
$$= \frac{n-1}{n}\left(\left[\frac{\cos^{n-3} x \sin x}{n-2}\right]_0^{\pi/2} + \frac{n-3}{n-2}\int_0^{\pi/2} \cos^{n-4} x\,dx\right)$$
$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\left(\left[\frac{\cos^{n-5} x \sin x}{n-4}\right]_0^{\pi/2} + \frac{n-5}{n-4}\int_0^{\pi/2} \cos^{n-6} x\,dx\right)$$
$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\int_0^{\pi/2} \cos^{n-6} x\,dx$$
$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\int_0^{\pi/2} \cos x\,dx$$
$$= \left[\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots(\sin x)\right]_0^{\pi/2}$$
$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots 1 \quad \text{(Reverse the order.)}$$
$$= (1)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right) = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$$

(b) n is even and $n \geq 2$.

$$\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\int_0^{\pi/2}\cos^2 x\,dx \qquad \text{(From part (a))}$$

$$= \left[\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\left(\frac{x}{2}+\frac{1}{4}\sin 2x\right)\right]_0^{\pi/2}$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{\pi}{4} \qquad \text{(Reverse the order.)}$$

$$= \left(\frac{\pi}{2}\cdot\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

88. $$\int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,dx = \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n}+\frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} = 0, \quad (m \neq n)$$

$$\int_{-\pi}^{\pi}\sin(mx)\sin(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\left[\cos(m-n)x - \cos(m+n)x\right]dx$$

$$= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n}-\frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0, \quad (m \neq n)$$

$$\int_{-\pi}^{\pi}\sin(mx)\cos(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\left[\sin(m+n)x + \sin(m-n)x\right]dx$$

$$= -\frac{1}{2}\left[\frac{\cos(m+n)x}{m+n}+\frac{\cos(m-n)x}{m-n}\right]_{-\pi}^{\pi}, \qquad (m \neq n)$$

$$= -\frac{1}{2}\left[\left(\frac{\cos(m+n)\pi}{m+n}+\frac{\cos(m-n)\pi}{m-n}\right)-\left(\frac{\cos(m+n)(-\pi)}{m+n}+\frac{\cos(m-n)(-\pi)}{m-n}\right)\right]$$

$$= 0, \text{ because } \cos(-\theta) = \cos\theta.$$

$$\int_{-\pi}^{\pi}\sin(mx)\cos(mx)\,dx = \frac{1}{m}\left[\frac{\sin^2(mx)}{2}\right]_{-\pi}^{\pi} = 0$$

89. $f(x) = \sum_{i=1}^{N} a_i \sin(ix)$

(a) $$f(x)\sin(nx) = \left[\sum_{i=1}^{N} a_i\sin(ix)\right]\sin(nx)$$

$$\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \int_{-\pi}^{\pi}\left[\sum_{i=1}^{N} a_i\sin(ix)\right]\sin(nx)\,dx$$

$$= \int_{-\pi}^{\pi} a_n\sin^2(nx)\,dx \qquad \text{(by Exercise 89)}$$

$$= \int_{-\pi}^{\pi} a_n\frac{1-\cos(2nx)}{2}\,dx = \left[\frac{a_n}{2}\left(x - \frac{\sin(2nx)}{2n}\right)\right]_{-\pi}^{\pi} = \frac{a_n}{2}(\pi+\pi) = a_n\pi$$

So, $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$

(b) $f(x) = x$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\,dx = 2$$

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin 2x\,dx = -1$$

$$a_3 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin 3x\,dx = \frac{2}{3}$$

Section 8.4 Trigonometric Substitution

1. (a) Use $x = 3 \tan \theta$.

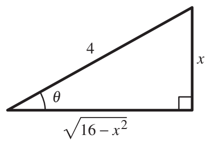
(b) Use $x = 2 \sin \theta$.

(c) Use $x = 5 \sin \theta$.

(d) Use $x = 5 \sec \theta$.

2. In order to use integration by trigonometric substitution, you need radicals of the form $\sqrt{a^2 - u^2}$, $\sqrt{a^2 + u^2}$, or $\sqrt{u^2 - a^2}$. Completing the square allows you to convert many integrands to one of these forms.

3. Let $x = 4 \sin \theta, dx = 4 \cos \theta \, d\theta, \sqrt{16 - x^2} = 4 \cos \theta$.



$$\int \frac{1}{\left(16 - x^2\right)^{3/2}} \, dx = \int \frac{4 \cos \theta}{(4 \cos \theta)^3} \, d\theta = \frac{1}{16} \int \sec^2 \theta \, d\theta = \frac{1}{16} \tan \theta + C = \frac{1}{16}\left(\frac{x}{\sqrt{16 - x^2}}\right) + C$$

4. Same substitution as in Exercise 3

$$\int \frac{4}{x^2\sqrt{16 - x^2}} \, dx = 4\int \frac{4 \cos \theta}{(4 \sin \theta)^2 (4 \cos \theta)} \, d\theta = \frac{1}{4}\int \csc^2 \theta \, d\theta = -\frac{1}{4} \cot \theta + C = -\frac{1}{4}\frac{\sqrt{16 - x^2}}{x} + C = \frac{-\sqrt{16 - x^2}}{4x} + C$$

5. Same substitution as in Exercise 5

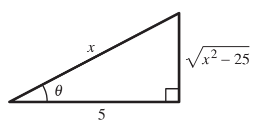
$$\begin{aligned}
\int \frac{\sqrt{16 - x^2}}{x} \, dx &= \int \frac{4 \cos \theta}{4 \sin \theta} 4 \cos \theta \, d\theta \\
&= 4\int \frac{\cos^2 \theta}{\sin \theta} \, d\theta \\
&= 4\int \frac{1 - \sin^2 \theta}{\sin \theta} \, d\theta \\
&= 4\int (\csc \theta - \sin \theta) \, d\theta \\
&= -4 \ln\left|\csc \theta + \cot \theta\right| + 4 \cos \theta + C \\
&= -4 \ln\left|\frac{4}{x} + \frac{\sqrt{16 - x^2}}{x}\right| + 4\frac{\sqrt{16 - x^2}}{4} + C \\
&= -4 \ln\left|\frac{4 + \sqrt{16 - x^2}}{x}\right| + \sqrt{16 - x^2} + C \\
&= 4 \ln\left|\frac{4 - \sqrt{16 - x^2}}{x}\right| + \sqrt{16 - x^2} + C
\end{aligned}$$

6. Same substitution as in Exercise 5

$$\int \frac{x^3}{\sqrt{16 - x^2}}\,dx = \int \frac{(4\sin\theta)^3}{4\cos\theta} 4\cos\theta\,d\theta$$
$$= 64\int \sin^3\theta\,d\theta$$
$$= 64\int (1 - \cos^2\theta)\sin\theta\,d\theta$$
$$= 64\left[-\cos\theta + \frac{\cos^3\theta}{3}\right] + C$$
$$= 64\left[-\frac{\sqrt{16 - x^2}}{4} + \frac{(16 - x^2)^{3/2}}{64(3)}\right] + C$$
$$= -16\sqrt{16 - x^2} + \frac{1}{3}(16 - x^2)^{3/2} + C$$
$$= -\frac{1}{3}\sqrt{16 - x^2}\left[48 - (16 - x^2)\right] + C$$
$$= -\frac{1}{3}\sqrt{16 - x^2}(32 + x^2) + C$$

7. Let $x = 5\sec\theta$, $dx = 5\sec\theta\tan\theta\,d\theta$,

$\sqrt{x^2 - 25} = 5\tan\theta.$



$$\int \frac{1}{\sqrt{x^2 - 25}}\,dx = \int \frac{5\sec\theta\tan\theta}{5\tan\theta}\,d\theta$$
$$= \int \sec\theta\,d\theta$$
$$= \ln\left|\sec\theta + \tan\theta\right| + C$$
$$= \ln\left|\frac{x}{5} + \frac{\sqrt{x^2 - 25}}{5}\right| + C$$
$$= \ln\left|x + \sqrt{x^2 - 25}\right| + C$$

8. Same substitution as in Exercise 9

$$\int \frac{\sqrt{x^2 - 25}}{x}\,dx = \int \frac{5\tan\theta}{5\sec\theta} 5\sec\theta\tan\theta\,d\theta$$
$$= 5\int \tan^2\theta\,d\theta$$
$$= 5\int (\sec^2\theta - 1)\,d\theta$$
$$= 5(\tan\theta - \theta) + C$$
$$= 5\left(\frac{\sqrt{x^2 - 25}}{5} - \operatorname{arcsec}\frac{x}{5}\right) + C$$
$$= \sqrt{x^2 - 25} - 5\operatorname{arcsec}\frac{x}{5} + C$$

$\left[\textbf{Note: } \operatorname{arcsec}\left(\frac{x}{5}\right) = \arctan\left(\frac{\sqrt{x^2 - 25}}{5}\right)\right]$

9. Same substitution as in Exercise 9

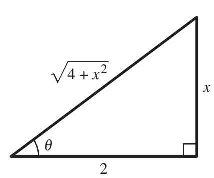
$$\begin{aligned}\int x^3\sqrt{x^2 - 25}\,dx &= \int (5\sec\theta)^3(5\tan\theta)(5\sec\theta\tan\theta)\,d\theta\\ &= 3125\int \sec^4\theta\tan^2\theta\,d\theta\\ &= 3125\int \left(1 + \tan^2\theta\right)\tan^2\theta\sec^2\theta\,d\theta\\ &= 3125\int\left(\tan^2\theta + \tan^4\theta\right)\sec^2\theta\,d\theta\\ &= 3125\left[\frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5}\right] + C\\ &= 3125\left[\frac{(x^2 - 25)^{3/2}}{125(3)} + \frac{(x^2 - 25)^{5/2}}{5^5(5)}\right] + C\\ &= \frac{1}{15}(x^2 - 25)^{3/2}\left[125 + 3(x^2 - 25)\right] + C\\ &= \frac{1}{15}(x^2 - 25)^{3/2}(50 + 3x^2) + C\end{aligned}$$

10. Same substitution as in Exercise 9

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2 - 25}}\,dx &= \int \frac{(5\sec\theta)^3}{5\tan\theta}5\sec\theta\tan\theta\,d\theta\\ &= 125\int\sec^4\theta\,d\theta\\ &= 125\int\left(\tan^2\theta + 1\right)\sec^2\theta\,d\theta\\ &= 125\left(\frac{\tan^3\theta}{3} + \tan\theta\right) + C\\ &= \frac{125}{3}\frac{(x^2 - 25)^{3/2}}{125} + 125\frac{\sqrt{x^2 - 25}}{5} + C\\ &= \frac{1}{3}(x^2 - 25)^{3/2} + 25(x^2 - 25)^{1/2} + C\\ &= \frac{1}{3}\sqrt{x^2 - 25}(x^2 - 25 + 75) + C\\ &= \frac{1}{3}\sqrt{x^2 - 25}(50 + x^2) + C\end{aligned}$$

11. Let $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$, $\sqrt{4 + x^2} = 2\sec\theta$.

$$\begin{aligned}\int \frac{x}{2}\sqrt{4 + x^2}\,dx &= \int \tan\theta(2\sec\theta)\left(2\sec^2\theta\right)d\theta\\ &= 4\int\sec^2\theta\,(\sec\theta\tan\theta)\,d\theta\\ &= \frac{4}{3}\sec^3\theta + C\\ &= \frac{4}{3}\cdot\frac{(4 + x^2)^{3/2}}{8} + C\\ &= \frac{1}{6}(4 + x^2)^{3/2} + C\end{aligned}$$



12. Same substitution as in Exercise 11

$$\begin{aligned}\int \frac{x^3}{4\sqrt{4 + x^2}}\,dx &= \int \frac{8\tan^3\theta}{4(2\sec\theta)}\left(2\sec^2\theta\right)d\theta \\ &= 2\int \tan^3\theta\sec\theta\,d\theta \\ &= 2\int \tan^2\theta\,(\tan\theta\sec\theta)\,d\theta \\ &= 2\int\left(\sec^2\theta - 1\right)(\tan\theta\sec\theta)\,d\theta \\ &= \frac{2}{3}\sec^3\theta - 2\sec\theta + C \\ &= \frac{2}{3}\cdot\frac{\left(4 + x^2\right)^{3/2}}{8} - 2\cdot\frac{\left(4 + x^2\right)^{1/2}}{2} + C \\ &= \frac{1}{12}\sqrt{4 + x^2}\left(4 + x^2 - 12\right) + C \\ &= \left(x^2 - 8\right)\frac{\sqrt{4 + x^2}}{12} + C\end{aligned}$$

13. Same substitution as in Exercise 11

$$\begin{aligned}\int \frac{4}{\left(4 + x^2\right)^2}\,dx &= \int \frac{4}{(2\sec\theta)^4}\left(2\sec^2\theta\right)d\theta \\ &= \frac{1}{2}\int\cos^2\theta\,d\theta \\ &= \frac{1}{2}\int\frac{1 + \cos 2\theta}{2}\,d\theta \\ &= \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta + C = \frac{1}{4}\theta + \frac{1}{4}\sin\theta\cos\theta + C \\ &= \frac{1}{4}\arctan\frac{x}{2} + \frac{1}{4}\left(\frac{x}{\sqrt{x^2 + 4}}\right)\left(\frac{2}{\sqrt{x^2 + 4}}\right) + C \\ &= \frac{1}{4}\left(\arctan\frac{x}{2} + \frac{2x}{x^2 + 4}\right) + C\end{aligned}$$

14. Same substitution as in Exercise 11

$$\begin{aligned}\int \frac{2x^2}{\left(4 + x^2\right)^2}\,dx &= \int \frac{2(2\tan\theta)^2}{(2\sec\theta)^4}\left(2\sec^2\theta\right)d\theta \\ &= \int\frac{\tan^2\theta}{\sec^2\theta}\,d\theta \\ &= \int\sin^2\theta\,d\theta \\ &= \int\frac{1 - \cos 2\theta}{2}\,d\theta \\ &= \frac{1}{2}\theta - \frac{\sin 2\theta}{4} + C \\ &= \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta + C \\ &= \frac{1}{2}\arctan\frac{x}{2} - \frac{1}{2}\left(\frac{x}{\sqrt{x^2 + 4}}\right)\left(\frac{2}{\sqrt{x^2 + 4}}\right) + C \\ &= \frac{1}{2}\arctan\frac{x}{2} - \frac{x}{x^2 + 4} + C\end{aligned}$$

15. Let $u = 4x$, $a = 7$, $du = 4\,dx$.

$$\int \sqrt{49 - 16x^2}\,dx = \frac{1}{4}\int \sqrt{7^2 - (4x)^2}\,4\,dx$$
$$= \frac{1}{4}\left(\frac{1}{2}\right)\left(4x\sqrt{49 - 16x^2} + 49 \arcsin \frac{4x}{4}\right) + C$$
$$= \frac{1}{2}x\sqrt{49 - 16x^2} + \frac{49}{8}\arcsin\frac{4x}{7} + C$$

16. Let $u = \sqrt{5}x$, $a = 1$, $du = \sqrt{5}\,dx$.

$$\int \sqrt{5x^2 - 1}\,dx = \frac{1}{\sqrt{5}}\int \sqrt{\left(\sqrt{5}x\right)^2 - 1}\,\sqrt{5}\,dx$$
$$= \frac{1}{\sqrt{5}}\left(\frac{1}{2}\right)\left(\sqrt{5}x\sqrt{5x^2 - 1} - \ln\left|\sqrt{5}x + \sqrt{5x^2 - 1}\right|\right) + C$$
$$= \frac{x}{2}\sqrt{5x^2 - 1} - \frac{\sqrt{5}}{10}\ln\left|\sqrt{5}x + \sqrt{5x^2 - 1}\right| + C$$

17. Let $u = \sqrt{5}\,x$, $du = \sqrt{5}\,dx$, $a = 6$.

$$\int \sqrt{36 - 5x^2}\,dx = \frac{1}{\sqrt{5}}\int \sqrt{36 - 5x^2}\left(\sqrt{5}\,dx\right)$$
$$= \frac{1}{2\sqrt{5}}\left(\sqrt{5}x\sqrt{36 - 5x^2} + 36 \arcsin\frac{\sqrt{5}x}{6}\right) + C$$

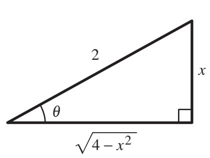
18. Let $u = 2x$, $du = 2\,dx$, $a = 3$.

$$\int \sqrt{9 + 4x^2}\,dx = \frac{1}{2}\int \sqrt{9 + 4x^2}\,2\,dx$$
$$= \frac{1}{2}\left(\frac{1}{2}\right)\left[2x\sqrt{9 + 4x^2} + 9\ln\left|2x + \sqrt{9 + 4x^2}\right|\right] + C$$
$$= \frac{x}{2}\sqrt{9 + 4x^2} + \frac{9}{4}\ln\left|2x + \sqrt{9 + 4x^2}\right| + C$$

19. Let $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$,

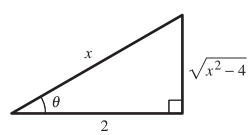
$\sqrt{4 - x^2} = 2\cos\theta$.

$$\int \sqrt{16 - 4x^2}\,dx = 2\int \sqrt{4 - x^2}\,dx$$
$$= 2\int 2\cos\theta(2\cos\theta\,d\theta)$$
$$= 8\int \cos^2\theta\,d\theta$$
$$= 4\int (1 + \cos 2\theta)\,d\theta$$
$$= 4\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$
$$= 4\theta + 4\sin\theta\cos\theta + C$$
$$= 4\arcsin\left(\frac{x}{2}\right) + x\sqrt{4 - x^2} + C$$



20. Let $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\,d\theta$,

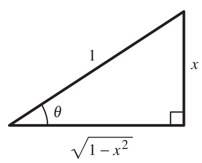
$\sqrt{x^2 - 4} = 2\tan\theta$.



$$\int \frac{1}{\sqrt{x^2 - 4}}\,dx = \int \frac{2\sec\theta\tan\theta}{2\tan\theta}\,d\theta$$
$$= \int \sec\theta\,d\theta$$
$$= \ln\left|\sec\theta + \tan\theta\right| + C$$
$$= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2 - 4}}{2}\right| + C$$
$$= \ln\left|x + \sqrt{x^2 - 4}\right| + C$$

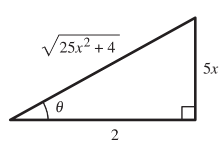
21. Let $x = \sin\theta$, $dx = \cos\theta\, d\theta$, $\sqrt{1 - x^2} = \cos\theta$.

$$\int \frac{\sqrt{1-x^2}}{x^4}\,dx = \int \frac{\cos\theta(\cos\theta\, d\theta)}{\sin^4\theta}$$
$$= \int \cot^2\theta \csc^2\theta\, d\theta$$
$$= -\frac{1}{3}\cot^3\theta + C$$
$$= -\frac{\left(1-x^2\right)^{3/2}}{3x^3} + C$$



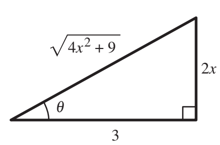
22. Let $5x = 2\tan\theta$, $5dx = 2\sec^2\theta\, d\theta$, $\sqrt{25x^2 + 4} = \sqrt{4\tan^2\theta + 4} = 2\sec\theta$.

$$\int \frac{\sqrt{25x^2+4}}{x^4}\,dx = \int \frac{2\sec\theta}{\left(\frac{2}{5}\tan\theta\right)^4}\left(\frac{2}{5}\sec^2\theta\right)d\theta$$
$$= \frac{125}{4}\int \frac{\cos\theta}{\sin^4\theta}\,d\theta$$
$$= \frac{125}{4}\left(\frac{1}{(-3)\sin^3\theta}\right) + C$$
$$= -\frac{125}{12}\csc^3\theta + C$$
$$= -\frac{125}{12}\left(\frac{\sqrt{25x^2+4}}{5x}\right)^3$$
$$= -\frac{\left(25x^2+4\right)^{3/2}}{12x^3} + C$$



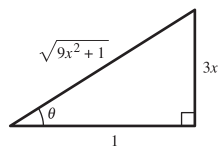
23. Let $2x = 3\tan\theta \Rightarrow x = \frac{3}{2}\tan\theta$, $dx = \frac{3}{2}\sec^2\theta\, d\theta$, $\sqrt{4x^2 + 9} = 3\sec\theta$.

$$\int \frac{1}{x\sqrt{4x^2+9}}\,dx = \int \frac{(3/2)\sec^2\theta\, d\theta}{(3/2)\tan\theta\, 3\sec\theta}$$
$$= \frac{1}{3}\int \csc\theta\, d\theta$$
$$= -\frac{1}{3}\ln\left|\csc\theta + \cot\theta\right| + C$$
$$= -\frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9}+3}{2x}\right| + C$$



24. Let $3x = \tan\theta, 3dx = \sec^2\theta\, d\theta, \sqrt{9x^2+1} = \sec\theta$.

$$\int \frac{1}{x\sqrt{9x^2+1}}\,dx = \int \frac{1}{\frac{1}{3}\tan\theta\sec\theta}\left(\frac{1}{3}\sec^2\theta\right)d\theta$$
$$= \int \frac{\sec\theta}{\tan\theta}\,d\theta$$
$$= \int \csc\theta\, d\theta$$
$$= \ln\left|\csc\theta - \cot\theta\right| + C$$
$$= \ln\left|\frac{\sqrt{9x^2+1}}{3x} - \frac{1}{3x}\right| + C$$
$$= \ln\left|\frac{\sqrt{9x^2+1}-1}{3x}\right| + C$$

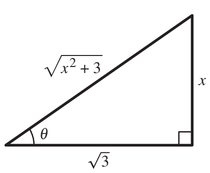


(**Note:** This equals $-\ln\left|\frac{\sqrt{9x^2+1}+1}{3x}\right| + C$.)

25. Let $x = \sqrt{3}\tan\theta, dx = \sqrt{3}\sec^2\theta\, d\theta$,

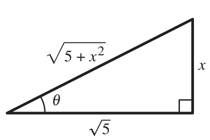
$x^2 + 3 = 3\tan^2\theta + 3 = 3\sec^2\theta$.

$$\int \frac{-3}{\left(x^2+3\right)^{3/2}}\,dx = \int \frac{-3}{3\sqrt{3}\sec^3\theta}\left(\sqrt{3}\sec^2\theta\right)d\theta$$
$$= -\int \cos\theta\, d\theta$$
$$= -\sin\theta + C$$
$$= -\frac{x}{\sqrt{x^2+3}} + C$$



26. Let $x = \sqrt{5}\tan\theta, dx = \sqrt{5}\sec^2\theta$,

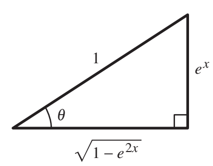
$x^2 + 5 = 5\sec^2\theta$.



$$\int \frac{1}{\left(x^2+5\right)^{3/2}}\,dx = \int \frac{\sqrt{5}\sec^2\theta}{\left(\sqrt{5}\sec\theta\right)^3}\,d\theta$$
$$= \frac{1}{5}\int \cos\theta\, d\theta$$
$$= \frac{1}{5}\sin\theta + C = \frac{x}{5\sqrt{5+x^2}} + C$$

27. Let $e^x = \sin\theta, e^x\,dx = \cos\theta\, d\theta, \sqrt{1-e^{2x}} = \cos\theta$.

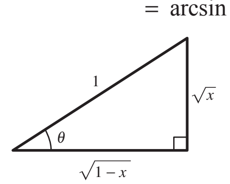
$$\int e^x\sqrt{1-e^{2x}}\,dx = \int \cos^2\theta\, d\theta$$
$$= \frac{1}{2}\int \left(1+\cos 2\theta\right)d\theta$$
$$= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)$$
$$= \frac{1}{2}\left(\theta + \sin\theta\cos\theta\right) + C$$
$$= \frac{1}{2}\left(\arcsin e^x + e^x\sqrt{1-e^{2x}}\right) + C$$



28. Let $\sqrt{x} = \sin\theta, x = \sin^2\theta, dx = 2\sin\theta\cos\theta\, d\theta$,

$\sqrt{1-x} = \cos\theta$.

$$\int \frac{\sqrt{1-x}}{\sqrt{x}}\,dx = \int \frac{\cos\theta(2\sin\theta\cos\theta\, d\theta)}{\sin\theta}$$
$$= 2\int \cos^2\theta\, d\theta$$
$$= \int \left(1+\cos 2\theta\right)d\theta$$
$$= \left(\theta + \sin\theta\cos\theta\right) + C$$
$$= \arcsin\sqrt{x} + \sqrt{x}\sqrt{1-x} + C$$

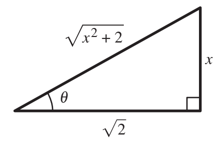


29. Let $x = \sqrt{2}\tan\theta, dx = \sqrt{2}\sec^2\theta\, d\theta,\ x^2 + 2 = 2\sec^2\theta.$

$$\int \frac{1}{4 + 4x^2 + x^4}\,dx = \int \frac{1}{\left(x^2 + 2\right)^2}\,dx = \int \frac{\sqrt{2}\sec^2\theta\, d\theta}{4\sec^4\theta}$$

$$= \frac{\sqrt{2}}{4}\int \cos^2\theta\, d\theta = \frac{\sqrt{2}}{4}\left(\frac{1}{2}\right)\int\left(1 + \cos 2\theta\right)d\theta$$

$$= \frac{\sqrt{2}}{8}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{\sqrt{2}}{8}\left(\theta + \sin\theta\cos\theta\right) + C$$



$$= \frac{\sqrt{2}}{8}\left(\arctan\frac{x}{\sqrt{2}} + \frac{x}{\sqrt{x^2 + 2}}\cdot\frac{\sqrt{2}}{\sqrt{x^2 + 2}}\right) = \frac{1}{4}\left(\frac{x}{x^2 + 2} + \frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}\right) + C$$

30. Let $x = \tan\theta, dx = \sec^2\theta\, d\theta, x^2 + 1 = \sec^2\theta.$

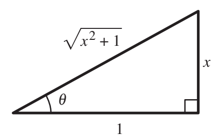
$$\int \frac{x^3 + x + 1}{x^4 + 2x^2 + 1}\,dx = \frac{1}{4}\int \frac{4x^3 + 4x}{x^4 + 2x^2 + 1}\,dx + \int \frac{1}{\left(x^2 + 1\right)^2}\,dx$$

$$= \frac{1}{4}\ln\left(x^4 + 2x^2 + 1\right) + \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta}$$

$$= \frac{1}{2}\ln\left(x^2 + 1\right) + \frac{1}{2}\int\left(1 + \cos 2\theta\right)d\theta$$

$$= \frac{1}{2}\ln\left(x^2 + 1\right) + \frac{1}{2}\left(\theta + \sin\theta\cos\theta\right) + C$$

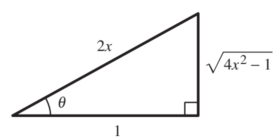
$$= \frac{1}{2}\left(\ln\left(x^2 + 1\right) + \arctan x + \frac{x}{x^2 + 1}\right) + C$$



31. Use integration by parts. Because $x > \frac{1}{2}$,

$$u = \operatorname{arcsec} 2x \Rightarrow du = \frac{1}{x\sqrt{4x^2 - 1}}\,dx, dv = dx \Rightarrow v = x$$

$$\int \operatorname{arcsec} 2x\, dx = x\operatorname{arcsec} 2x - \int \frac{1}{\sqrt{4x^2 - 1}}\,dx$$



$$2x = \sec\theta, dx = \frac{1}{2}\sec\theta\tan\theta\, d\theta, \sqrt{4x^2 - 1} = \tan\theta$$

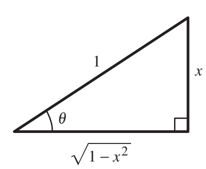
$$\int \operatorname{arcsec} 2x\, dx = x\operatorname{arcsec} 2x - \int \frac{(1/2)\sec\theta\tan\theta\, d\theta}{\tan\theta} = x\operatorname{arcsec} 2x - \frac{1}{2}\int \sec\theta\, d\theta$$

$$= x\operatorname{arcsec} 2x - \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C = x\operatorname{arcsec} 2x - \frac{1}{2}\ln\left|2x + \sqrt{4x^2 - 1}\right| + C.$$

32. $u = \arcsin x \Rightarrow du = \dfrac{1}{\sqrt{1 - x^2}}\,dx, dv = x\,dx \Rightarrow v = \dfrac{x^2}{2}$

$$\int x\arcsin x\, dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1 - x^2}}\,dx$$

$$x = \sin\theta, dx = \cos\theta\, d\theta, \sqrt{1 - x^2} = \cos\theta$$



$$\int x\arcsin x\, dx = \frac{x^2}{2}\arcsin x = \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta}\cos\theta\, d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{4}\int\left(1 - \cos 2\theta\right)d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C = \frac{x^2}{2}\arcsin x - \frac{1}{4}\left(\theta - \sin\theta\cos\theta\right) + C$$

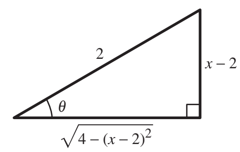
$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left(\arcsin x - x\sqrt{1 - x^2}\right) + C = \frac{1}{4}\left[\left(2x^2 - 1\right)\arcsin x + x\sqrt{1 - x^2}\right] + C$$

33. $\int \frac{x}{\sqrt{4x - x^2}}\,dx = \int \frac{x}{\sqrt{4 - (x-2)^2}}\,dx$

Let $x - 2 = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$,

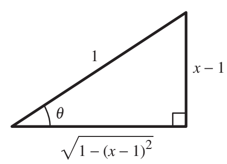
$\sqrt{4 - (x-2)^2} = \sqrt{4 - 4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta.$

$$\begin{aligned}\int \frac{x}{\sqrt{4-(x-2)^2}}\,dx &= \int \frac{2 + 2\sin\theta}{2\cos\theta} 2\cos\theta\,d\theta\\ &= \int (2 + 2\sin\theta)\,d\theta\\ &= 2\theta - 2\cos\theta + C\\ &= 2\arcsin\frac{x-2}{2} - 2\frac{\sqrt{4-(x-2)^2}}{2} + C\\ &= 2\arcsin\frac{x-2}{2} - \sqrt{4x - x^2} + C\end{aligned}$$



34. Let $x - 1 = \sin\theta$, $dx = \cos\theta\,d\theta$, $\sqrt{1-(x-1)^2} = \sqrt{2x - x^2} = \cos\theta.$

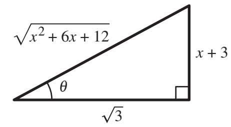
$$\begin{aligned}\int \frac{x^2}{\sqrt{2x - x^2}}\,dx &= \int \frac{x^2}{\sqrt{1-(x-1)^2}}\,dx\\ &= \int \frac{(1+\sin\theta)^2(\cos\theta\,d\theta)}{\cos\theta}\\ &= \int (1 + 2\sin\theta + \sin^2\theta)\,d\theta\\ &= \int\left(\frac{3}{2} + 2\sin\theta - \frac{1}{2}\cos 2\theta\right)d\theta\\ &= \frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta + C\\ &= \frac{3}{2}\theta - 2\cos\theta - \frac{1}{2}\sin\theta\cos\theta + C\\ &= \frac{3}{2}\arcsin(x-1) - 2\sqrt{2x-x^2} - \frac{1}{2}(x-1)\sqrt{2x-x^2} + C\\ &= \frac{3}{2}\arcsin(x-1) - \frac{1}{2}\sqrt{2x-x^2}(x+3) + C\end{aligned}$$



35. $x^2 + 6x + 12 = x^2 + 6x + 9 + 3 = (x+3)^2 + (\sqrt{3})^2$

Let $x + 3 = \sqrt{3}\tan\theta$, $dx = \sqrt{3}\sec^2\theta\,d\theta$.

$\sqrt{x^2 + 6x + 12} = \sqrt{(x+3)^2 + (\sqrt{3})^2} = \sqrt{3}\sec\theta$



$$\begin{aligned}\int \frac{x}{\sqrt{x^2 + 6x + 12}}\,dx &= \int \frac{\sqrt{3}\tan\theta - 3}{\sqrt{3}\sec\theta}\sqrt{3}\sec^2\theta\,d\theta\\ &= \int \sqrt{3}\sec\theta\tan\theta\,d\theta - 3\int\sec\theta\,d\theta\\ &= \sqrt{3}\sec\theta - 3\ln|\sec\theta + \tan\theta| + C\\ &= \sqrt{3}\left(\frac{\sqrt{x^2+6x+12}}{\sqrt{3}}\right) - 3\ln\left|\frac{\sqrt{x^2+6x+12}}{\sqrt{3}} + \frac{x+3}{\sqrt{3}}\right| + C\\ &= \sqrt{x^2+6x+12} - 3\ln\left|\sqrt{x^2+6x+12} + (x+3)\right| + C\end{aligned}$$

36. Let $x - 3 = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\,d\theta$, $\sqrt{(x-3)^2 - 4} = 2\tan\theta$.

$$\int \frac{x}{\sqrt{x^2 - 6x + 5}}\,dx = \int \frac{x}{\sqrt{(x-3)^2 - 4}}\,dx$$

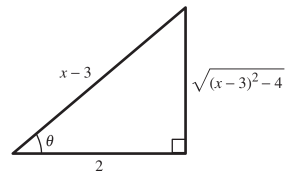
$$= \int \frac{(2\sec\theta + 3)}{2\tan\theta}(2\sec\theta\tan\theta)\,d\theta$$

$$= \int \left(2\sec^2\theta + 3\sec\theta\right)d\theta$$

$$= 2\tan\theta + 3\ln\left|\sec\theta + \tan\theta\right| + C_1$$

$$= 2\left[\frac{\sqrt{(x-3)^3 - 4}}{2}\right] + 3\ln\left|\frac{x-3}{2} + \frac{\sqrt{(x-3)^2 - 4}}{2}\right| + C_1$$

$$= \sqrt{x^2 - 6x + 5} + 3\ln\left|(x-3) + \sqrt{x^2 - 6x + 5}\right| + C$$



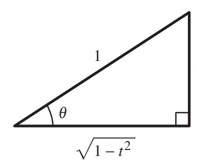
37. Let $t = \sin\theta$, $dt = \cos\theta\,d\theta$, $1 - t^2 = \cos^2\theta$.

(a) $\displaystyle\int \frac{t^2}{\left(1 - t^2\right)^{3/2}}\,dt = \int \frac{\sin^2\theta\cos\theta\,d\theta}{\cos^3\theta} = \int \tan^2\theta\,d\theta = \int\left(\sec^2\theta - 1\right)d\theta = \tan\theta - \theta + C = \frac{t}{\sqrt{1 - t^2}} - \arcsin t + C$

So, $\displaystyle\int_0^{\sqrt{3}/2} \frac{t^2}{\left(1 - t^2\right)^{3/2}}\,dt = \left[\frac{t}{\sqrt{1 - t^2}} - \arcsin t\right]_0^{\sqrt{3}/2} = \frac{\sqrt{3}/2}{\sqrt{1/4}} - \arcsin\frac{\sqrt{3}}{2} = \sqrt{3} - \frac{\pi}{3} \approx 0.685.$

(b) When $t = 0$, $\theta = 0$. When $t = \sqrt{3}/2$, $\theta = \pi/3$. So,

$$\int_0^{\sqrt{3}/2} \frac{t^2}{\left(1 - t^2\right)^{3/2}}\,dt = \left[\tan\theta - \theta\right]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3} \approx 0.685.$$



38. Same substitution as in Exercise 37

(a) $\displaystyle\int \frac{1}{\left(1 - t^2\right)^{5/2}}\,dt = \int \frac{\cos\theta\,d\theta}{\cos^5\theta} = \int \sec^4\theta\,d\theta = \int\left(\tan^2\theta + 1\right)\sec^2\theta\,d\theta$

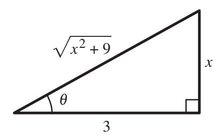
$$= \frac{1}{3}\tan^3\theta + \tan\theta + C = \frac{1}{3}\left(\frac{t}{\sqrt{1 - t^2}}\right)^3 + \frac{t}{\sqrt{1 - t^2}} + C$$

So, $\displaystyle\int_0^{\sqrt{3}/2} \frac{1}{\left(1 - t^2\right)^{5/2}}\,dt = \left[\frac{t^3}{3\left(1 - t^2\right)^{3/2}} + \frac{t}{\sqrt{1 - t^2}}\right]_0^{\sqrt{3}/2} = \frac{3\sqrt{3}/8}{3(1/4)^{3/2}} + \frac{\sqrt{3}/2}{\sqrt{1/4}} = \sqrt{3} + \sqrt{3} = 2\sqrt{3} \approx 3.464.$

(b) When $t = 0$, $\theta = 0$. When $t = \sqrt{3}/2$, $\theta = \pi/3$. So,

$$\int_0^{\sqrt{3}/2} \frac{1}{\left(1 - t^2\right)^{5/2}}\,dt = \left[\frac{1}{3}\tan^3\theta + \tan\theta\right]_0^{\pi/3} = \frac{1}{3}\left(\sqrt{3}\right)^3 + \sqrt{3} = 2\sqrt{3} \approx 3.464.$$

39. (a) Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$, $\sqrt{x^2+9} = 3\sec\theta$.



$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \frac{(27\tan^3\theta)(3\sec^2\theta\,d\theta)}{3\sec\theta}$$
$$= 27\int(\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta$$
$$= 27\left[\frac{1}{3}\sec^3\theta - \sec\theta\right] + C = 9\left[\sec^3\theta - 3\sec\theta\right] + C$$
$$= 9\left[\left(\frac{\sqrt{x^2+9}}{3}\right)^3 - 3\left(\frac{\sqrt{x^2+9}}{3}\right)\right] + C = \frac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9} + C$$

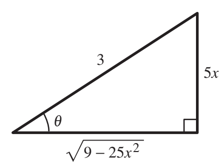
So, $$\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,dx = \left[\frac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9}\right]_0^3$$
$$= \left(\frac{1}{3}(54\sqrt{2}) - 27\sqrt{2}\right) - (9-27) = 18 - 9\sqrt{2} = 9(2-\sqrt{2}) \approx 5.272.$$

(b) When $x = 0$, $\theta = 0$. When $x = 3$, $\theta = \pi/4$. So,

$$\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,dx = 9\left[\sec^3\theta - 3\sec\theta\right]_0^{\pi/4} = 9(2\sqrt{2} - 3\sqrt{2}) - 9(1-3) = 9(2-\sqrt{2}) \approx 5.272.$$

40. (a) Let $5x = 3\sin\theta$, $dx = \frac{3}{5}\cos\theta\,d\theta$, $\sqrt{9-25x^2} = 3\cos\theta$.

$$\int\sqrt{9-25x^2}\,dx = \int(3\cos\theta)\frac{3}{5}\cos\theta\,d\theta$$
$$= \frac{9}{5}\int\frac{1+\cos 2\theta}{2}\,d\theta$$
$$= \frac{9}{10}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$
$$= \frac{9}{10}(\theta + \sin\theta\cos\theta) + C$$
$$= \frac{9}{10}\left(\arcsin\frac{5x}{3} + \frac{5x}{3}\cdot\frac{\sqrt{9-25x^2}}{3}\right) + C$$

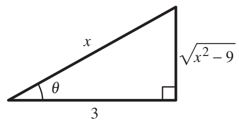


So, $$\int_0^{3/5}\sqrt{9-25x^2}\,dx = \frac{9}{10}\left[\arcsin\frac{5x}{3} + \frac{5x\sqrt{9-25x^2}}{9}\right]_0^{3/5} = \frac{9}{10}\left[\frac{\pi}{2}\right] = \frac{9\pi}{20}.$$

(b) When $x = 0$, $\theta = 0$. When $x = \frac{3}{5}$, $\theta = \frac{\pi}{2}$.

So, $$\int_0^{3/5}\sqrt{9-25x^2}\,dx = \left[\frac{9}{10}(\theta + \sin\theta\cos\theta)\right]_0^{\pi/2} = \frac{9}{10}\left(\frac{\pi}{2}\right) = \frac{9\pi}{20}.$$

41. (a) Let $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-9} = 3\tan\theta$.



$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2-9}}\,dx &= \int \frac{9\sec^2\theta}{3\tan\theta}3\sec\theta\tan\theta\,d\theta\\ &= 9\int \sec^3\theta\,d\theta\\ &= 9\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int\sec\theta\,d\theta\right) \quad \text{(8.3 Exercise 102 or Example 5, Section 8.2)}\\ &= \frac{9}{2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)\\ &= \frac{9}{2}\left(\frac{x}{3}\cdot\frac{\sqrt{x^2-9}}{3} + \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|\right)\end{aligned}$$

So,

$$\begin{aligned}\int_4^6 \frac{x^2}{\sqrt{x^2-9}}\,dx &= \frac{9}{2}\left[\frac{x\sqrt{x^2-9}}{9} + \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|\right]_4^6\\ &= \frac{9}{2}\left[\left(\frac{6\sqrt{27}}{9} + \ln\left|2 + \frac{\sqrt{27}}{3}\right|\right) - \left(\frac{4\sqrt{7}}{9} + \ln\left|\frac{4}{3} + \frac{\sqrt{7}}{3}\right|\right)\right]\\ &= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\left[\ln\left(\frac{6+\sqrt{27}}{3}\right) - \ln\left(\frac{4+\sqrt{7}}{3}\right)\right]\\ &= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{6+3\sqrt{3}}{4+\sqrt{7}}\right) \approx 12.644.\end{aligned}$$

(b) When $x = 4$, $\theta = \operatorname{arcsec}\left(\frac{4}{3}\right)$. When $x = 6$, $\theta = \operatorname{arcsec}(2) = \frac{\pi}{3}$.

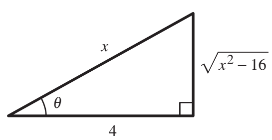
$$\begin{aligned}\int_4^6 \frac{x^2}{\sqrt{x^2-9}}\,dx &= \frac{9}{2}\left[\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right]_{\operatorname{arcsec}(4/3)}^{\pi/3}\\ &= \frac{9}{2}\left(2\cdot\sqrt{3} + \ln\left|2 + \sqrt{3}\right|\right) - \frac{9}{2}\left(\frac{4}{3}\left(\frac{\sqrt{7}}{3}\right) + \ln\left|\frac{4}{3} + \frac{\sqrt{7}}{3}\right|\right)\\ &= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{6+3\sqrt{3}}{4+\sqrt{7}}\right) \approx 12.644\end{aligned}$$

42. (a) Let $x = 4\sec\theta$, $dx = 4\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-16} = 4\tan\theta$.

$$\begin{aligned}\int \frac{\sqrt{x^2-16}}{x^2}\,dx &= \int \frac{4\tan\theta}{16\sec^2\theta}\left(4\sec\theta\tan\theta\right)d\theta \\ &= \int \frac{\tan^2\theta}{\sec\theta}\,d\theta \\ &= \int \frac{\sin^2\theta}{\cos\theta}\,d\theta \\ &= \int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta \\ &= \int \sec\theta\,d\theta - \int \cos\theta\,d\theta \\ &= \ln\left|\sec\theta + \tan\theta\right| - \sin\theta + C \\ &= \ln\left|\frac{x}{4} + \frac{\sqrt{x^2-16}}{4}\right| - \frac{\sqrt{x^2-16}}{x} + C\end{aligned}$$

So,

$$\begin{aligned}\int_4^8 \frac{\sqrt{x^2-16}}{x^2}\,dx &= \left[\ln\left|\frac{x}{4} + \frac{\sqrt{x^2-16}}{4}\right| - \frac{\sqrt{x^2-16}}{x}\right]_4^8 \\ &= \left[\ln\left(2 + \frac{\sqrt{48}}{4}\right) - \frac{\sqrt{48}}{8}\right] - \left[\ln(1)\right] \\ &= \ln\left(2+\sqrt{3}\right) - \frac{\sqrt{3}}{2}.\end{aligned}$$



(b) When $x = 4$, $\theta = 0$, and when $x = 8$, $\theta = \frac{\pi}{3}$. So,

$$\begin{aligned}\int_4^8 \frac{\sqrt{x^2-16}}{x^2}\,dx &= \left[\ln\left|\sec\theta + \tan\theta\right| - \sin\theta\right]_0^{\pi/3} \\ &= \ln\left|2+\sqrt{3}\right| - \frac{\sqrt{3}}{2}.\end{aligned}$$

43. Substitution: $u = x^2 + 1$, $du = 2x\,dx$

44. Trigonometric substitution: $x = \sec\theta$

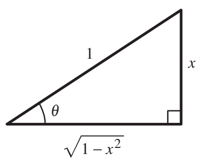
45. (a) u-substitution: Let $u = 1 - x^2$, $du = -2x\,dx$.

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \left(1-x^2\right)^{-1/2}(-2x)\,dx$$
$$= -\frac{1}{2}\left(1-x^2\right)^{1/2}(2) + C = -\sqrt{1-x^2} + C$$

Trigonometric substitution

Let $x = \sin\theta$, $dx = \cos\theta\,d\theta$, $a = 1$, $\sqrt{1-x^2} = \cos\theta$.

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{\sin\theta}{\cos\theta}\cos\theta\,d\theta = \int \sin\theta\,d\theta$$
$$= -\cos\theta + C = -\sqrt{1-x^2} + C$$



The answers are equivalent.

(b) $$\int \frac{x^2}{x^2+9}\,dx = \int \frac{x^2+9-9}{x^2+9}\,dx = \int\left(1 - \frac{9}{x^2+9}\right)dx = x - 3\arctan\left(\frac{x}{3}\right) + C$$

Let $x = 3\tan\theta$, $x^2 + 9 = 9\sec^2\theta$, $dx = 3\sec^2\theta\,d\theta$.

$$\int \frac{x^2}{x^2+9}\,dx = \int \frac{9\tan^2\theta}{9\sec^2\theta}3\sec^2\theta\,d\theta$$
$$= 3\int \tan^2\theta\,d\theta = 3\int\left(\sec^2\theta - 1\right)d\theta$$
$$= 3\tan\theta - 3\theta + C_1$$
$$= x - 3\arctan\left(\frac{x}{3}\right) + C_1$$

The answers are equivalent.

46. (a) The graph of f is increasing when $f' > 0 : 0 < x < \infty$.

The graph of f is decreasing when $f' < 0 : -\infty < x < 0$.

(b) The graph of f is concave upward when the graph of f' is increasing. There are no such intervals.

The graph of f is concave downward when the graph of f' is decreasing:

$-\infty < x < 0$ and $0 < x < \infty$.

47. True

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta\,d\theta}{\cos\theta} = \int d\theta$$

48. False

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\tan\theta}{\sec\theta}(\sec\theta\tan\theta\,d\theta) = \int \tan^2\theta\,d\theta$$

49. False

$$\int_0^{\sqrt{3}} \frac{dx}{\left(\sqrt{1+x^2}\right)^3} = \int_0^{\pi/3} \frac{\sec^2\theta\,d\theta}{\sec^3\theta} = \int_0^{\pi/3} \cos\theta\,d\theta$$

50. True

$$\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx = 2\int_0^1 x^2\sqrt{1-x^2}\,dx$$
$$= 2\int_0^{\pi/2}\left(\sin^2\theta\right)(\cos\theta)(\cos\theta\,d\theta)$$
$$= 2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

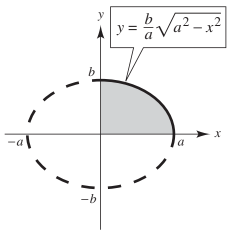
51. $A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$

$= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$

$= \left[\frac{4b}{a}\left(\frac{1}{2}\right)\left(a^2 \arcsin\frac{x}{a} + x\sqrt{a^2 - x^2}\right)\right]_0^a$

$= \frac{2b}{a}\left(a^2\left(\frac{\pi}{2}\right)\right) = \pi ab$

Note: See Theorem 8.2 for $\int\sqrt{a^2 - x^2}\,dx$.



52. $x^2 + y^2 = a^2$

$x = \pm\sqrt{a^2 - y^2}$

$A = 2\int_h^a \sqrt{a^2 - y^2}\,dy$

$= \left[a^2 \arcsin\left(\frac{y}{a}\right) + y\sqrt{a^2 - y^2}\right]_h^a$ (Theorem 8.2)

$= \left(a^2\frac{\pi}{2}\right) - \left(a^2 \arcsin\left(\frac{h}{a}\right) + h\sqrt{a^2 - h^2}\right)$

$= \frac{a^2\pi}{2} - a^2 \arcsin\left(\frac{h}{a}\right) - h\sqrt{a^2 - h^2}$

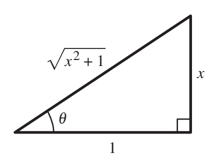
53. $y = \ln x,\ y' = \frac{1}{x},\ 1 + (y')^2 = 1 + \frac{1}{x^2} = \frac{x^2 + 1}{x^2}$

Let $x = \tan\theta,\ dx = \sec^2\theta\,d\theta,\ \sqrt{x^2 + 1} = \sec\theta$.

$s = \int_1^5 \sqrt{\frac{x^2 + 1}{x^2}}\,dx = \int_1^5 \frac{\sqrt{x^2 + 1}}{x}\,dx$

$= \int_a^b \frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta = \int_a^b \frac{\sec\theta}{\tan\theta}\left(1 + \tan^2\theta\right)d\theta$

$= \int_a^b (\csc\theta + \sec\theta\tan\theta)\,d\theta = \left[-\ln\left|\csc\theta + \cot\theta\right| + \sec\theta\right]_a^b$



$= \left[-\ln\left|\frac{\sqrt{x^2 + 1}}{x} + \frac{1}{x}\right| + \sqrt{x^2 + 1}\right]_1^5$

$= \left[-\ln\left(\frac{\sqrt{26} + 1}{5}\right) + \sqrt{26}\right] - \left[-\ln\left(\sqrt{2} + 1\right) + \sqrt{2}\right]$

$= \ln\left[\frac{5\left(\sqrt{2} + 1\right)}{\sqrt{26} + 1}\right] + \sqrt{26} - \sqrt{2} \approx 4.367$ or $\ln\left[\frac{\sqrt{26} - 1}{5\left(\sqrt{2} - 1\right)}\right] + \sqrt{26} - \sqrt{2}$

54. $y = \frac{x^2}{4} - 2x,\ y' = \frac{x}{2} - 2,$

$1 + (y')^2 = 1 + \left(\frac{x}{2} - 2\right)^2 = 1 + \frac{x^2}{4} - 2x + 4 = \frac{x^2}{4} - 2x + 5 = \frac{1}{4}\left(x^2 - 8x + 20\right)$

$s = \int_4^8 \sqrt{\frac{1}{4}\left(x^2 - 8x + 20\right)}\,dx = \frac{1}{2}\int_4^8 \sqrt{x^2 - 8x + 16 + 4}\,dx$

$= \frac{1}{2}\int_4^8 \sqrt{(x - 4)^2 + 4}\,dx$ $(u = x - 4,\ a = 2)$

$= \frac{1}{2}\left(\frac{1}{2}\right)\left[(x - 4)\sqrt{(x - 4)^2 + 4} + 4\ln\left|(x - 4) + \sqrt{(x - 4)^2 + 4}\right|\right]_4^8$

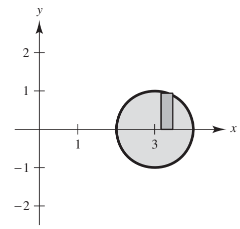
$= \frac{1}{4}\left[\left(4\sqrt{20} + 4\ln\left(4 + \sqrt{20}\right)\right) - (0 + 4\ln 2)\right]$

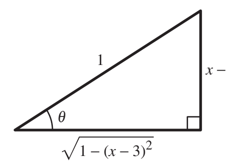
≈ 5.916

55. Let $x - 3 = \sin\theta$, $dx = \cos\theta\,d\theta$, $\sqrt{1 - (x - 3)^2} = \cos\theta$.

Shell Method:

$$V = 4\pi\int_2^4 x\sqrt{1 - (x - 3)^2}\,dx$$
$$= 4\pi\int_{-\pi/2}^{\pi/2}(3 + \sin\theta)\cos^2\theta\,d\theta$$
$$= 4\pi\left[\frac{3}{2}\int_{-\pi/2}^{\pi/2}(1 + \cos 2\theta)\,d\theta + \int_{-\pi/2}^{\pi/2}\cos^2\theta\sin\theta\,d\theta\right]$$
$$= 4\pi\left[\frac{3}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) - \frac{1}{3}\cos^3\theta\right]_{-\pi/2}^{\pi/2} = 6\pi^2$$

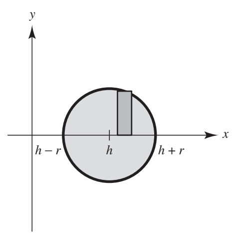


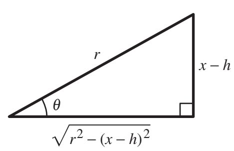


56. Let $x - h = r\sin\theta$, $dx = r\cos\theta\,d\theta$, $\sqrt{r^2 - (x - h)^2} = r\cos\theta$.

Shell Method:

$$V = 4\pi\int_{h-r}^{h+r} x\sqrt{r^2 - (x - h)^2}\,dx$$
$$= 4\pi\int_{-\pi/2}^{\pi/2}(h + r\sin\theta)r\cos\theta(r\cos\theta)\,d\theta = 4\pi r^2\int_{-\pi/2}^{\pi/2}(h + r\sin\theta)\cos^2\theta\,d\theta$$
$$= 4\pi r^2\left[\frac{h}{2}\int_{-\pi/2}^{\pi/2}(1 + \cos 2\theta)\,d\theta + r\int_{-\pi/2}^{\pi/2}\sin\theta\cos^2\theta\,d\theta\right]$$
$$= 2\pi r^2 h\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} - \left[4\pi r^3\left(\frac{\cos^3\theta}{3}\right)\right]_{-\pi/2}^{\pi/2} = 2\pi^2 r^2 h$$





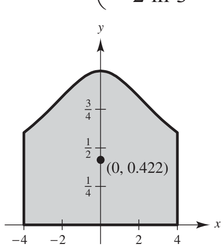
57. Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$, $\sqrt{x^2 + 9} = 3\sec\theta$.

$$A = 2\int_0^4 \frac{3}{\sqrt{x^2 + 9}}\,dx = 6\int_0^4 \frac{dx}{\sqrt{x^2 + 9}} = 6\int_a^b \frac{3\sec^2\theta\,d\theta}{3\sec\theta}$$
$$= 6\int_a^b \sec\theta\,d\theta = \Big[6\ln|\sec\theta + \tan\theta|\Big]_a^b = \left[6\ln\left|\frac{\sqrt{x^2 + 9} + x}{3}\right|\right]_0^4 = 6\ln 3$$

$\bar{x} = 0$ (by symmetry)

$$\bar{y} = \frac{1}{2}\left(\frac{1}{A}\right)\int_{-4}^4\left(\frac{3}{\sqrt{x^2 + 9}}\right)^2 dx = \frac{9}{12\ln 3}\int_{-4}^4 \frac{1}{x^2 + 9}\,dx = \frac{3}{4\ln 3}\left[\frac{1}{3}\arctan\frac{x}{3}\right]_{-4}^4 = \frac{2}{4\ln 3}\arctan\frac{4}{3} \approx 0.422$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{1}{2\ln 3}\arctan\frac{4}{3}\right) \approx (0, 0.422)$$



58. First find where the curves intersect.

$$y^2 = 16 - (x-4)^2 = \frac{1}{16}x^4$$

$$16^2 - 16(x-4)^2 = x^4$$

$$16^2 - 16x^2 + 128x - 16^2 = x^4$$

$$x^4 + 16x^2 - 128x = 0$$

$$x(x-4)(x^2 + 4x + 32) = 0$$

$$\Rightarrow x = 0, 4$$

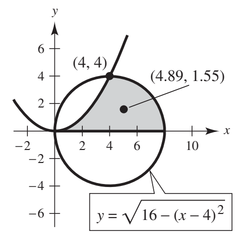
$$A = \int_0^4 \frac{1}{4}x^2\,dx + \frac{1}{4}\pi(4)^2 = \left[\frac{1}{12}x^3\right]_0^4 + 4\pi = \frac{16}{3} + 4\pi$$

$$M_y = \int_0^4 x\left(\frac{1}{4}x^2\right)dx + \int_4^8 x\sqrt{16-(x-4)^2}\,dx$$

$$= \left[\frac{x^4}{16}\right]_0^4 + \int_4^8 (x-4)\sqrt{16-(x-4)^2}\,dx + \int_4^8 4\sqrt{16-(x-4)^2}\,dx$$

$$= 16 + \left[\frac{-1}{3}\left(16-(x-4)^2\right)^{3/2}\right]_4^8 + 2\left[16\arcsin\frac{x-4}{4} + (x-4)\sqrt{16-(x-4)^2}\right]_4^8$$

$$= 16 + \frac{1}{3}\left(16^{3/2}\right) + 2\left[16\left(\frac{\pi}{2}\right)\right] = 16 + \frac{64}{3} + 16\pi = \frac{112}{3} + 16\pi$$



$$M_x = \int_0^4 \frac{1}{2}\left(\frac{1}{4}x^2\right)^2 dx + \int_4^8 \frac{1}{2}\left(16-(x-4)^2\right)dx = \left[\frac{1}{32}\cdot\frac{x^5}{5}\right]_0^4 + \left[8x - \frac{(x-4)^3}{6}\right]_4^8 = \frac{32}{5} + \left(64 - \frac{64}{6}\right) - 32 = \frac{416}{15}$$

$$\bar{x} = \frac{M_y}{A} = \frac{112/3 + 16\pi}{16/3 + 4\pi} = \frac{112 + 48\pi}{16 + 12\pi} = \frac{28 + 12\pi}{4 + 3\pi} \approx 4.89$$

$$\bar{y} = \frac{M_x}{A} = \frac{416/15}{(16/3) + 4\pi} = \frac{104}{5(4 + 3\pi)} \approx 1.55$$

$$(\bar{x}, \bar{y}) \approx (4.89, 1.55)$$

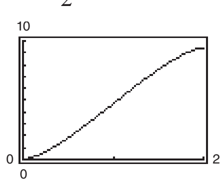
59. (a) Place the center of the circle at $(0, 1)$; $x^2 + (y-1)^2 = 1$. The depth d satisfies $0 \le d \le 2$. The volume is

$$V = 3\cdot 2\int_0^d \sqrt{1-(y-1)^2}\,dy = 6\cdot\frac{1}{2}\left[\arcsin(y-1) + (y-1)\sqrt{1-(y-1)^2}\right]_0^d \quad \text{(Theorem 8.2 (1))}$$

$$= 3\left[\arcsin(d-1) + (d-1)\sqrt{1-(d-1)^2} - \arcsin(-1)\right]$$

$$= \frac{3\pi}{2} + 3\arcsin(d-1) + 3(d-1)\sqrt{2d - d^2}.$$

(b)



(c) The full tank holds $3\pi \approx 9.4248$ cubic meters. The horizontal lines

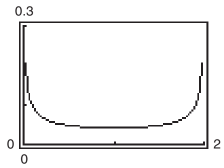
$$y = \frac{3\pi}{4},\ y = \frac{3\pi}{2},\ y = \frac{9\pi}{4}$$

intersect the curve at $d = 0.596, 1.0, 1.404$. The dipstick would have these markings on it.

(d) $V = 6\int_0^d \sqrt{1 - (y-1)^2}\, dy$

$$\frac{dV}{dt} = \frac{dV}{dd}\cdot\frac{dd}{dt} = 6\sqrt{1-(d-1)^2}\cdot d'(t) = \frac{1}{4} \Rightarrow d'(t) = \frac{1}{24\sqrt{1-(d-1)^2}}$$

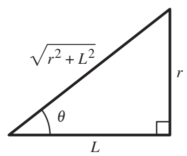
(e)



The minimum occurs at $d = 1$, which is the widest part of the tank.

60. Let $r = L\tan\theta$, $dr = L\sec^2\theta\, d\theta$, $r^2 + L^2 = L^2\sec^2\theta$.

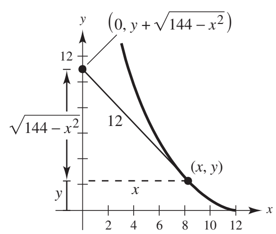
$$\frac{1}{R}\int_0^R \frac{2mL}{\left(r^2+L^2\right)^{3/2}}\, dr = \frac{2mL}{R}\int_a^b \frac{L\sec^2\theta\, d\theta}{L^3\sec^3\theta} = \frac{2m}{RL}\int_a^b \cos\theta\, d\theta = \left[\frac{2m}{RL}\sin\theta\right]_a^b = \left[\frac{2m}{RL}\frac{r}{\sqrt{r^2+L^2}}\right]_0^R = \frac{2m}{L\sqrt{R^2+L^2}}$$



61. (a) $m = \dfrac{dy}{dx}$

$$= \frac{y - \left(y + \sqrt{144-x^2}\right)}{x - 0}$$

$$= -\frac{\sqrt{144-x^2}}{x}$$



(b) $y = -\int \dfrac{\sqrt{144-x^2}}{x}\, dx$

Let $x = 12\sin\theta$, $dx = 12\cos\theta\, d\theta$, $\sqrt{144-x^2} = 12\cos\theta$.

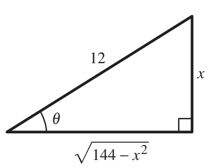
$$y = -\int \frac{12\cos\theta}{12\sin\theta} 12\cos\theta\, d\theta = -12\int \frac{1-\sin^2\theta}{\sin\theta}\, d\theta$$

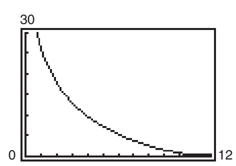
$$= -12\int(\csc\theta - \sin\theta)\, d\theta = -12\ln\left|\csc\theta - \cot\theta\right| - 12\cos\theta + C$$

$$= -12\ln\left|\frac{12}{x} - \frac{\sqrt{144-x^2}}{x}\right| - 12\left(\frac{\sqrt{144-x^2}}{12}\right) + C = -12\ln\left|\frac{12-\sqrt{144-x^2}}{x}\right| - \sqrt{144-x^2} + C$$

When $x = 12$, $y = 0 \Rightarrow C = 0$. So, $y = -12\ln\left(\dfrac{12-\sqrt{144-x^2}}{x}\right) - \sqrt{144-x^2}$.

Note: $\dfrac{12-\sqrt{144-x^2}}{x} > 0$ for $0 < x \le 12$





(c) Vertical asymptote: $x = 0$

(d) $y + \sqrt{144 - x^2} = 12 \Rightarrow y = 12 - \sqrt{144 - x^2}$

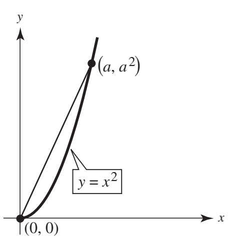
So,
$$12 - \sqrt{144 - x^2} = -12 \ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$$
$$-1 = \ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right)$$
$$xe^{-1} = 12 - \sqrt{144 - x^2}$$
$$\left(xe^{-1} - 12\right)^2 = \left(-\sqrt{144 - x^2}\right)^2$$
$$x^2e^{-2} - 24xe^{-1} + 144 = 144 - x^2$$
$$x^2\left(e^{-2} + 1\right) - 24xe^{-1} = 0$$
$$x\left[x\left(e^{-2} + 1\right) - 24e^{-1}\right] = 0$$
$$x = 0 \text{ or } x = \frac{24e^{-1}}{e^{-2} + 1} \approx 7.77665.$$

Therefore, $s = \int_{7.77665}^{12} \sqrt{1 + \left(-\frac{\sqrt{144 - x^2}}{x}\right)^2}\,dx = \int_{7.77665}^{12} \sqrt{\frac{x^2 + \left(144 - x^2\right)}{x^2}}\,dx$

$$= \int_{7.77665}^{12} \frac{12}{x}\,dx = \Big[12 \ln|x|\Big]_{7.77665}^{12} = 12\left(\ln 12 - \ln 7.77665\right) \approx 5.2 \text{ meters.}$$

62. (a) Distance along line:

$$d = \sqrt{\left(a^2 - 0\right)^2 + \left(a - 0\right)^2} = \sqrt{a^4 + a^2} = a\sqrt{a^2 + 1}$$



Distance along $y = x^2$:

$$y' = 2x, 1 + \left(y'\right)^2 = 1 + 4x^2$$

$$d = \int_0^a \sqrt{1 + 4x^2}\,dx = \frac{1}{4}\left[\ln\left(\sqrt{4a^2 + 1} + 2a\right) + 2a\sqrt{4a^2 + 1}\right]$$

(b)

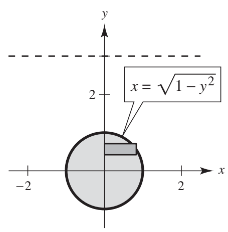
a	Line	Parabola
1	$\sqrt{2} \approx 1.4142$	1.4789
10	100.4988	101.0473
100	10,000.5	10,001.6

(c) The difference between the distances approaches 0 as a increases.

63. (a) Area of representative rectangle: $2\sqrt{1-y^2}\,\Delta y$

Force: $2(62.4)(3-y)\sqrt{1-y^2}\,\Delta y$

$$F = 124.8\int_{-1}^{1}(3-y)\sqrt{1-y^2}\,dy$$
$$= 124.8\left[3\int_{-1}^{1}\sqrt{1-y^2}\,dy - \int_{-1}^{1}y\sqrt{1-y^2}\,dy\right]$$
$$= 124.8\left[\frac{3}{2}\left(\arcsin y + y\sqrt{1-y^2}\right) + \frac{1}{2}\left(\frac{2}{3}\right)\left(1-y^2\right)^{3/2}\right]_{-1}^{1} = (62.4)3\left[\arcsin 1 - \arcsin(-1)\right] = 187.2\pi \text{ lb}$$



(b) $$F = 124.8\int_{-1}^{1}(d-y)\sqrt{1-y^2}\,dy = 124.8d\int_{-1}^{1}\sqrt{1-y^2}\,dy - 124.8\int_{-1}^{1}y\sqrt{1-y^2}\,dy$$
$$= 124.8\left(\frac{d}{2}\right)\left[\arcsin y + y\sqrt{1-y^2}\right]_{-1}^{1} - 124.8(0) = 62.4\pi d \text{ lb}$$

64. (a) $$F_{\text{inside}} = 48\int_{-1}^{0.8}(0.8-y)(2)\sqrt{1-y^2}\,dy$$
$$= 96\left[0.8\int_{-1}^{0.8}\sqrt{1-y^2}\,dy - \int_{-1}^{0.8}y\sqrt{1-y^2}\,dy\right]$$
$$= 96\left[\frac{0.8}{2}\left(\arcsin y + y\sqrt{1-y^2}\right) + \frac{1}{3}\left(1-y^2\right)^{3/2}\right]_{-1}^{0.8}$$
$$\approx 96(1.263) \approx 121.3 \text{ lb}$$

(b) $$F_{\text{outside}} = 64\int_{-1}^{0.4}(0.4-y)(2)\sqrt{1-y^2}\,dy$$
$$= 128\left[0.4\int_{-1}^{0.4}\sqrt{1-y^2}\,dy - \int_{-1}^{0.4}y\sqrt{1-y^2}\,dy\right]$$
$$= 128\left[\frac{0.4}{2}\left(\arcsin y + y\sqrt{1-y^2}\right) + \frac{1}{3}\left(1-y^2\right)^{3/2}\right]_{-1}^{0.4} \approx 92.98$$

65. Let $u = a\sin\theta$, $du = a\cos\theta\,d\theta$, $\sqrt{a^2 - u^2} = a\cos\theta$.

$$\int\sqrt{a^2 - u^2}\,du = \int a^2\cos^2\theta\,d\theta = a^2\int\frac{1 + \cos 2\theta}{2}\,d\theta$$
$$= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{a^2}{2}(\theta + \sin\theta\cos\theta) + C$$
$$= \frac{a^2}{2}\left[\arcsin\frac{u}{a} + \left(\frac{u}{a}\right)\left(\frac{\sqrt{a^2 + u^2}}{a}\right)\right] + C = \frac{1}{2}\left(a^2\arcsin\frac{u}{a} + u\sqrt{a^2 - u^2}\right) + C$$

Let $u = a\sec\theta$, $du = a\sec\theta\tan\theta\,d\theta$, $\sqrt{u^2 - a^2} = a\tan\theta$.

$$\int\sqrt{u^2 - a^2}\,du = \int a\tan\theta(a\sec\theta\tan\theta)\,d\theta = a^2\int\tan^2\theta\sec\theta\,d\theta$$
$$= a^2\int(\sec^2\theta - 1)\sec\theta\,d\theta = a^2\int(\sec^3\theta - \sec\theta)\,d\theta$$
$$= a^2\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int\sec\theta\,d\theta\right] - a^2\int\sec\theta\,d\theta = a^2\left[\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right|\right]$$
$$= \frac{a^2}{2}\left[\frac{u}{a}\cdot\frac{\sqrt{u^2 - a^2}}{a} - \ln\left|\frac{u}{a} + \frac{\sqrt{u^2 - a^2}}{a}\right|\right] + C_1 = \frac{1}{2}\left[u\sqrt{u^2 - a^2} - a^2\ln\left|u + \sqrt{u^2 - a^2}\right|\right] + C$$

Let $u = a\tan\theta$, $du = a\sec^2\theta\,d\theta$, $\sqrt{u^2 + a^2} = a\sec\theta$.

$$\int\sqrt{u^2 + a^2}\,du = \int(a\sec\theta)(a\sec^2\theta)\,d\theta$$
$$= a^2\int\sec^3\theta\,d\theta = a^2\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right|\right] + C_1$$
$$= \frac{a^2}{2}\left[\frac{\sqrt{u^2 + a^2}}{a}\cdot\frac{u}{a} + \ln\left|\frac{\sqrt{u^2 + a^2}}{a} + \frac{u}{a}\right|\right] + C_1 = \frac{1}{2}\left[u\sqrt{u^2 + a^2} + a^2\ln\left|u + \sqrt{u^2 + a^2}\right|\right] + C$$

66. $y = \sin x$ on $[0, 2]$

$y' = \cos x$

$s_1 = 2\int_0^{\pi}\sqrt{1 + \cos^2 x}\,dx \quad (\approx 3.820197789)$

Ellipse: $x^2 + 2y^2 = 2$

Upper half: $y = \sqrt{1 - \frac{1}{2}x^2}, \; -\sqrt{2} \le x \le \sqrt{2}$

$$y' = \frac{-x}{2\sqrt{1 - (1/2)x^2}}$$

$$s_2 = 2\int_{-\sqrt{2}}^{\sqrt{2}}\sqrt{1 + \frac{x^2}{4(1 - (1/2)x^2)}}\,dx = 2\int_{-\sqrt{2}}^{\sqrt{2}}\sqrt{1 + \frac{x^2}{4 - 2x^2}}\,dx$$

Let $x = \sqrt{2}\sin\theta$, $dx = \sqrt{2}\cos\theta\,d\theta$, $x^2 = 2\sin^2\theta$, $4 - 2x^2 = 4 - 4\sin^2\theta = 4\cos^2\theta$.

$$s_2 = 2\int_{-\pi/2}^{\pi/2}\sqrt{1 + \frac{2\sin^2\theta}{4\cos^2\theta}}\sqrt{2}\cos\theta\,d\theta$$
$$= 2\int_{-\pi/2}^{\pi/2}\frac{\sqrt{4\cos^2\theta + 2\sin^2\theta}}{2\cos\theta}\sqrt{2}\cos\theta\,d\theta$$
$$= 2\int_{-\pi/2}^{\pi/2}\frac{\sqrt{2 + 2\cos^2\theta}}{\sqrt{2}}\,d\theta = 2\int_{-\pi/2}^{\pi/2}\sqrt{1 + \cos^2\theta}\,d\theta = 2\int_0^{\pi}\sqrt{1 + \cos^2\theta}\,d\theta = s_1$$

67. Large circle: $x^2 + y^2 = 25$

$$y = \sqrt{25 - x^2}, \quad \text{upper half}$$

From the right triangle, the center of the small circle is $(0, 4)$.

$$x^2 + (y - 4)^2 = 9$$

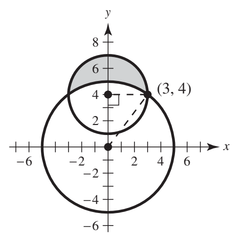
$$y = 4 + \sqrt{9 - x^2}, \quad \text{upper half}$$

$$A = 2\int_0^3 \left[\left(4 + \sqrt{9 - x^2}\right) - \sqrt{25 - x^2}\right] dx$$

$$= 2\left[4x + \frac{1}{2}\left[9 \arcsin\left(\frac{x}{3}\right) + x\sqrt{9 - x^2}\right] - \frac{1}{2}\left[25 \arcsin\left(\frac{x}{5}\right) + x\sqrt{25 - x^2}\right]\right]_0^3$$

$$= 2\left[12 + \frac{9}{2}\arcsin(1) - \frac{25}{2}\arcsin\frac{3}{5} - 6\right]$$

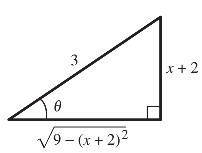
$$= 12 + \frac{9\pi}{2} - 25 \arcsin\frac{3}{5} \approx 10.050$$



68. The left circle has equation $(x + 2)^2 + y^2 = 9$. The shaded area is four times the area in the first quadrant, under the curve $y = \sqrt{9 - (x + 2)^2}$.

$$A = 4\int_0^1 \sqrt{9 - (x + 2)^2}\, dx$$

Let $x + 2 = 3 \sin \theta$, $dx = 3 \cos \theta\, d\theta$, $\sqrt{9 - (x + 2)^2} = 3 \cos \theta$



$$\int \sqrt{9 - (x + 2)^2}\, dx = \int 3 \cos \theta(3 \cos \theta)\, d\theta = 9\int \frac{1 + \cos 2\theta}{2}\, d\theta$$

$$= \frac{9}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C = \frac{9}{2}(\theta + \sin \theta \cos \theta) + C$$

$$= \frac{9}{2}\left(\arcsin\left(\frac{x + 2}{3}\right) + \left(\frac{x + 2}{3}\right)\left(\frac{\sqrt{9 - (x + 2)^2}}{3}\right)\right) + C$$

$$A = 4 \cdot \frac{9}{2}\left[\arcsin\left(\frac{x + 2}{3}\right) + \left(\frac{x + 2}{3}\right)\left(\frac{\sqrt{9 - (x + 2)^2}}{3}\right)\right]_0^1 = 18\left[\left(\frac{\pi}{2} + 0\right) - \left(\arcsin\frac{2}{3} + \frac{2}{3}\frac{\sqrt{5}}{3}\right)\right] = 9\pi - 18 \arcsin\frac{2}{3} - 4\sqrt{5}$$

69. Let $I = \int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$

Let $x = \frac{1-u}{1+u}, \quad dx = \frac{-2}{(1+u)^2}\,du$

$x + 1 = \frac{2}{1+u}, \quad x^2 + 1 = \frac{2+2u^2}{(1+u)^2}$

$$I = \int_1^0 \frac{\ln\left(\frac{2}{1+u}\right)}{\left(\frac{2+2u^2}{(1+u)^2}\right)}\left(\frac{-2}{(1+u)^2}\right)du$$

$$= \int_1^0 \frac{-\ln\left(\frac{2}{1+u}\right)}{1+u^2}\,du = \int_0^1 \frac{\ln\left(\frac{2}{1+u}\right)}{1+u^2}\,du = \int_0^1 \frac{\ln 2}{1+u^2} - \int_0^1 \frac{\ln(1+u)}{1+u^2}\,du = (\ln 2)[\arctan u]_0^1 - I$$

$$\Rightarrow 2I = \ln 2\left(\frac{\pi}{4}\right)$$

$$I = \frac{\pi}{8}\ln 2 \approx 0.272198$$

Section 8.5 Partial Fractions

1. (a) $\frac{4}{x^2 - 8x} = \frac{4}{x(x-8)} = \frac{A}{x} + \frac{B}{x-8}$

(b) $\frac{2x^2+1}{(x-3)^3} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3}$

(c) $\frac{2x-3}{x^3+10x} = \frac{2x-3}{x(x^2+10)} = \frac{A}{x} + \frac{Bx+C}{x^2+10}$

(d) $\frac{2x-1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

2. For a basic equation involving quadratic factors, you will have to solve a system of linear equations.

3. $\frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)} = \frac{A}{x+3} + \frac{B}{x-3}$

$1 = A(x-3) + B(x+3)$

When $x = 3$, $1 = 6B \Rightarrow B = \frac{1}{6}$.

When $x = -3$, $1 = -6A \Rightarrow A = -\frac{1}{6}$.

$$\int \frac{1}{x^2-9}\,dx = -\frac{1}{6}\int \frac{1}{x+3}\,dx + \frac{1}{6}\int \frac{1}{x-3}\,dx$$

$$= -\frac{1}{6}\ln|x+3| + \frac{1}{6}\ln|x-3| + C$$

$$= \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C$$

4. $\frac{2}{9x^2-1} = \frac{2}{(3x-1)(3x+1)} = \frac{A}{3x-1} + \frac{B}{3x+1}$

$2 = A(3x+1) + B(3x-1)$

When $x = \frac{1}{3}$, $2 = 2A \Rightarrow A = 1$.

When $x = -\frac{1}{3}$, $2 = -2B \Rightarrow B = -1$.

$$\int \frac{2}{9x^2-1}\,dx = \int \frac{1}{3x-1}\,dx + \int \frac{-1}{3x+1}\,dx$$

$$= \frac{1}{3}\ln|3x-1| - \frac{1}{3}\ln|3x+1| + C$$

$$= \frac{1}{3}\ln\left|\frac{3x-1}{3x+1}\right| + C$$

5. $\frac{5}{x^2+3x-4} = \frac{5}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$

$5 = A(x-1) + B(x+4)$

When $x = 1$, $5 = 5B \Rightarrow B = 1$.

When $x = -4$, $5 = -5A \Rightarrow A = -1$.

$$\int \frac{5}{x^2+3x-4}\,dx = \int \frac{-1}{x+4}\,dx + \int \frac{1}{x-1}\,dx$$

$$= -\ln|x+4| + \ln|x-1| + C$$

$$= \ln\left|\frac{x-1}{x+4}\right| + C$$

6. $\dfrac{3-x}{3x^2-2x-1} = \dfrac{3-x}{(3x+1)(x-1)} = \dfrac{A}{3x+1} + \dfrac{B}{x-1}$

$$3 - x = A(x-1) + B(3x+1)$$

When $x = 1$, $2 = 4B \Rightarrow B = \dfrac{1}{2}$.

When $x = -\dfrac{1}{3}$, $\dfrac{10}{3} = -\dfrac{4}{3}A \Rightarrow A = -\dfrac{5}{2}$.

$$\int \frac{3-x}{3x^2-2x-1}\,dx = -\frac{5}{2}\int \frac{1}{3x+1}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx$$
$$= \frac{-5}{6}\ln|3x+1| + \frac{1}{2}\ln|x-1| + C$$

7. $\dfrac{x^2+12x+12}{x(x+2)(x-2)} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-2}$

$$x^2 + 12x + 12 = A(x+2)(x-2) + Bx(x-2) + Cx(x+2)$$

When $x = 0, 12 = -4A \Rightarrow A = -3$.

When $x = -2, -8 = 8B \Rightarrow B = -1$.

When $x = 2, 40 = 8C \Rightarrow C = 5$.

$$\int \frac{x^2+12x+12}{x^3-4x}\,dx = 5\int \frac{1}{x-2}\,dx - \int \frac{1}{x+2}\,dx - 3\int \frac{1}{x}\,dx = 5\ln|x-2| - \ln|x+2| - 3\ln|x| + C$$

8. $\dfrac{x^3-x+3}{x^2+x-2} = x - 1 + \dfrac{2x+1}{(x+2)(x-1)} = x - 1 + \dfrac{A}{x+2} + \dfrac{B}{x-1}$

$$2x + 1 = A(x-1) + B(x+2)$$

When $x = -2, -3 = -3A \Rightarrow A = 1$.

When $x = 1, 3 = 3B \Rightarrow B = 1$.

$$\int \frac{x^3-x+3}{x^2+x-2}\,dx = \int\left(x - 1 + \frac{1}{x+2} + \frac{1}{x-1}\right)dx = \frac{x^2}{2} - x + \ln|x+2| + \ln|x-1| + C = \frac{x^2}{2} - x + \ln\left|x^2+x-2\right| + C$$

9. $\dfrac{2x^3-4x^2-15x+5}{x^2-2x-8} = 2x + \dfrac{x+5}{(x-4)(x+2)} = 2x + \dfrac{A}{x-4} + \dfrac{B}{x+2}$

$$x + 5 = A(x+2) + B(x-4)$$

When $x = 4, 9 = 6A \Rightarrow A = \dfrac{3}{2}$.

When $x = -2, 3 = -6B \Rightarrow B = -\dfrac{1}{2}$.

$$\int \frac{2x^3-4x^2-15x+5}{x^2-2x-8}\,dx = \int\left(2x + \frac{3/2}{x-4} - \frac{1/2}{x+2}\right)dx = x^2 + \frac{3}{2}\ln|x-4| - \frac{1}{2}\ln|x+2| + C$$

10. $\frac{x+2}{x^2+5x} = \frac{x+2}{x(x+5)} = \frac{A}{x} + \frac{B}{x+5}$

$x + 2 = A(x+5) + Bx$

When $x = -5$, $-3 = -5B \Rightarrow B = \frac{3}{5}$.

When $x = 0$, $2 = 5A \Rightarrow A = \frac{2}{5}$.

$$\int \frac{x+2}{x^2+5x}\,dx = \frac{2}{5}\int \frac{1}{x}\,dx + \frac{3}{5}\int \frac{1}{x+5}\,dx$$
$$= \frac{2}{5}\ln|x| + \frac{3}{5}\ln|x+5| + C$$

11. $\frac{4x^2+2x-1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$4x^2 + 2x - 1 = Ax(x+1) + B(x+1) + Cx^2$

When $x = 0$, $B = -1$.

When $x = -1$, $C = 1$.

When $x = 1$, $A = 3$.

$$\int \frac{4x^2+2x-1}{x^3+x^2}\,dx = \int\left(\frac{3}{x} - \frac{1}{x^2} + \frac{1}{x+1}\right)dx$$
$$= 3\ln|x| + \frac{1}{x} + \ln|x+1| + C$$
$$= \frac{1}{x} + \ln|x^4 + x^3| + C$$

12. $\frac{5x-2}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$

$5x - 2 = A(x-2) + B$

When $x = 2$, $8 = B$.

When $x = 0$, $-2 = -2A + B = -2A + 8 \Rightarrow A = 5$.

$$\int \frac{5x-2}{(x-2)^2}\,dx = \int \frac{5}{x-2}\,dx + \int \frac{8}{(x-2)^2}\,dx$$
$$= 5\ln|x-2| - \frac{8}{x-2} + C$$

13. $\frac{x^2-6x+2}{x^3+2x^2+x} = \frac{x^2-6x+2}{x(x^2+2x+1)} = \frac{x^2-6x+2}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$x^2 - 6x + 2 = A(x+1)^2 + Bx(x+1) + Cx$

When $x = 0$, $A = 2$.

When $x = -1$, $C = -9$.

When $x = 1$, $-3 = 2(4) + 2B - 9 \Rightarrow B = -1$.

$$\int \frac{x^2-6x+2}{x^3+2x^2+x}\,dx = \int\left(\frac{2}{x} - \frac{1}{x+1} - \frac{9}{(x+1)^2}\right)dx$$
$$= 2\ln|x| - \ln|x+1| + \frac{9}{x+1} + C$$
$$= \ln\left|\frac{x^2}{x+1}\right| + \frac{9}{x+1} + C$$

14. $\dfrac{8x}{x^3 + x^2 - x - 1} = \dfrac{8x}{x^2(x+1) - (x+1)} = \dfrac{8x}{(x+1)(x-1)(x+1)}$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$8x = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

When $x = 1$, $8 = 4A \Rightarrow A = 2$.

When $x = -1$, $-8 = -2C \Rightarrow C = 4$.

When $x = 0$, $0 = A - B - C = 2 - B - 4 \Rightarrow B = -2$.

$$\int \frac{8x}{x^3 + x^2 - x - 1}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{-2}{x+1}\,dx + \int \frac{4}{(x+1)^2}\,dx$$

$$= 2\ln|x-1| - 2\ln|x+1| - \frac{4}{x+1} + C$$

15. $\dfrac{9 - x^2}{7x^3 + x} = \dfrac{9 - x^2}{x(7x^2 + 1)} = \dfrac{A}{x} + \dfrac{Bx + C}{7x^2 + 1}$

$$9 - x^2 = A(7x^2 + 1) + Bx^2 + Cx$$

When $x = 0$, $A = 9$.

When $x = 1$, $8 = 9(8) + B + C \Rightarrow B + C = -64$.

When $x = -1$, $8 = 9(8) + B - C \Rightarrow B - C = -64$

$2B = -128$, $B = -64$ and $C = 0$.

$$\int \frac{9 - x^2}{7x^3 + x}\,dx = \int \left(\frac{9}{x} - \frac{64x}{7x^2 + 1}\right)dx$$

$$= 9\ln|x| - \frac{32}{7}\ln(7x^2 + 1) + C$$

16. $\dfrac{6x}{x^3 - 8} = \dfrac{6x}{(x-2)(x^2 + 2x + 4)} = \dfrac{A}{x-2} + \dfrac{Bx + C}{x^2 + 2x + 4}$

$$6x = A(x^2 + 2x + 4) + (Bx + C)(x - 2)$$

When $x = 2$, $12 = 12A \Rightarrow A = 1$.

When $x = 0$, $0 = 4 - 2C \Rightarrow C = 2$.

When $x = 1$, $6 = 7 + (B + 2)(-1) \Rightarrow B = -1$.

$$\int \frac{6x}{x^3 - 8}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{-x+2}{x^2 + 2x + 4}\,dx$$

$$= \int \frac{1}{x-2}\,dx + \int \frac{-x-1}{x^2 + 2x + 4}\,dx + \int \frac{3}{(x^2 + 2x + 1) + 3}\,dx$$

$$= \ln|x-2| - \frac{1}{2}\ln|x^2 + 2x + 4| + \frac{3}{\sqrt{3}}\arctan\left(\frac{x+1}{\sqrt{3}}\right) + C$$

$$= \ln|x-2| - \frac{1}{2}\ln|x^2 + 2x + 4| + \sqrt{3}\arctan\left(\frac{\sqrt{3}(x+1)}{3}\right) + C$$

17. $\dfrac{x^2}{x^4 - 2x^2 - 8} = \dfrac{A}{x - 2} + \dfrac{B}{x + 2} + \dfrac{Cx + D}{x^2 + 2}$

$$x^2 = A(x + 2)(x^2 + 2) + B(x - 2)(x^2 + 2) + (Cx + D)(x + 2)(x - 2)$$

When $x = 2, 4 = 24A$.

When $x = -2, 4 = -24B$.

When $x = 0, 0 = 4A - 4B - 4D$.

When $x = 1, 1 = 9A - 3B - 3C - 3D$.

Solving these equations you have $A = \frac{1}{6}, B = -\frac{1}{6}, C = 0, D = \frac{1}{3}$.

$$\int \frac{x^2}{x^4 - 2x^2 - 8}\,dx = \frac{1}{6}\left(\int \frac{1}{x - 2}\,dx - \int \frac{1}{x + 2}\,dx + 2\int \frac{1}{x^2 + 2}\,dx\right) = \frac{1}{6}\left(\ln\left|\frac{x - 2}{x + 2}\right| + \sqrt{2}\arctan\frac{x}{\sqrt{2}}\right) + C$$

18. $\dfrac{x}{(2x - 1)(2x + 1)(4x^2 + 1)} = \dfrac{A}{2x - 1} + \dfrac{B}{2x + 1} + \dfrac{Cx + D}{4x^2 + 1}$

$$x = A(2x + 1)(4x^2 + 1) + B(2x - 1)(4x^2 + 1) + (Cx + D)(2x - 1)(2x + 1)$$

When $x = \frac{1}{2}, \frac{1}{2} = 4A$.

When $x = -\frac{1}{2}, -\frac{1}{2} = -4B$.

When $x = 0, 0 = A - B - D$.

When $x = 1, 1 = 15A + 5B + 3C + 3D$.

Solving these equations you have $A = \frac{1}{8}, B = \frac{1}{8}, C = -\frac{1}{2}, D = 0$.

$$\int \frac{x}{16x^4 - 1}\,dx = \frac{1}{8}\left(\int \frac{1}{2x - 1}\,dx + \int \frac{1}{2x + 1}\,dx - 4\int \frac{x}{4x^2 + 1}\,dx\right) = \frac{1}{16}\ln\left|\frac{4x^2 - 1}{4x^2 + 1}\right| + C$$

19. $\dfrac{x^2 + 5}{(x + 1)(x^2 - 2x + 3)} = \dfrac{A}{x + 1} + \dfrac{Bx + C}{x^2 - 2x + 3}$

$$\begin{aligned} x^2 + 5 &= A(x^2 - 2x + 3) + (Bx + C)(x + 1) \\ &= (A + B)x^2 + (-2A + B + C)x + (3A + C) \end{aligned}$$

When $x = -1, A = 1$.

By equating coefficients of like terms, you have $A + B = 1, -2A + B + C = 0, 3A + C = 5$.

Solving these equations you have $A = 1, B = 0, C = 2$.

$$\int \frac{x^2 + 5}{x^3 - x^2 + x + 3}\,dx = \int \frac{1}{x + 1}\,dx + 2\int \frac{1}{(x - 1)^2 + 2}\,dx = \ln|x + 1| + \sqrt{2}\arctan\left(\frac{x - 1}{\sqrt{2}}\right) + C$$

20. $\dfrac{x^2+6x+4}{x^4+8x^2+16} = \dfrac{x^2+6x+4}{\left(x^2+4\right)^2} = \dfrac{Ax+B}{x^2+4} + \dfrac{Cx+D}{\left(x^2+4\right)^2}$

$$\begin{aligned} x^2+6x+4 &= (Ax+B)\left(x^2+4\right)+Cx+D \\ &= Ax^3+Bx^2+(4A+C)x+4B+D \end{aligned}$$

By equating coefficients of like terms, you have

$A = 0, \quad B = 1, \quad 4A + C = 6, \quad 4B + D = 4.$

Solving these equations you have $A = 0, B = 1, C = 6, D = 0.$

$$\begin{aligned} \int \frac{x^2+6x+4}{x^4+8x^2+16}\,dx &= \int \frac{1}{x^2+4}\,dx + \int \frac{6x}{\left(x^2+4\right)^2}\,dx \\ &= \frac{1}{2}\arctan\frac{x}{2} - \frac{3}{x^2+4} + C \end{aligned}$$

21. $\dfrac{3}{4x^2+5x+1} = \dfrac{3}{(4x+1)(x+1)} = \dfrac{A}{4x+1} + \dfrac{B}{x+1}$

$$3 = A(x+1) + B(4x+1)$$

When $x = -1, 3 = -3B \Rightarrow B = -1.$

When $-\dfrac{1}{4}, 3 = \dfrac{3}{4}A \Rightarrow A = 4.$

$$\begin{aligned} \int_0^2 \frac{3}{4x^2+5x+1}\,dx &= \int_0^2 \frac{4}{4x+1}\,dx + \int_0^2 \frac{-1}{x+1}\,dx \\ &= \Big[\ln|4x+1| - \ln|x+1|\Big]_0^2 \\ &= \ln 9 - \ln 3 \\ &= 2\ln 3 - \ln 3 = \ln 3 \end{aligned}$$

22. $\dfrac{x-1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

$$x - 1 = Ax(x+1) + B(x+1) + Cx^2$$

When $x = 0, B = -1.$

When $x = -1, C = -2.$

When $x = 1, 0 = 2A + 2B + C.$

Solving these equations you have
$A = 2, B = -1, C = -2.$

$$\begin{aligned} \int_1^5 \frac{x-1}{x^2(x+1)}\,dx &= 2\int_1^5 \frac{1}{x}\,dx - \int_1^5 \frac{1}{x^2}\,dx - 2\int_1^5 \frac{1}{x+1}\,dx \\ &= \left[2\ln|x| + \frac{1}{x} - 2\ln|x+1|\right]_1^5 \\ &= \left[2\ln\left|\frac{x}{x+1}\right| + \frac{1}{x}\right]_1^5 \\ &= 2\ln\frac{5}{3} - \frac{4}{5} \end{aligned}$$

23. $\dfrac{x+1}{x\left(x^2+1\right)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

$$x + 1 = A\left(x^2+1\right) + (Bx+C)x$$

When $x = 0, A = 1.$

When $x = 1, 2 = 2A + B + C.$

When $x = -1, 0 = 2A + B - C.$

Solving these equations we have
$A = 1, B = -1, C = 1.$

$$\begin{aligned} \int_1^2 \frac{x+1}{x\left(x^2+1\right)}\,dx &= \int_1^2 \frac{1}{x}\,dx - \int_1^2 \frac{x}{x^2+1}\,dx + \int_1^2 \frac{1}{x^2+1}\,dx \\ &= \left[\ln|x| - \frac{1}{2}\ln\left(x^2+1\right) + \arctan x\right]_1^2 \\ &= \frac{1}{2}\ln\frac{8}{5} - \frac{\pi}{4} + \arctan 2 \\ &\approx 0.557 \end{aligned}$$

24. $$\begin{aligned} \int_0^1 \frac{x^2-x}{x^2+x+1}\,dx &= \int_0^1 dx - \int_0^1 \frac{2x+1}{x^2+x+1}\,dx \\ &= \Big[x - \ln\left|x^2+x+1\right|\Big]_0^1 \\ &= 1 - \ln 3 \end{aligned}$$

25. Let $u = \cos x$, $du = -\sin x\, dx$.

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

$$1 = A(u+1) + Bu$$

When $u = 0$, $A = 1$.

When $u = -1$, $B = -1$.

$$\int \frac{\sin x}{\cos x + \cos^2 x}\, dx = -\int \frac{1}{u(u+1)}\, du$$
$$= \int \frac{1}{u+1}\, du - \int \frac{1}{u}\, du$$
$$= \ln|u+1| - \ln|u| + C$$
$$= \ln\left|\frac{u+1}{u}\right| + C$$
$$= \ln\left|\frac{\cos x + 1}{\cos x}\right| + C$$
$$= \ln|1 + \sec x| + C$$

26. $$\int \frac{5\cos x}{\sin^2 x + 3\sin x - 4}\, dx = 5\int \frac{1}{u^2 + 3u - 4}\, du$$
$$= \ln\left|\frac{u-1}{u+4}\right| + C$$
$$= \ln\left|\frac{-1 + \sin x}{4 + \sin x}\right| + C$$

(From Exercise 5 with $u = \sin x$, $du = \cos x\, dx$)

27. Let $u = \tan x$, $du = \sec^2 x\, dx$.

$$\frac{1}{u^2 + 5u + 6} = \frac{1}{(u+3)(u+2)} = \frac{A}{u+3} + \frac{B}{u+2}$$

$$1 = A(u+2) + B(u+3)$$

When $u = -2$, $1 = B$.

When $u = -3$, $1 = -A \Rightarrow A = -1$.

$$\int \frac{\sec^2 x}{\tan^2 x + 5\tan x + 6}\, dx = \int \frac{1}{u^2 + 5u + 6}\, du$$
$$= \int \frac{-1}{u+3}\, du + \int \frac{1}{u+2}\, du$$
$$= -\ln|u+3| + \ln|u+2| + C$$
$$= \ln\left|\frac{\tan x + 2}{\tan x + 3}\right| + C$$

28. $\dfrac{1}{u(u+1)} = \dfrac{A}{u} + \dfrac{B}{u+1}$, $u = \tan x$, $du = \sec^2 x\, dx$

$$1 = A(u+1) + Bu$$

When $u = 0$, $A = 1$.

When $u = -1$, $1 = -B \Rightarrow B = -1$.

$$\int \frac{\sec^2 x\, dx}{\tan x(\tan x + 1)} = \int \frac{1}{u(u+1)}\, du$$
$$= \int \left(\frac{1}{u} - \frac{1}{u+1}\right) du$$
$$= \ln|u| - \ln|u+1| + C$$
$$= \ln\left|\frac{u}{u+1}\right| + C$$
$$= \ln\left|\frac{\tan x}{\tan x + 1}\right| + C$$

29. Let $u = e^x$, $du = e^x\, dx$.

$$\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4}$$

$$1 = A(u+4) + B(u-1)$$

When $u = 1$, $A = \frac{1}{5}$.

When $u = -4$, $B = -\frac{1}{5}$.

$$\int \frac{e^x}{(e^x - 1)(e^x + 4)}\, dx = \int \frac{1}{(u-1)(u+4)}\, du$$
$$= \frac{1}{5}\left(\int \frac{1}{u-1}\, du - \int \frac{1}{u+4}\, du\right)$$
$$= \frac{1}{5}\ln\left|\frac{u-1}{u+4}\right| + C$$
$$= \frac{1}{5}\ln\left|\frac{e^x - 1}{e^x + 4}\right| + C$$

30. Let $u = e^x$, $du = e^x\,dx$.

$$\frac{1}{(u^2+1)(u-1)} = \frac{A}{u-1} + \frac{Bu+C}{u^2+1}$$

$$1 = A(u^2+1) + (Bu+C)(u-1)$$

When $u = 1$, $A = \frac{1}{2}$.

When $u = 0$, $1 = A - C$.

When $u = -1$, $1 = 2A + 2B - 2C$.

Solving these equations you have $A = \frac{1}{2}$, $B = -\frac{1}{2}$, and $C = -\frac{1}{2}$.

$$\begin{aligned}\int \frac{e^x}{(e^{2x}+1)(e^x-1)}\,dx &= \int \frac{1}{(u^2+1)(u-1)}\,du \\ &= \frac{1}{2}\left(\int \frac{1}{u-1}\,du - \int \frac{u+1}{u^2+1}\,du\right) \\ &= \frac{1}{2}\left(\ln|u-1| - \frac{1}{2}\ln|u^2+1| - \arctan u\right) + C \\ &= \frac{1}{4}\left(2\ln|e^x-1| - \ln|e^{2x}+1| - 2\arctan e^x\right) + C\end{aligned}$$

31. Let $u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$.

$$\int \frac{\sqrt{x}}{x-4}\,dx = \int \frac{u(2u)du}{u^2-4} = \int\left(\frac{2u^2-8}{u^2-4} + \frac{8}{u^2-4}\right)du = \int\left(2 + \frac{8}{u^2-4}\right)du$$

$$\frac{8}{u^2-4} = \frac{8}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2}$$

$$8 = A(u+2) + B(u-2)$$

When $u = -2$, $8 = -4B \Rightarrow B = -2$.

When $u = 2$, $8 = 4A \Rightarrow A = 2$.

$$\begin{aligned}\int\left(2 + \frac{8}{u^2-4}\right)du &= 2u + \int\left(\frac{2}{u-2} - \frac{2}{u+2}\right)du \\ &= 2u + 2\ln|u-2| - 2\ln|u+2| + C \\ &= 2\sqrt{x} + 2\ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right| + C\end{aligned}$$

32. Let $u = \sqrt{x}, u^2 = x, 2u\,du = dx$.

$$\int \frac{1}{x\left(\sqrt{3} - \sqrt{x}\right)}\,dx = \int \frac{2u\,du}{u^2\left(\sqrt{3} - u\right)} = \int \frac{2}{u\left(\sqrt{3} - u\right)}\,du$$

$$\frac{2}{u\left(\sqrt{3} - u\right)} = \frac{A}{u} + \frac{B}{\sqrt{3} - u}$$

$$2 = A\left(\sqrt{3} - u\right) + Bu$$

When $u = 0, 2 = \sqrt{3}A \Rightarrow A = \dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$.

When $u = \sqrt{3}, 2 = B\sqrt{3} \Rightarrow B = \dfrac{2\sqrt{3}}{3}$.

$$\int \frac{2}{u\left(\sqrt{3} - u\right)}\,du = \frac{2\sqrt{3}}{3}\int\left(\frac{1}{u} + \frac{1}{\sqrt{3} - u}\right)du$$

$$= \frac{2\sqrt{3}}{3}\left(\ln|u| - \ln\left|u - \sqrt{3}\right|\right) + C$$

$$\int \frac{1}{x\left(\sqrt{3} - \sqrt{x}\right)}\,dx = \frac{2\sqrt{3}}{3}\left(\ln\sqrt{x} - \ln\left|\sqrt{x} - 3\right|\right) + C$$

33. $$\frac{1}{x(a + bx)} = \frac{A}{x} + \frac{B}{a + bx}$$

$$1 = A(a + bx) + Bx$$

When $x = 0, 1 = aA \Rightarrow A = 1/a$.

When $x = -a/b, 1 = -(a/b)B \Rightarrow B = -b/a$.

$$\int \frac{1}{x(a + bx)}\,dx = \frac{1}{a}\int\left(\frac{1}{x} - \frac{b}{a + bx}\right)dx$$

$$= \frac{1}{a}\left(\ln|x| - \ln|a + bx|\right) + C$$

$$= \frac{1}{a}\ln\left|\frac{x}{a + bx}\right| + C$$

34. $$\frac{1}{a^2 - x^2} = \frac{A}{a - x} + \frac{B}{a + x}$$

$$1 = A(a + x) + B(a - x)$$

When $x = a, 1 = 2aA \Rightarrow A = 1/2a$.

When $x = -a, 1 = 2aB \Rightarrow B = 1/2a$.

$$\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\int\left(\frac{1}{a - x} + \frac{1}{a + x}\right)dx$$

$$= \frac{1}{2a}\left(-\ln|a - x| + \ln|a + x|\right) + C$$

$$= \frac{1}{2a}\ln\left|\frac{a + x}{a - x}\right| + C$$

35. $$\frac{x}{(a + bx)^2} = \frac{A}{a + bx} + \frac{B}{(a + bx)^2}$$

$$x = A(a + bx) + B$$

When $x = -a/b, B = -a/b$.

When $x = 0, 0 = aA + B \Rightarrow A = 1/b$.

$$\int \frac{x}{(a + bx)^2}\,dx = \int\left(\frac{1/b}{a + bx} + \frac{-a/b}{(a + bx)^2}\right)dx$$

$$= \frac{1}{b}\int \frac{1}{a + bx}\,dx - \frac{a}{b}\int \frac{1}{(a + bx)^2}\,dx$$

$$= \frac{1}{b^2}\ln|a + bx| + \frac{a}{b^2}\left(\frac{1}{a + bx}\right) + C$$

$$= \frac{1}{b^2}\left(\frac{a}{a + bx} + \ln|a + bx|\right) + C$$

36. $\dfrac{1}{x^2(a+bx)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{a+bx}$

$$1 = Ax(a+bx) + B(a+bx) + Cx^2$$

When $x = 0, 1 = Ba \Rightarrow B = 1/a$. When $x = -a/b$,

$1 = C(a^2/b^2) \Rightarrow C = b^2/a^2$. When $x = 1$,

$1 = (a+b)A + (a+b)B + C \Rightarrow A = -b/a^2$.

$$\int \frac{1}{x^2(a+bx)}\,dx = \int\left(\frac{-b/a^2}{x} + \frac{1/a}{x^2} + \frac{b^2/a^2}{a+bx}\right)dx$$

$$= -\frac{b}{a^2}\ln|x| - \frac{1}{ax} + \frac{b}{a^2}\ln|a+bx| + C$$

$$= -\frac{1}{ax} + \frac{b}{a^2}\ln\left|\frac{a+bx}{x}\right| + C$$

$$= -\frac{1}{ax} - \frac{b}{a^2}\ln\left|\frac{x}{a+bx}\right| + C$$

37. Substitution: $u = x^2 + 2x - 8$

38. Partial fractions

39. Trigonometric substitution (tan) or inverse tangent rule

40. (a) Yes. Because $f' > 0$ on $(0, 5)$, f is increasing, and $f(3) > f(2)$. Therefore, $f(3) - f(2) > 0$.

(b) The area under the graph of f' is greater on the interval $[1, 2]$ because the graph is decreasing on $[1, 4]$.

41. $\dfrac{12}{x^2+5x+6} = \dfrac{12}{(x+2)(x+3)} = \dfrac{A}{x+2} + \dfrac{B}{x+3}$

$$12 = A(x+3) + B(x+2)$$

When $x = -3$, $B = -12$.

When $x = -2$, $A = 12$.

$$A = \int_0^4 \frac{12}{x^2+5x+6}\,dx$$

$$= \int_0^4 \left(\frac{12}{x+2} - \frac{12}{x+3}\right)dx$$

$$= 12\Big[\ln|x+2| - \ln|x+3|\Big]_0^4$$

$$= 12\big[(\ln 6 - \ln 7) - (\ln 2 - \ln 3)\big]$$

$$= 12\ln\left(\frac{6(3)}{7(2)}\right) = 12\ln\frac{9}{7} \approx 3.016$$

42. $\dfrac{15}{x^2+7x+12} = \dfrac{15}{(x+3)(x+4)} = \dfrac{A}{x+3} + \dfrac{B}{x+4}$

$$15 = A(x+4) + B(x+3)$$

When $x = -3$, $A = 15$.

When $x = -4$, $B = -15$.

$$A = \int_1^2 \frac{15}{x^2+7x+12}\,dx$$

$$= \int_1^2 \left(\frac{15}{x+3} - \frac{15}{x+4}\right)dx$$

$$= 15\Big[\ln(x+3) - \ln(x+4)\Big]_1^2$$

$$= 15\big[(\ln 5 - \ln 6) - (\ln 4 - \ln 5)\big]$$

$$= 15\ln\frac{5(5)}{6(4)} = 15\ln\frac{25}{24} \approx 0.612$$

43. $\dfrac{15}{9-x^2} = \dfrac{-15}{(x-3)(x+3)} = \dfrac{A}{x-3} + \dfrac{B}{x+3}$

$$-15 = A(x+3) + B(x-3)$$

When $x = -3, -15 = -6B \Rightarrow B = \dfrac{5}{2}$.

When $x = 3, -15 = 6A \Rightarrow A = -\dfrac{5}{2}$.

$$A = \int_0^2 \frac{15}{9-x^2}\,dx$$

$$= \int_0^2 \left(\frac{-5/2}{x-3} + \frac{5/2}{x+3}\right)dx$$

$$= \frac{5}{2}\Big[\ln|x+3| - \ln|x-3|\Big]_0^2$$

$$= \frac{5}{2}\big[(\ln 5 - \ln 1) - (\ln 3 - \ln 3)\big] = \frac{5}{2}\ln 5 \approx 4.024$$

44. $\dfrac{7}{16-x^2} = \dfrac{-7}{(x-4)(x+4)} = \dfrac{A}{x-4} + \dfrac{B}{x+4}$

$$-7 = A(x+4) + B(x-4)$$

When $x = 4, -7 = 8A \Rightarrow A = -\dfrac{7}{8}$.

When $x = -4, -7 = -8B \Rightarrow B = \dfrac{7}{8}$.

$$A = \int_1^3 \frac{7}{16-x^2}\,dx = \int_1^3 \left(\frac{-7/8}{x-4} + \frac{7/8}{x+4}\right)dx$$

$$= \frac{7}{8}\Big[\ln|x+4| - \ln|x-4|\Big]_1^3$$

$$= \frac{7}{8}\big[(\ln 7 - \ln 1) - (\ln 5 - \ln 3)\big]$$

$$= \frac{7}{8}\ln\frac{21}{5} \approx 1.256$$

45. Average cost $= \dfrac{1}{80-75}\displaystyle\int_{75}^{80} \frac{124p}{(10+p)(100-p)}\,dp$

$$= \frac{1}{5}\int_{75}^{80}\left(\frac{-124}{(10+p)11} + \frac{1240}{(100-p)11}\right)dp$$

$$= \frac{1}{5}\left[\frac{-124}{11}\ln(10+p) - \frac{1240}{11}\ln(100-p)\right]_{75}^{80}$$

$$\approx \frac{1}{5}(24.51) = 4.9$$

Approximately \$490,000

46. $\dfrac{1}{4x^2-1} = \dfrac{1}{(2x+1)(2x-1)} = \dfrac{A}{2x+1} + \dfrac{B}{2x-1}$

$$1 = A(2x-1) + B(2x+1)$$

When $x = -\dfrac{1}{2}, 1 = -2A \Rightarrow A = -\dfrac{1}{2}.$

When $x = \dfrac{1}{2}, 1 = 2B \Rightarrow B = \dfrac{1}{2}.$

$$\int_1^4 \frac{1}{4x^2-1}\,dx = \int_1^4\left(\frac{-1/2}{2x+1} + \frac{1/2}{2x-1}\right)dx$$

$$= \frac{1}{2}\left[\frac{1}{2}\ln|2x-1| - \frac{1}{2}\ln|2x+1|\right]_1^4$$

$$= \frac{1}{4}\left[(\ln 7 - \ln 9) - (\ln 1 - \ln 3)\right]$$

$$= \frac{1}{4}\ln\frac{7}{3}$$

Average value $= \dfrac{1}{4-1}\displaystyle\int_1^4 \frac{1}{4x^2-1}\,dx$

$$= \frac{1}{3}\left(\frac{1}{4}\ln\frac{7}{3}\right) = \frac{1}{12}\ln\frac{7}{3}$$

47. (a) $V = \pi\displaystyle\int_0^3\left(\frac{2x}{x^2+1}\right)^2 dx = 4\pi\int_0^3 \frac{x^2}{(x^2+1)^2}\,dx$

$$= 4\pi\int_0^3\left(\frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}\right)dx \qquad \text{(partial fractions)}$$

$$= 4\pi\left[\arctan x - \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right)\right]_0^3 \qquad \text{(trigonometric substitution)}$$

$$= 2\pi\left[\arctan x - \frac{x}{x^2+1}\right]_0^3 = 2\pi\left(\arctan 3 - \frac{3}{10}\right) \approx 5.963$$

(b) $A = \int_0^3 \frac{2x}{x^2+1}\,dx = \Big[\ln(x^2+1)\Big]_0^3 = \ln 10$

$$\bar{x} = \frac{1}{A}\int_0^3 \frac{2x^2}{x^2+1}\,dx = \frac{1}{\ln 10}\int_0^3 \left(2 - \frac{2}{x^2+1}\right)dx = \frac{1}{\ln 10}\big[2x - 2\arctan x\big]_0^3 = \frac{2}{\ln 10}(3 - \arctan 3) \approx 1.521$$

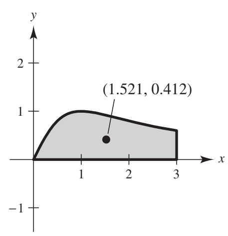
$$\bar{y} = \frac{1}{A}\left(\frac{1}{2}\right)\int_0^3 \left(\frac{2x}{x^2+1}\right)^2 dx = \frac{2}{\ln 10}\int_0^3 \frac{x^2}{\left(x^2+1\right)^2}\,dx$$

$$= \frac{2}{\ln 10}\int_0^3 \left(\frac{1}{x^2+1} - \frac{1}{\left(x^2+1\right)^2}\right)dx \qquad \text{(partial fractions)}$$

$$= \frac{2}{\ln 10}\left[\arctan x - \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right)\right]_0^3 \qquad \text{(trigonometric substitution)}$$

$$= \frac{2}{\ln 10}\left[\frac{1}{2}\arctan x - \frac{x}{2\left(x^2+1\right)}\right]_0^3 = \frac{1}{\ln 10}\left[\arctan x - \frac{x}{x^2+1}\right]_0^3 = \frac{1}{\ln 10}\left(\arctan 3 - \frac{3}{10}\right) \approx 0.412$$

$(\bar{x}, \bar{y}) \approx (1.521, 0.412)$



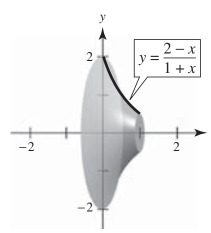
48. $y^2 = \dfrac{(2-x)^2}{(1+x)^2}, \quad [0, 1]$

$$V = \int_0^1 \pi\frac{(2-x)^2}{(1+x)^2}\,dx$$

$$= \pi\left[\int_0^1 \frac{4}{(1+x)^2}\,dx - \int_0^1 \frac{4x}{(1+x)^2}\,dx + \int_0^1 \frac{x^2}{(1+x)^2}\,dx\right]$$

$$= \pi\left[2 - (4\ln 2 - 2) + \frac{3}{2} - 2\ln 2\right]$$

$$= \pi\left(\frac{11}{2} - 6\ln 2\right) = \frac{\pi}{2}(11 - 12\ln 2)$$



49. $$\frac{1}{(x+1)(n-x)} = \frac{A}{x+1} + \frac{B}{n-x},\ A = B = \frac{1}{n+1}$$

$$\frac{1}{n+1}\int\left(\frac{1}{x+1} + \frac{1}{n-x}\right)dx = kt + C$$

$$\frac{1}{n+1}\ln\left|\frac{x+1}{n-x}\right| = kt + C$$

When $t = 0, x = 0, C = \frac{1}{n+1}\ln\frac{1}{n}$.

$$\frac{1}{n+1}\ln\left|\frac{x+1}{n-x}\right| = kt + \frac{1}{n+1}\ln\frac{1}{n}$$

$$\frac{1}{n+1}\left[\ln\left|\frac{x+1}{n-x}\right| - \ln\frac{1}{n}\right] = kt$$

$$\ln\frac{nx+n}{n-x} = (n+1)kt$$

$$\frac{nx+n}{n-x} = e^{(n+1)kt}$$

$$x = \frac{n\left[e^{(n+1)kt} - 1\right]}{n + e^{(n+1)kt}}$$

Note: $\lim_{t\to\infty} x = n$

50. (a) $$\frac{1}{(y_0 - x)(z_0 - x)} = \frac{A}{y_0 - x} + \frac{B}{z_0 - x},$$

$$A = \frac{1}{z_0 - y_0},\ B = -\frac{1}{z_0 - y_0}, \qquad (\text{Assume } y_0 \neq z_0.)$$

$$\frac{1}{z_0 - y_0}\int\left(\frac{1}{y_0 - x} - \frac{1}{z_0 - x}\right)dx = kt + C$$

$$\frac{1}{z_0 - y_0}\ln\left|\frac{z_0 - x}{y_0 - x}\right| = kt + C, \text{ when } t = 0, x = 0$$

$$C = \frac{1}{z_0 - y_0}\ln\frac{z_0}{y_0}$$

$$\frac{1}{z_0 - y_0}\left[\ln\left|\frac{z_0 - x}{y_0 - x}\right| - \ln\left(\frac{z_0}{y_0}\right)\right] = kt$$

$$\ln\left[\frac{y_0(z_0 - x)}{z_0(y_0 - x)}\right] = (z_0 - y_0)kt$$

$$\frac{y_0(z_0 - x)}{z_0(y_0 - x)} = e^{(z_0 - y_0)kt}$$

$$x = \frac{y_0 z_0\left[e^{(z_0 - y_0)kt} - 1\right]}{z_0 e^{(z_0 - y_0)kt} - y_0}$$

(b) (1) If $y_0 < z_0$, $\lim_{t\to\infty} x = y_0$.

(2) If $y_0 > z_0$, $\lim_{t\to\infty} x = z_0$.

(3) If $y_0 = z_0$, then the original equation is:

$$\int\frac{1}{(y_0 - x)^2}\,dx = \int k\,dt$$

$$(y_0 - x)^{-1} = kt + C_1$$

$$x = 0 \text{ when } t = 0 \Rightarrow \frac{1}{y_0} = C_1$$

$$\frac{1}{y_0 - x} = kt + \frac{1}{y_0} = \frac{kty_0 + 1}{y_0}$$

$$y_0 - x = \frac{y_0}{kty_0 + 1}$$

$$x = y_0 - \frac{y_0}{kty_0 + 1}$$

As $t \to \infty$, $x \to y_0 = z_0$.

51. $\dfrac{x}{1+x^4} = \dfrac{Ax+B}{x^2+\sqrt{2}x+1} + \dfrac{Cx+D}{x^2-\sqrt{2}x+1}$

$$x = (Ax+B)\left(x^2-\sqrt{2}x+1\right) + (Cx+D)\left(x^2+\sqrt{2}x+1\right)$$

$$= (A+C)x^3 + \left(B+D-\sqrt{2}A+\sqrt{2}C\right)x^2 + \left(A+C-\sqrt{2}B+\sqrt{2}D\right)x + (B+D)$$

$0 = A + C \Rightarrow C = -A$

$0 = B + D - \sqrt{2}A + \sqrt{2}C \qquad -2\sqrt{2}A = 0 \Rightarrow A = 0 \text{ and } C = 0$

$1 = A + C - \sqrt{2}B + \sqrt{2}D \qquad -2\sqrt{2}B = 1 \Rightarrow B = -\dfrac{\sqrt{2}}{4} \text{ and } D = \dfrac{\sqrt{2}}{4}$

$0 = B + D \Rightarrow D = -B$

So,

$$\int_0^1 \frac{x}{1+x^4}\,dx = \int_0^1 \left(\frac{-\sqrt{2}/4}{x^2+\sqrt{2}x+1} + \frac{\sqrt{2}/4}{x^2-\sqrt{2}x+1}\right)dx$$

$$= \frac{\sqrt{2}}{4}\int_0^1 \left[\frac{-1}{\left[x+\left(\sqrt{2}/2\right)\right]^2+(1/2)} + \frac{1}{\left[x-\left(\sqrt{2}/2\right)\right]^2+(1/2)}\right]dx$$

$$= \frac{\sqrt{2}}{4}\cdot\frac{1}{1/\sqrt{2}}\left[-\arctan\left(\frac{x+\left(\sqrt{2}/2\right)}{1/\sqrt{2}}\right) + \arctan\left(\frac{x-\left(\sqrt{2}/2\right)}{1/\sqrt{2}}\right)\right]_0^1$$

$$= \frac{1}{2}\left[-\arctan\left(\sqrt{2}x+1\right) + \arctan\left(\sqrt{2}x-1\right)\right]_0^1$$

$$= \frac{1}{2}\left[\left(-\arctan\left(\sqrt{2}+1\right) + \arctan\left(\sqrt{2}-1\right)\right) - \left(-\arctan 1 + \arctan(-1)\right)\right]$$

$$= \frac{1}{2}\left[\arctan\left(\sqrt{2}-1\right) - \arctan\left(\sqrt{2}+1\right) + \frac{\pi}{4} + \frac{\pi}{4}\right].$$

Because $\arctan x - \arctan y = \arctan\left[(x-y)/(1+xy)\right]$, you have:

$$\int_0^1 \frac{x}{1+x^4}\,dx = \frac{1}{2}\left[\arctan\left(\frac{\left(\sqrt{2}-1\right)-\left(\sqrt{2}+1\right)}{1+\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}\right) + \frac{\pi}{2}\right] = \frac{1}{2}\left[\arctan\left(\frac{-2}{2}\right) + \frac{\pi}{2}\right] = \frac{1}{2}\left(-\frac{\pi}{4} + \frac{\pi}{2}\right) = \frac{\pi}{8}$$

52. The partial fraction decomposition is:

$$\frac{x^4(1-x)^4}{1+x^2} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}$$

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \left[\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4}{3}x^3 + 4x - 4\arctan x\right]_0^1$$

$$= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - 4\left(\frac{\pi}{4}\right)$$

$$= \frac{22}{7} - \pi$$

Note: You can easily verify this calculation with a graphing utility.

53. The answer is 3984. Use the division algorithm to write $p(x) = (x^3 - x)\,q(x) + r(x)$, where the degree of $r(x)$ is less than 3, and the degree of $q(x)$ is less than 1989. Hence,

$$\frac{d^{1992}}{dx^{1992}}\left[\frac{p(x)}{x^3 - 3}\right] = \frac{d^{1992}}{dx^{1992}}\left[\frac{(x^3 - x)\,q(x) + r(x)}{x^3 - x}\right]$$
$$= \frac{d^{1992}}{dx^{1992}}\left[\frac{r(x)}{x^3 - x}\right].$$

Using partial fractions,

$$\frac{r(x)}{x^3 - x} = \frac{r(x)}{(x-1)x(x+1)} = \frac{A}{x-1} + \frac{B}{x} + \frac{C}{x+1}$$

Since $p(x)$ has no common factors with $x^3 - x$, then $r(x)$ does not either. Hence, A, B, and C are all non zero.

$$\frac{d^{1992}}{dx^{1992}}\left[\frac{r(x)}{x^3 - x}\right] = \frac{d^{1992}}{dx^{1992}}\left[\frac{A}{x-1} + \frac{B}{x} + \frac{C}{x+1}\right]$$
$$= 1992!\left[\frac{A}{(x-1)^{1993}} + \frac{B}{x^{1993}} + \frac{C}{(x+1)^{1993}}\right]$$
$$= 1992!\left[\frac{Ax^{1993}(x+1)^{1993} + B(x-1)^{1993}(x+1)^{1993} + Cx^{1993}(x-1)^{1993}}{(x^3 - x)^{1993}}\right]$$

Now expand the numerator to obtain an expression of the form

$(A + B + C)x^{3986} + 1993(A - C)x^{3985} + 1993(996A - B + 996C)x^{3984} + \dots$

From $A = C = 1$ and $B = -2$, you see that the degree could be 3984. A lower degree would imply that $A + B + C = 0$, $A - C = 0$, and $996A - B + 996C = 0$, which means $A = B = C$, a contradiction.

Section 8.6 Numerical Integration

1. No. The integral can easily be evaluated using basic integration rules:

$$\int_0^2 (e^x + 5x)\,dx = \left[e^x + \frac{5}{2}x^2\right]_0^2 = e^2 + 10 - 1 = e^2 + 9.$$

2. You can decrease the error by making Δx smaller.

3. Exact: $\int_0^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^2 = \frac{8}{3} \approx 2.6667$

Trapezoidal: $\int_0^2 x^2\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)^2 + 2(1)^2 + 2\left(\frac{3}{2}\right)^2 + (2)^2\right] = \frac{11}{4} = 2.7500$

Simpson's: $\int_0^2 x^2\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^2 + 2(1)^2 + 4\left(\frac{3}{2}\right)^2 + (2)^2\right] = \frac{8}{3} \approx 2.6667$

4. Exact: $\int_1^2 \left(\frac{x^2}{4} + 1\right)dx = \left[\frac{x^3}{12} + x\right]_1^2 = \frac{19}{12} \approx 1.5833$

Trapezoidal: $\int_1^2 \left(\frac{x^2}{4} + 1\right)dx \approx \frac{1}{8}\left[\left(\frac{1^2}{4} + 1\right) + 2\left(\frac{(5/4)^2}{4} + 1\right) + 2\left(\frac{(3/2)^2}{4} + 1\right) + 2\left(\frac{(7/4)^2}{4} + 1\right) + \left(\frac{2^2}{4} + 1\right)\right] = \frac{203}{128} \approx 1.5859$

Simpson's: $\int_0^1 \left(\frac{x^2}{4} + 1\right)dx \approx \frac{1}{12}\left[\left(\frac{1^2}{4} + 1\right) + 4\left(\frac{(5/4)^2}{4} + 1\right) + 2\left(\frac{(3/2)^2}{4} + 1\right) + 4\left(\frac{(7/4)^2}{4} + 1\right) + \left(\frac{2^2}{4} + 1\right)\right] = \frac{19}{12} \approx 1.5833$

5. Exact: $\int_3^4 \frac{1}{x-2}\,dx = \ln|x-2|\Big]_3^4 = \ln 2 - \ln 1 = \ln 2 - 0 = \ln 2 \approx 0.6931$

Trapezoidal Rule: $\int_3^4 \frac{1}{x-2}\,dx \approx \frac{1}{8}\left[1 + 2\left(\frac{4}{5}\right) + 2\left(\frac{2}{3}\right) + 2\left(\frac{4}{7}\right) + \frac{1}{2}\right] \approx 0.6970$

Simpson's Rule: $\int_3^4 \frac{1}{x-2}\,dx \approx \frac{1}{12}\left[1 + 4\left(\frac{4}{5}\right) + 2\left(\frac{2}{3}\right) + 4\left(\frac{4}{7}\right) + \frac{1}{2}\right] \approx 0.6933$

6. Exact: $\int_2^3 \frac{2}{x^2}\,dx = \left[-\frac{2}{x}\right]_2^3 = -\frac{2}{3} + \frac{2}{2} = \frac{1}{3}$

Trapezoidal: $\int_2^3 \frac{2}{x^2}\,dx \approx \frac{1}{8}\left[\frac{2}{2^2} + 2\left(\frac{2}{(9/4)^2}\right) + 2\left(\frac{2}{(10/4)^2}\right) + 2\left(\frac{2}{(11/4)^2}\right) + \frac{2}{3^2}\right] \approx 0.3352$

Simpson's: $\int_2^3 \frac{2}{x^2}\,dx \approx \frac{1}{12}\left[\frac{2}{2^2} + 4\left(\frac{2}{(9/4)^2}\right) + 2\left(\frac{2}{(10/4)^2}\right) + 4\left(\frac{2}{(11/4)^2}\right) + \frac{2}{3^2}\right] \approx 0.3334$

7. Exact: $\int_1^3 x^3\,dx = \left[\frac{x^4}{4}\right]_1^3 = \frac{81}{4} - \frac{1}{4} = 20$

Trapezoidal: $\int_1^3 x^3\,dx \approx \frac{1}{6}\left[1 + 2\left(\frac{4}{3}\right)^3 + 2\left(\frac{5}{3}\right)^3 + 2(2)^3 + 2\left(\frac{7}{3}\right)^3 + 2\left(\frac{8}{3}\right)^3 + 27\right] \approx 20.2222$

Simpson's: $\int_1^3 x^3\,dx \approx \frac{1}{9}\left[1 + 4\left(\frac{4}{3}\right)^3 + 2\left(\frac{5}{3}\right)^3 + 4(2)^3 + 2\left(\frac{7}{3}\right)^3 + 4\left(\frac{8}{3}\right)^3 + 27\right] = 20.0000$

8. Exact: $\int_0^8 \sqrt[3]{x}\,dx = \left[\frac{3}{4}x^{4/3}\right]_0^8 = 12.0000$

Trapezoidal: $\int_0^8 \sqrt[3]{x}\,dx \approx \frac{1}{2}\left[0 + 2 + 2\sqrt[3]{2} + 2\sqrt[3]{3} + 2\sqrt[3]{4} + 2\sqrt[3]{5} + 2\sqrt[3]{6} + 2\sqrt[3]{7} + 2\right] \approx 11.7296$

Simpson's: $\int_0^8 \sqrt[3]{x}\,dx \approx \frac{1}{3}\left[0 + 4 + 2\sqrt[3]{2} + 4\sqrt[3]{3} + 2\sqrt[3]{4} + 4\sqrt[3]{5} + 2\sqrt[3]{6} + 4\sqrt[3]{7} + 2\right] \approx 11.8632$

9. Exact: $\int_4^9 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_4^9 = 18 - \frac{16}{3} = \frac{38}{3} \approx 12.6667$

Trapezoidal: $\int_4^9 \sqrt{x}\,dx \approx \frac{5}{16}\left[2 + 2\sqrt{\frac{37}{8}} + 2\sqrt{\frac{21}{4}} + 2\sqrt{\frac{47}{8}} + 2\sqrt{\frac{26}{4}} + 2\sqrt{\frac{57}{8}} + 2\sqrt{\frac{31}{4}} + 2\sqrt{\frac{67}{8}} + 3\right] \approx 12.6640$

Simpson's: $\int_4^9 \sqrt{x}\,dx \approx \frac{5}{24}\left[2 + 4\sqrt{\frac{37}{8}} + \sqrt{21} + 4\sqrt{\frac{47}{8}} + \sqrt{26} + 4\sqrt{\frac{57}{8}} + \sqrt{31} + 4\sqrt{\frac{67}{8}} + 3\right] \approx 12.6667$

10. Exact: $\int_1^4 \left(4 - x^2\right)dx = \left[4x - \frac{x^3}{3}\right]_1^4 = -\frac{16}{3} - \frac{11}{3} = -9$

Trapezoidal: $\int_1^4 \left(4 - x^2\right)dx \approx \frac{1}{4}\left\{3 + 2\left[4 - \left(\frac{3}{2}\right)^2\right] + 2(0) + 2\left[4 - \left(\frac{5}{2}\right)^2\right] + 2(-5) + 2\left[4 - \left(\frac{7}{2}\right)^2\right] - 12\right\} \approx -9.1250$

Simpson's: $\int_1^4 \left(4 - x^2\right)dx \approx \frac{1}{6}\left[3 + 4\left(4 - \frac{9}{4}\right) + 0 + 4\left(4 - \frac{25}{4}\right) - 10 + 4\left(4 - \frac{49}{4}\right) - 12\right] = -9$

11. Exact: $\int_0^1 \frac{2}{(x+2)^2}\,dx = \left[\frac{-2}{(x+2)}\right]_0^1 = \frac{-2}{3} + \frac{2}{2} = \frac{1}{3}$

Trapezoidal: $\int_0^1 \frac{2}{(x+2)^2}\,dx \approx \frac{1}{8}\left[\frac{1}{2} + 2\left(\frac{2}{((1/4)+2)^2}\right) + 2\left(\frac{2}{((1/2)+2)^2}\right) + 2\left(\frac{2}{((3/4)+2)^2}\right) + \frac{2}{9}\right]$

$= \frac{1}{8}\left[\frac{1}{2} + 2\left(\frac{32}{81}\right) + 2\left(\frac{8}{25}\right) + 2\left(\frac{32}{121}\right) + \frac{2}{9}\right] \approx 0.3352$

Simpson's: $\int_0^1 \frac{2}{(x+2)^2}\,dx \approx \frac{1}{12}\left[\frac{1}{2} + 4\left(\frac{2}{((1/4)+2)^2}\right) + 2\left(\frac{2}{((1/2)+2)^2}\right) + 4\left(\frac{2}{((3/4)+2)^2}\right) + \frac{2}{9}\right]$

$= \frac{1}{12}\left[\frac{1}{2} + 4\left(\frac{32}{81}\right) + 2\left(\frac{8}{25}\right) + 4\left(\frac{32}{121}\right) + \frac{2}{9}\right] \approx 0.3334$

12. Exact: $\int_0^2 x\sqrt{x^2+1}\,dx = \frac{1}{3}\left[\left(x^2+1\right)^{3/2}\right]_0^2 = \frac{1}{3}\left(5^{3/2} - 1\right) \approx 3.393$

Trapezoidal: $\int_0^2 x\sqrt{x^2+1}\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)\sqrt{\left(\frac{1}{2}\right)^2+1} + 2(1)\sqrt{1^2+1} + 2\left(\frac{3}{2}\right)\sqrt{\left(\frac{3}{2}\right)^2+1} + 2\sqrt{2^2+1}\right] \approx 3.4567$

Simpson's: $\int_0^2 x\sqrt{x^2+1}\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)\sqrt{\left(\frac{1}{2}\right)^2+1} + 2(1)\sqrt{1^2+1} + 4\left(\frac{3}{2}\right)\sqrt{\left(\frac{3}{2}\right)^2+1} + 2\sqrt{2^2+1}\right] \approx 3.3922$

13. Exact: $\int_0^2 xe^{-x}\,dx = -\left[e^{-x}(x+1)\right]_0^2 = -3e^{-2} + 1 \approx 0.5940$

Trapezoidal: $\int_0^2 xe^{-x}\,dx \approx \frac{1}{4}\left[0 + e^{-1/2} + 2e^{-1} + 3e^{-3/2} + 2e^{-2}\right] \approx \frac{2.2824}{4} \approx 0.5706$

Simpson's: $\int_0^2 2xe^{-x}\,dx \approx \frac{1}{6}\left[0 + 2e^{-1/2} + 2e^{-1} + 6e^{-3/2} + 2e^{-2}\right] \approx \frac{3.5583}{6} \approx 0.5930$

14. Exact: $\int_0^2 x\ln(x+1)\,dx = \frac{1}{4}\left[\left(2x^2-2\right)\ln(x+1) - x^2 + 2x\right]_0^2 = \frac{3}{2}\ln 3 \approx 1.6479$

Trapezoidal: $\int_0^2 x\ln(x+1)\,dx \approx \frac{1}{4}\left[0 + 2(0.5)\ln(1.5) + 2\ln(2) + 2(1.5)\ln(2.5) + 2\ln(3)\right] \approx 1.6845$

Simpson's: $\int_0^2 x\ln(x+1)\,dx \approx \frac{1}{6}\left[0 + 4(0.5)\ln(1.5) + 2\ln(2) + 4(1.5)\ln(2.5) + 2\ln(3)\right] \approx 1.6487$

15. Trapezoidal: $\int_0^2 \sqrt{1+x^3}\,dx \approx \frac{1}{4}\left[1 + 2\sqrt{1+\left(\frac{1}{8}\right)} + 2\sqrt{2} + 2\sqrt{1+\left(\frac{27}{8}\right)} + 3\right] \approx 3.2833$

Simpson's: $\int_0^2 \sqrt{1+x^3}\,dx \approx \frac{1}{6}\left[1 + 4\sqrt{1+\left(\frac{1}{8}\right)} + 2\sqrt{2} + 4\sqrt{1+\left(\frac{27}{8}\right)} + 3\right] \approx 3.2396$

Graphing utility: 3.2413

16. $\int_0^1 \sqrt{x}\sqrt{1-x}\,dx = \int_0^1 \sqrt{x(1-x)}\,dx$

Trapezoidal: $\int_0^1 \sqrt{x(1-x)}\,dx \approx \frac{1}{8}\left[0 + 2\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 2\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)}\right] \approx 0.3415$

Simpson's: $\int_0^1 \sqrt{x(1-x)}\,dx \approx \frac{1}{12}\left[0 + 4\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 4\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)}\right] \approx 0.3720$

Graphing utility: 0.3927

17. Trapezoidal: $\int_0^1 \frac{1}{1+x^2}\,dx \approx \frac{1}{8}\left[\frac{1}{1+0} + \frac{2}{1+\left(\frac{1}{4}\right)^2} + \frac{2}{1+\left(\frac{1}{2}\right)^2} + \frac{2}{1+\left(\frac{3}{4}\right)^2} + \frac{1}{1+1^2}\right] \approx 0.7828$

Simpson's: $\int_0^1 \frac{1}{1+x^2}\,dx \approx \frac{1}{12}\left[\frac{1}{1+0} + \frac{4}{1+\left(\frac{1}{4}\right)^2} + \frac{2}{1+\left(\frac{1}{2}\right)^2} + \frac{4}{1+\left(\frac{3}{4}\right)^2} + \frac{1}{1+1^2}\right] \approx 0.7854$

Graphing utility: 0.7854

18. Trapezoidal: $\int_0^2 \frac{1}{\sqrt{1+x^3}}\,dx \approx \frac{1}{4}\left[1 + 2\left(\frac{1}{\sqrt{1+(1/2)^3}}\right) + 2\left(\frac{1}{\sqrt{1+1^3}}\right) + 2\left(\frac{1}{\sqrt{1+(3/2)^3}}\right) + \frac{1}{3}\right] \approx 1.3973$

Simpson's: $\int_0^2 \frac{1}{\sqrt{1+x^3}}\,dx \approx \frac{1}{6}\left[1 + 4\left(\frac{1}{\sqrt{1+(1/2)^3}}\right) + 2\left(\frac{1}{\sqrt{1+1^3}}\right) + 4\left(\frac{1}{\sqrt{1+(3/2)^3}}\right) + \frac{1}{3}\right] \approx 1.4052$

Graphing utility: 1.4022

19. Trapezoidal: $\int_0^4 \sqrt{x}e^x\,dx = \frac{1}{2}\left[0 + 2e + 2\sqrt{2}e^2 + 2\sqrt{3}e^3 + 2e^4\right] \approx 102.5553$

Simpson's: $\int_0^4 \sqrt{x}e^x\,dx = \frac{1}{3}\left[0 + 4e + 2\sqrt{2}e^2 + 4\sqrt{3}e^3 + 2e^4\right] \approx 93.3752$

Graphing utility: 92.7437

20. Trapezoidal: $\int_1^3 \ln x\,dx = \frac{1}{4}\left[0 + 2\ln\frac{3}{2} + 2\ln 2 + 2\ln\frac{5}{2} + \ln 3\right] \approx 1.2821$

Simpson's: $\int_1^3 \ln x\,dx = \frac{1}{6}\left[0 + 4\ln\frac{3}{2} + 2\ln 2 + 4\ln\frac{5}{2} + \ln 3\right] \approx 1.2953$

Graphing utility: 1.2958

21. Trapezoidal:

$\int_0^{\sqrt{\pi/2}} \sin(x^2)\,dx \approx \frac{\sqrt{\pi/2}}{8}\left[\sin 0 + 2\sin\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2\sin\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 2\sin\left(\frac{3\sqrt{\pi/2}}{4}\right)^2 + \sin\left(\sqrt{\frac{\pi}{2}}\right)^2\right] \approx 0.5495$

Simpson's: $\int_0^{\sqrt{\pi/2}} \sin(x^2)\,dx \approx \frac{\sqrt{\pi/2}}{12}\left[\sin 0 + 4\sin\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2\sin\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 4\sin\left(\frac{3\sqrt{\pi/2}}{4}\right)^2 + \sin\left(\sqrt{\frac{\pi}{2}}\right)^2\right] \approx 0.5483$

Graphing utility: 0.5493

22. Trapezoidal: $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx \approx \frac{\pi}{16}\left[\sqrt{\frac{\pi}{2}}(1) + 2\sqrt{\frac{5\pi}{8}}\sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}}\sin\left(\frac{3\pi}{4}\right) + 2\sqrt{\frac{7\pi}{8}}\sin\left(\frac{7\pi}{8}\right) + 0\right] \approx 1.4299$

Simpson's: $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx \approx \frac{\pi}{24}\left[\sqrt{\frac{\pi}{2}} + 4\sqrt{\frac{5\pi}{8}}\sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}}\sin\left(\frac{3\pi}{4}\right) + 4\sqrt{\frac{7\pi}{8}}\sin\left(\frac{7\pi}{8}\right) + 0\right] \approx 1.4583$

Graphing utility: 1.4579

23. Trapezoidal: $\int_0^{\pi/4} x \tan x \, dx \approx \frac{\pi}{32}\left[0 + 2\left(\frac{\pi}{16}\right)\tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right)\tan\left(\frac{2\pi}{16}\right) + 2\left(\frac{3\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4}\right] \approx 0.1940$

Simpson's: $\int_0^{\pi/4} x \tan x \, dx \approx \frac{\pi}{48}\left[0 + 4\left(\frac{\pi}{16}\right)\tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right)\tan\left(\frac{2\pi}{16}\right) + 4\left(\frac{3\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4}\right] \approx 0.1860$

Graphing utility: 0.1858

24. Trapezoidal: $\int_0^{\pi} \frac{\sin x}{x} \, dx \approx \frac{\pi}{8}\left[1 + \frac{2\sin(\pi/4)}{\pi/4} + \frac{2\sin(\pi/2)}{\pi/2} + \frac{2\sin(3\pi/4)}{3\pi/4} + 0\right] \approx 1.8355$

Simpson's: $\int_0^{\pi} \frac{\sin x}{x} \, dx \approx \frac{\pi}{12}\left[1 + \frac{4\sin(\pi/4)}{\pi/4} + \frac{2\sin(\pi/2)}{\pi/2} + \frac{4\sin(3\pi/4)}{3\pi/4} + 0\right] \approx 1.8522$

Graphing utility: 1.8519

25. $f(x) = x^2 + 2x$

$f'(x) = 2x + 2$

$f''(x) = 2$

$f'''(x) = 0$

$f^{(4)}(x) = 0$

(a) Trapezoidal Rule: Because $|f''(x)|$ is maximum for all x in $[0, 2]$ and $|f''(x)| = 2$, you have

$|\text{Error}| \le \frac{(2-0)^3}{12(4)^2}(2) = \frac{1}{12} \approx 0.0833.$

(b) Simpson's Rule: Because $|f^{(4)}(x)|$ is maximum for all x in $[0, 2]$ and $f^{(4)}(x) = 0$, you have $|\text{Error}| \le \frac{(2-0)^5}{180(4)^4}(0) = 0.$

26. $f(x) = 2x^3$

$f'(x) = 6x^2$

$f''(x) = 12x$

$f'''(x) = 12$

$f^{(4)}(x) = 0$

(a) Trapezoidal: Error $\le \frac{(3-1)^3}{12(4^2)}(36) = 1.5$ because

$|f''(x)|$ is maximum in [1, 3] when $x = 3$.

(b) Simpson's: Error $\le \frac{(3-1)^5}{180(4^4)}(0) = 0$ because

$f^{(4)}(x) = 0.$

27. $f(x) = (x-1)^{-2}$

$f'(x) = -2(x-1)^{-3}$

$f''(x) = 6(x-1)^{-4}$

$f'''(x) = -24(x-1)^{-5}$

$f^{(4)}(x) = 120(x-1)^{-6}$

(a) Trapezoidal: Error $\le \frac{(4-2)^3}{12(4^2)}(6) = \frac{1}{4}$ because

$|f''(x)|$ is a maximum of 6 at $x = 2$.

(b) Simpson's: Error $\le \frac{(4-2)^5}{180(4^4)}(120) = \frac{1}{12}$ because

$|f^{(4)}(x)|$ is a maximum of 120 at $x = 2$.

28. $f(x) = e^{x^3}$

$f'(x) = 3x^2e^{x^3}$

$f''(x) = 3(3x^4 + 2x)e^{x^3}$

$f'''(x) = 3(9x^6 + 18x^3 + 2)e^{x^3}$

$f^{(4)}(x) = 9(9x^8 + 36x^5 + 20x^2)e^{x^3}$

(a) Trapezoidal Rule: Because $|f''(x)|$ is maximum in $[0, 1]$ when $x = 1$ and $|f''(1)| = 15e$, you have

$$|\text{Error}| \le \frac{(1-0)^3}{12(4)^2}(15e) = \frac{5e}{64} \approx 0.212.$$

(b) Simpson's Rule: Because $|f^{(4)}(x)|$ is maximum in $[0, 1]$ when $x = 1$ and $|f^{(4)}(1)| = 585e$, you have

$$|\text{Error}| \le \frac{(1-0)^5}{180(4)^4}(585e) = \frac{13e}{1024} \approx 0.035.$$

29. $f(x) = x^{-1}, \quad 1 \le x \le 3$

$f'(x) = -x^{-2}$

$f''(x) = 2x^{-3}$

$f'''(x) = -6x^{-4}$

$f^{(4)}(x) = 24x^{-5}$

(a) Maximum of $|f''(x)| = |2x^{-3}|$ is 2.

Trapezoidal:

Error $\le \dfrac{2^3}{12n^2}(2) \le 0.00001$, $n^2 \ge 133{,}333.33$, $n \ge 365.15$ Let $n = 366$.

(b) Maximum of $|f^{(4)}(x)| = |24x^{-5}|$ is 24.

Simpson's: Error $\le \dfrac{2^5}{180n^4}(24) \le 0.00001$, $n^4 \ge 426{,}666.67$, $n \ge 25.56$ Let $n = 26$.

30. $f(x) = (1 + x)^{-1}, \quad 0 \le x \le 1$

$f'(x) = -(1 + x)^{-2}$

$f''(x) = 2(1 + x)^{-3}$

$f'''(x) = -6(1 + x)^{-4}$

$f^{(4)}(x) = 24(1 + x)^{-5}$

(a) Maximum of $|f''(x)| = |2(1 + x)^{-3}|$ is 2.

Trapezoidal:

$$\text{Error} \le \frac{1}{12n^2}(2) \le 0.00001$$

$$n^2 \ge 16{,}666.67$$

$$n \ge 129.10. \text{ Let } n = 130.$$

(b) Maximum of $|f^{(4)}(x)| = |24(1 + x)^{-5}|$ is 24.

Simpson's:

$$\text{Error} \le \frac{1}{180n^4}(24) \le 0.00001$$

$$n^4 \ge 13{,}333.33$$

$$n \ge 10.75$$

Let $n = 12$. (In Simpson's Rule n must be even.)

31. $f(x) = (x + 2)^{1/2}, \quad 0 \le x \le 2$

$f'(x) = \frac{1}{2}(x + 2)^{-1/2}$

$f''(x) = -\frac{1}{4}(x + 2)^{-3/2}$

$f'''(x) = \frac{3}{8}(x + 2)^{-5/2}$

$f^{(4)}(x) = \frac{-15}{16}(x + 2)^{-7/2}$

(a) Maximum of $|f''(x)| = \left|\dfrac{-1}{4(x + 2)^{3/2}}\right|$ is $\dfrac{\sqrt{2}}{16} \approx 0.0884$.

Trapezoidal:

$$\text{Error} \le \frac{(2-0)^3}{12n^2}\left(\frac{\sqrt{2}}{16}\right) \le 0.00001$$

$$n^2 \ge \frac{8\sqrt{2}}{12(16)}10^5 = \frac{\sqrt{2}}{24}10^5$$

$$n \ge 76.8. \text{ Let } n = 77.$$

(b) Maximum of $|f^{(4)}(x)| = \left|\dfrac{-15}{16(x + 2)^{7/2}}\right|$ is $\dfrac{15\sqrt{2}}{256} \approx 0.0829$.

Simpson's:

$$\text{Error} \le \frac{2^5}{180n^4}\left(\frac{15\sqrt{2}}{256}\right) \le 0.00001$$

$$n^4 \ge \frac{32(15)\sqrt{2}}{180(256)}10^5 = \frac{\sqrt{2}}{96}10^5$$

$$n \ge 6.2. \text{ Let } n = 8 \text{ (even)}.$$

32. $f(x) = e^{2x}, 1 \le x \le 3$

$f'(x) = 2e^{2x}$

$f''(x) = 4e^{2x}$

$f'''(x) = 8e^{2x}$

$f^{(4)}(x) = 16e^{2x}$

(a) Maximum of $\left|f''(x)\right| = 4e^{2x}$ is $4e^{2(3)} \approx 1613.7152$

Trapezoidal: Error $\le \dfrac{(3-1)^3}{12n^2}(1613.7152) \le 0.00001$

$$n^2 \ge \frac{8}{12}(1613.7152)\,10^5$$
$$n \ge 10{,}372.1.$$

Let $n = 10{,}373$.

(b) Maximum of $\left|f^{(4)}(x)\right| = 16e^{2x}$ is $16e^{2(3)} \approx 6454.8607$

Simpson's: Error $\le \dfrac{(3-1)^5}{180n^4}(6454.8607) \le 0.00001$

$$n^4 \ge \frac{32}{180}(6454.8607)\,10^5$$
$$n \ge 103.5.$$

Let $n = 104$(even).

33. $f(x) = \tan(x^2)$

(a) $f''(x) = 2\sec^2(x^2)\left[1 + 4x^2\tan(x^2)\right]$ in $[0, 1]$.

$\left|f''(x)\right|$ is maximum when $x = 1$ and $\left|f''(1)\right| \approx 49.5305$.

Trapezoidal: Error $\le \dfrac{(1-0)^3}{12n^2}(49.5305) \le 0.00001$, $n^2 \ge 412{,}754.17$, $n \ge 642.46$; let $n = 643$.

(b) $f^{(4)}(x) = 8\sec^2(x^2)\left[12x^2 + (3 + 32x^4)\tan(x^2) + 36x^2\tan^2(x^2) + 48x^4\tan^3(x^2)\right]$ in $[0, 1]$

$\left|f^{(4)}(x)\right|$ is maximum when $x = 1$ and $\left|f^{(4)}(1)\right| \approx 9184.4734$.

Simpson's: Error $\le \dfrac{(1-0)^5}{180n^4}(9184.4734) \le 0.00001$, $n^4 \ge 5{,}102{,}485.22$, $n \ge 47.53$; let $n = 48$.

34. $f(x) = (x+1)^{2/3}$

(a) $f''(x) = -\dfrac{2}{9(x+1)^{4/3}}$ in $[0, 2]$.

$\left|f''(x)\right|$ is maximum when $x = 0$ and $\left|f''(0)\right| = \dfrac{2}{9}$.

Trapezoidal: Error $\le \dfrac{8}{12n^4}\left(\dfrac{2}{9}\right) \le 0.00001$,

$n^2 \ge 14{,}814.81$, $n \ge 121.72$; let $n = 122$.

(b) $f^{(4)}(x) = -\dfrac{56}{81(x+1)^{10/3}}$ in $[0, 2]$.

$\left|f^{(4)}(x)\right|$ is maximum when $x = 0$ and $\left|f^{(4)}(0)\right| = \dfrac{56}{81}$.

Simpson's: Error $\le \dfrac{32}{180n^4}\left(\dfrac{56}{81}\right) \le 0.00001$,

$n^4 \ge 12{,}290.81$, $n \ge 10.53$; let $n = 12$. (In Simpson's Rule n must be even.)

35. $n = 4,\ b - a = 4 - 0 = 4$

Trapezoidal: $\int_0^4 f(x)\,dx \approx \frac{4}{8}\left[3 + 2(7) + 2(9) + 2(7) + 0\right] = \frac{1}{2}(49) = \frac{49}{2} = 24.5$

Simpson's: $\int_0^4 f(x)\,dx \approx \frac{4}{12}\left[3 + 4(7) + 2(9) + 4(7) + 0\right] = \frac{77}{3} \approx 25.67$

36. $n = 8,\ b - a = 8 - 0 = 8$

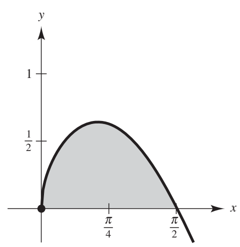
Trapezoidal: $\int_0^8 f(x)\,dx \approx \frac{8}{16}\left[0 + 2(1.5) + 2(3) + 2(5.5) + 2(9) + 2(10) + 2(9) + 2(6) + 0\right] = \frac{1}{2}(88) = 44$

Simpson's: $\int_0^8 f(x)\,dx \approx \frac{8}{24}\left[0 + 4(1.5) + 2(3) + 4(5.5) + 2(9) + 4(10) + 2(9) + 4(6) + 0\right] = \frac{1}{3}(134) = \frac{134}{3}$

37. $A = \int_0^{\pi/2} \sqrt{x}\cos x\,dx$

Simpson's Rule: $n = 14$

$$\int_0^{\pi/2} \sqrt{x}\cos x\,dx \approx \frac{\pi}{84}\left[\sqrt{0}\cos 0 + 4\sqrt{\frac{\pi}{28}}\cos\frac{\pi}{28} + 2\sqrt{\frac{\pi}{14}}\cos\frac{\pi}{14} + 4\sqrt{\frac{3\pi}{28}}\cos\frac{3\pi}{28} + \cdots + \sqrt{\frac{\pi}{2}}\cos\frac{\pi}{2}\right] \approx 0.701$$



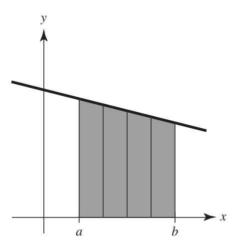
38. (a) The integral $\int_0^2 f(x)dx$ would be overestimated because the trapezoids would be above the curve. Similarly, the integral $\int_0^2 g(x)dx$ would be underestimated.

(b) Simpson's Rule would be more accurate because it takes into account the curvature of the graph.

39. The Trapezoidal Rule is the average of the left-hand Riemann Sum and the right-hand Rieman Sum,

$T_n = \frac{1}{2}(L_n + R_n).$

40. For a linear function, the Trapezoidal Rule is exact. The error formula says that $E \le \frac{(b-a)^3}{12n^2}\left[\max\left|f''(x)\right|\right]$ and $f''(x) = 0$ for a linear function. Geometrically, a linear function is approximated exactly by trapezoids:



41. Area $\approx \frac{1000}{2(10)}\left[125 + 2(125) + 2(120) + 2(112) + 2(90) + 2(90) + 2(95) + 2(88) + 2(75) + 2(35)\right] = 89{,}250\text{ m}^2$

42. Simpson's Rule: $n = 8$

$$8\sqrt{3}\int_0^{\pi/2}\sqrt{1 - \frac{2}{3}\sin^2\theta\,d\theta} \approx \frac{\sqrt{3}\pi}{6}\left[\sqrt{1 - \frac{2}{3}\sin^2 0} + 4\sqrt{1 - \frac{2}{3}\sin^2\frac{\pi}{16}} + 2\sqrt{1 - \frac{2}{3}\sin^2\frac{\pi}{8}} + \cdots + \sqrt{1 - \frac{2}{3}\sin^2\frac{\pi}{2}}\right]$$

$$\approx 17.476$$

43. $W = \int_0^5 100x\sqrt{125 - x^3}\, dx$

Simpson's Rule: $n = 12$

$$\int_0^5 100x\sqrt{125 - x^3}\, dx \approx \frac{5}{3(12)}\left[0 + 400\left(\frac{5}{12}\right)\sqrt{125 - \left(\frac{5}{12}\right)^3} + 200\left(\frac{10}{12}\right)\sqrt{125 - \left(\frac{10}{12}\right)^3} + 400\left(\frac{15}{12}\right)\sqrt{125 - \left(\frac{15}{12}\right)^3} + \cdots + 0\right] \approx 10{,}233.58 \text{ ft-lb}$$

44. (a) Trapezoidal:

$$\int_0^2 f(x)\, dx \approx \frac{2}{2(8)}\left[4.32 + 2(4.36) + 2(4.58) + 2(5.79) + 2(6.14) + 2(7.25) + 2(7.64) + 2(8.08) + 8.14\right] \approx 12.518$$

Simpson's:

$$\int_0^2 f(x)\, dx \approx \frac{2}{3(8)}\left[4.32 + 4(4.36) + 2(4.58) + 4(5.79) + 2(6.14) + 4(7.25) + 2(7.64) + 4(8.08) + 8.14\right] \approx 12.592$$

(b) Using a graphing utility,

$y = -1.37266x^3 + 4.0092x^2 - 0.620x + 4.28$. Integrating, $\int_0^2 y\, dx \approx 12.521$.

45. $\int_0^t \sin\sqrt{x}\, dx = 2,\ n = 10$

By trial and error, you obtain $t \approx 2.477$.

46. Let $f(x) = Ax^3 + Bx^2 + Cx + D$. Then $f^{(4)}(x) = 0$.

Simpson's: Error $\le \dfrac{(b-a)^5}{180n^4}(0) = 0$

So, Simpson's Rule is exact when approximating the integral of a cubic polynomial.

Example: $\int_0^1 x^3\, dx = \dfrac{1}{6}\left[0 + 4\left(\dfrac{1}{2}\right)^3 + 1\right] = \dfrac{1}{4}$

This is the exact value of the integral.

47. The quadratic polynomial

$$p(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}y_1 + \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}y_2 + \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}y_3$$

passes through the three points.

Section 8.7 Integration by Tables and Other Integration Techniques

1. Use Formula 40.

2. A reduction formula reduces an integral to the sum of a function and a simpler integral. For example, see Formulas 50, 54.

3. By Formula 6: $(a = 5, b = 1)$

$$\int \frac{x^2}{5 + x}\, dx = \left[-\frac{x}{2}(10 - x) + 25\ln|5 + x|\right] + C$$

4. By Formula 13: $(a = 4, b = 3)$

$$\int \frac{2}{x^2(4 + 3x)^2}\, dx = 2\left(\frac{-1}{16}\right)\left[\frac{4 + 6x}{x(4 + 3x)} + \frac{6}{4}\ln\left|\frac{x}{4 + 3x}\right|\right] + C$$

$$= -\frac{(2 + 3x)}{4x(3x + 4)} - \frac{3}{16}\ln\left|\frac{x}{4 + 3x}\right| + C$$

5. By Formula 44: $\int \frac{1}{x^2\sqrt{1-x^2}}\,dx = -\frac{\sqrt{1-x^2}}{x} + C$

6. Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{\sqrt{64-x^4}}{x}\,du = \frac{1}{2}\int \frac{\sqrt{64-x^4}}{x^2}(2x\,dx)$$
$$= \frac{1}{2}\int \frac{\sqrt{64-u^2}}{u}\,du$$

By Formula 39: $(a = 8)$

$$\int \frac{\sqrt{64-u^2}}{u}\,dx = \frac{1}{2}\left[\sqrt{64-u^2} - 8\ln\left|\frac{8+\sqrt{64-u^2}}{u}\right|\right] + C$$
$$= \frac{1}{2}\sqrt{64-x^4} - 4\ln\left|\frac{8+\sqrt{64-x^4}}{x^2}\right| + C$$

7. By Formulas 51 and 49:

$$\int \cos^4 3x\,dx = \frac{1}{3}\int \cos^4 3x\,(3)\,dx$$
$$= \frac{1}{3}\left[\frac{\cos^3 3x \sin 3x}{4} + \frac{3}{4}\int \cos^2 3x\,dx\right]$$
$$= \frac{1}{12}\cos^3 3x \sin 3x + \frac{1}{4}\cdot\frac{1}{3}\int \cos^2 3x\,(3)\,dx$$
$$= \frac{1}{12}\cos^3 3x \sin 3x + \frac{1}{12}\cdot\frac{1}{2}(3x + \sin 3x \cos 3x) + C$$
$$= \frac{1}{24}\left(2\cos^3 3x \sin 3x + 3x + \sin 3x \cos 3x\right) + C$$

8. Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$.

$$\int \frac{\sin^4\sqrt{x}}{\sqrt{x}}\,dx = 2\int \sin^4 u\,du$$
$$= 2\left[-\frac{\sin^3 u \cos u}{4} + \frac{3}{4}\int \sin^2 u\,du\right] \qquad \text{(Formula 50, } n = 4\text{)}$$
$$= 2\left[-\frac{\sin^3 u \cos u}{4} + \frac{3}{4}\cdot\frac{1}{2}(u - \sin u \cos u)\right] + C \qquad \text{(Formula 48)}$$
$$= -\frac{1}{2}\sin^3 u \cos u + \frac{3}{4}u - \frac{3}{4}\sin u \cos u + C$$
$$= -\frac{1}{2}\sin^3\sqrt{x}\,\cos\sqrt{x} + \frac{3}{4}\sqrt{x} - \frac{3}{4}\sin\sqrt{x}\cos\sqrt{x} + C$$

9. By Formula 57: $\int \frac{1}{\sqrt{x}\left(1-\cos\sqrt{x}\right)}\,dx = 2\int \frac{1}{1-\cos\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)dx = -2\left(\cot\sqrt{x} + \csc\sqrt{x}\right) + C$

$u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$

10. Let $u = 4x$, $du = 4\,dx$.

By Formula 72:

$$\int \frac{1}{1 + \cot 4x}\,dx = \frac{1}{4}\int \frac{1}{1 + \cot 4x}(4\,dx)$$
$$= \frac{1}{4}\cdot\frac{1}{2}\left(4x - \ln\left|\sin 4x + \cos 4x\right|\right) + C$$
$$= \frac{1}{2}x - \frac{1}{8}\ln\left|\sin 4x + \cos 4x\right| + C$$

11. By Formula 84:

$$\int \frac{1}{1 + e^{2x}}\,dx = 2x - \frac{1}{2}\ln\left(1 + e^{2x}\right) + C$$

12. By Formula 85: $(a = -4, b = 3)$

$$\int e^{-4x}\sin 3x\,dx = \frac{e^{-4x}}{(-4)^2 + 3^2}(-4\sin 3x - 3\cos 3x) + C$$
$$= \frac{e^{-4x}}{25}(-4\sin 3x - 3\cos 3x) + C$$

13. By Formula 89: $(n = 6)$

$$\int x^6 \ln x\,dx = \frac{x^7}{49}(-1 + 7\ln x) + C$$

14. By Formulas 90 and 91: $\int (\ln x)^3\,dx = x(\ln x)^3 - 3\int(\ln x)^2\,dx$

$$= x(\ln x)^3 - 3x\left[2 - 2\ln x + (\ln x)^2\right] + C$$
$$= x\left[(\ln x)^3 - 3(\ln x)^2 + 6\ln x - 6\right] + C$$

15. (a) Let $u = \frac{1}{3}x \Rightarrow du = \frac{1}{3}\,dx$

$$\int \ln\frac{x}{3}\,dx = 3\int \ln u\,du$$

By Formula 87:

$$\int \ln\frac{x}{3}\,dx = 3\left[\frac{1}{3}x\left(-1 + \ln\frac{x}{3}\right)\right] + C$$
$$= x\left(\ln\frac{x}{3} - 1\right) + C$$

(b) Integration by parts:

$dv = dx \quad\Rightarrow\quad v = \int dx = x$

$u = \ln\frac{x}{3} \quad\Rightarrow\quad du = \frac{1}{x/3}\cdot\frac{1}{3} = \frac{1}{x}\,dx$

$$\int \ln\frac{x}{3}\,dx = x\ln\frac{x}{3} - \int x\cdot\frac{1}{x}\,dx$$
$$= x\ln\frac{x}{3} - \int dx$$
$$= x\ln\frac{x}{3} - x + C$$
$$= x\left(\ln\frac{x}{3} - 1\right) + C$$

16. Let $u = 3x, du = 3\,dx$.

(a) By Formula 48:

$$\int \sin^2 3x\,dx = \frac{1}{2}\cdot\frac{1}{3}(3x - \sin 3x\cos 3x) + C$$
$$= \frac{1}{6}(3x - \sin 3x \cos 3x) + C$$

(b) Power-reducing formula:

$$\int \sin^2 3x\,dx = \frac{1}{3}\int \sin^2 3x\,(3\,dx)$$
$$= \frac{1}{3}\left[-\frac{\sin 3x\cos 3x}{2} + \frac{1}{2}\int 3\,dx\right]$$
$$= -\frac{1}{6}\sin 3x\cos 3x + \frac{1}{2}x + C$$

17. (a) By Formula 12: $(a = -1, b = 1)$

$$\int \frac{1}{x^2(x-1)}\,dx = \frac{-1}{(-1)}\left(\frac{1}{x} + \frac{1}{(-1)}\ln\left|\frac{x}{-1+x}\right|\right) + C$$
$$= \frac{1}{x} - \ln\left|\frac{x}{x-1}\right| + C$$
$$= \frac{1}{x} + \ln\left|\frac{x-1}{x}\right| + C$$

(b) Partial fractions:

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$
$$1 = Ax(x-1) + B(x-1) + Cx^2$$

When $x = 1, C = 1$.

When $x = 0, B = -1$.

When $x = -1, 1 = 2A + 2 + 1 \Rightarrow A = -1$.

$$\int \frac{1}{x^2(x-1)}\,dx = \int\left(\frac{-1}{x} + \frac{-1}{x^2} + \frac{1}{x-1}\right)dx$$
$$= -\ln|x| + \frac{1}{x} + \ln|x-1| + C$$
$$= \frac{1}{x} + \ln\left|\frac{x-1}{x}\right| + C$$

18. (a) By Formula 36: $(n = 2)$

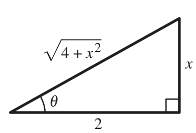
$$\int \frac{dx}{(4+x^2)^{3/2}} = \frac{x}{4\sqrt{4+x^2}} + C$$

(b) Trigonometric substitution:

$x = 2\tan\theta, dx = 2\sec^2\theta\,d\theta$

$4 + x^2 = 4\sec^2\theta$

$$\int \frac{dx}{(4+x^2)^{3/2}} = \int \frac{2\sec^2\theta\,d\theta}{8\sec^3\theta}$$
$$= \frac{1}{4}\int \cos\theta\,d\theta$$
$$= \frac{1}{4}\sin\theta + C = \frac{x}{4\sqrt{4+x^2}} + C$$



19. By Formula 80:

$$\int x\,\text{arccsc}(x^2+1)\,dx = \frac{1}{2}\int \text{arccsc}(x^2+1)(2x)\,dx$$
$$= \frac{1}{2}\left[(x^2+1)\text{arccsc}(x^2+1) + \ln\left|x^2+1+\sqrt{(x^2+1)^2-1}\right|\right] + C$$
$$= \frac{1}{2}(x^2+1)\text{arccsc}(x^2+1) + \frac{1}{2}\ln\left(x^2+1+\sqrt{x^4+2x^2}\right) + C$$

20. By Formula 78: $u = 4x - 5$, $du = 4\,dx$

$$\int \operatorname{arccot}(4x - 5)\,dx = \frac{1}{4}\int \operatorname{arccot}(4x - 5)(4\,dx)$$
$$= \frac{1}{4}\left[(4x - 5)\operatorname{arccot}(4x - 5) + \ln\sqrt{1 + (4x - 5)^2}\right] + C$$

21. Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{2}{x^3\sqrt{x^4 - 1}}\,dx = \int \frac{2x}{x^4\sqrt{x^4 - 1}}\,dx = \int \frac{du}{u^2\sqrt{u^2 - 1}}$$

By Formula 35:

$$\int \frac{du}{u^2\sqrt{u^2 - 1}} = \frac{\sqrt{u^2 - 1}}{u} = \frac{\sqrt{x^4 - 1}}{x^2} + C$$

22. By Formula 14: $a = 8$, $b = 4$, $c = 1$, $b^2 < 4ac$

$$\int \frac{1}{x^2 + 4x + 8}\,dx = \frac{2}{\sqrt{16}}\arctan\frac{2x + 4}{\sqrt{16}} + C$$
$$= \frac{1}{2}\arctan\left(\frac{x + 2}{2}\right) + C$$

23. By Formula 4: $a = 7$, $b = -6$

$$\int \frac{x}{(7 - 6x)^2}\,dx = \frac{1}{(-6)^2}\left(\frac{7}{7 - 6x} + \ln|7 - 6x|\right) + C$$
$$= \frac{1}{36}\left(\frac{7}{7 - 6x} + \ln|7 - 6x|\right) + C$$

24. By Formula 56: $u = \theta^4$, $du = 4\theta^3\,d\theta$

$$\int \frac{\theta^3}{1 + \sin\theta^4}\,d\theta = \frac{1}{4}\int \frac{1}{1 + \sin\theta^4}4\theta^3\,d\theta$$
$$= \frac{1}{4}\left(\tan\theta^4 - \sec\theta^4\right) + C$$

25. By Formula 76: $u = e^x$, $du = e^x\,dx$

$$\int e^x \arccos e^x\,dx = e^x \arccos e^x - \sqrt{1 - e^{2x}} + C$$

26. By Formula 71: $u = e^x$, $du = e^x\,dx$

$$\int \frac{e^x}{1 - \tan e^x}\,dx = \frac{1}{2}\left(e^x - \ln\left|\cos e^x - \sin e^x\right|\right) + C$$

27. By Formula 73:

$$\int \frac{x}{1 - \sec x^2}\,dx = \frac{1}{2}\int \frac{2x}{1 - \sec x^2}\,dx$$
$$= \frac{1}{2}\left(x^2 + \cot x^2 + \csc x^2\right) + C$$

28. By Formula 23: $u = \ln t$, $du = \frac{1}{t}\,dt$

$$\int \frac{1}{t\left[1 + (\ln t)^2\right]}\,dt = \int \frac{1}{1 + (\ln t)^2}\left(\frac{1}{t}\right)dt = \arctan(\ln t) + C$$

29. By Formula 14: $u = \sin\theta$, $du = \cos\theta\,d\theta$

$$\int \frac{\cos\theta}{3 + 2\sin\theta + \sin^2\theta}\,d\theta = \frac{\sqrt{2}}{2}\arctan\left(\frac{1 + \sin\theta}{\sqrt{2}}\right) + C \quad \left(b^2 = 4 < 12 = 4ac\right)$$

30. $\int x^2\sqrt{3 + 25x^2}\,dx$, $u = 5x$, $du = 5\,dx$

$$\frac{1}{125}\int (5x)^2\sqrt{3 + (5x)^2}\,(5\,dx) = \frac{1}{125}\int u^2\sqrt{3 + u^2}\,du \quad \left(a = \sqrt{3}\right)$$

By Formula 27:

$$\frac{1}{125}\int u^2\sqrt{3 + u^2}\,du = \frac{1}{125}\cdot\frac{1}{8}\left[u\left(2u^2 + 3\right)\sqrt{u^2 + 3} - 9\ln\left|u + \sqrt{u^2 + 3}\right|\right] + C$$
$$= \frac{1}{1000}\left[5x\left(50x^2 + 3\right)\sqrt{25x^2 + 3} - 9\ln\left|5x + \sqrt{25x^2 + 3}\right|\right] + C$$

31. By Formula 35: $\int \frac{1}{x^2\sqrt{2 + 9x^2}}\,dx = 3\int \frac{3}{(3x)^2\sqrt{\left(\sqrt{2}\right)^2 + (3x)^2}}\,dx = -\frac{3\sqrt{2 + 9x^2}}{6x} + C = -\frac{\sqrt{2 + 9x^2}}{2x} + C$

32. By Formula 77: $\int \sqrt{x}\arctan\left(x^{3/2}\right)dx = \frac{2}{3}\int \arctan\left(x^{3/2}\right)\left(\frac{3}{2}\sqrt{x}\right)dx = \frac{2}{3}\left[x^{3/2}\arctan\left(x^{3/2}\right) - \ln\sqrt{1 + x^3}\right] + C$

33. By Formula 3: $u = \ln x$, $du = \frac{1}{x}\,dx$

$$\int \frac{\ln x}{x(3 + 2\ln x)}\,dx = \frac{1}{4}\left(2\ln|x| - 3\ln\left|3 + 2\ln|x|\right|\right) + C$$

34. By Formula 45: $u = e^x$, $du = e^x\,dx$

$$\int \frac{e^x}{\left(1 - e^{2x}\right)^{3/2}}\,dx = \frac{e^x}{\sqrt{1 - e^{2x}}} + C$$

35. By Formulas 1, 23, and 35: $\int \frac{x}{\left(x^2 - 6x + 10\right)^2}\,dx = \frac{1}{2}\int \frac{2x - 6 + 6}{\left(x^2 - 6x + 10\right)^2}\,dx$

$$= \frac{1}{2}\int \left(x^2 - 6x + 10\right)^{-2}(2x - 6)\,dx + 3\int \frac{1}{\left[(x - 3)^2 + 1\right]^2}\,dx$$

$$= -\frac{1}{2\left(x^2 - 6x + 10\right)} + \frac{3}{2}\left[\frac{x - 3}{x^2 - 6x + 10} + \arctan(x - 3)\right] + C$$

$$= \frac{3x - 10}{2\left(x^2 - 6x + 10\right)} + \frac{3}{2}\arctan(x - 3) + C$$

36. By Formula 41:

$$\int \sqrt{\frac{5 - x}{5 + x}}\,dx = \int \frac{\sqrt{5 - x}}{\sqrt{5 + x}} \cdot \frac{\sqrt{5 - x}}{\sqrt{5 - x}}\,dx$$

$$= \int \frac{5 - x}{\sqrt{25 - x^2}}\,dx$$

$$= \int \frac{5\,dx}{\sqrt{25 - x^2}} - \int \frac{x}{\sqrt{25 - x^2}}\,dx$$

$$= 5\arcsin\left(\frac{x}{5}\right) + \sqrt{25 - x^2} + C$$

37. By Formula 31: $u = x^2 - 3$, $du = 2x\,dx$

$$\int \frac{x}{\sqrt{x^4 - 6x^2 + 5}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{\left(x^2 - 3\right)^2 - 4}}\,dx = \frac{1}{2}\ln\left|x^2 - 3 + \sqrt{x^4 - 6x^2 + 5}\right| + C$$

38. By Formula 31: $u = \sin x$, $du = \cos x\,dx$

$$\int \frac{\cos x}{\sqrt{\sin^2 x + 1}}\,dx = \ln\left|\sin x + \sqrt{\sin^2 x + 1}\right| + C$$

39. By Formula 8: $u = e^x, du = e^x\,dx$

$$\int \frac{e^{3x}}{\left(1 + e^x\right)^3}\,dx = \int \frac{\left(e^x\right)^2}{\left(1 + e^x\right)^3}\left(e^x\right)dx$$
$$= \frac{2}{1 + e^x} - \frac{1}{2\left(1 + e^x\right)^2} + \ln\left|1 + e^x\right| + C$$

40. By Formulas 64 and 68:

$$\int \cot^4 \theta\,d\theta = -\frac{\cot^3 \theta}{3} - \int \cot^2 \theta\,d\theta$$
$$= -\frac{\cot^3 \theta}{3} + \theta + \cot\theta + C$$

41. By Formula 21: $a = 1, b = 1, u = x, du = dx$

$$\int_0^1 \frac{x}{\sqrt{1 + x}}\,dx = \left[\frac{-2(2 - x)}{3(1)^2}\sqrt{1 + x}\right]_0^1$$
$$= \left[\frac{2}{3}(x - 2)\sqrt{1 + x}\right]_0^1$$
$$= \frac{2}{3}(-1)\sqrt{2} - \frac{2}{3}(-2)\sqrt{1}$$
$$= -\frac{2}{3}\left(\sqrt{2} - 2\right)$$
$$\approx 0.3905$$

42. Let $u = x^2, du = 2x\,dx$

$$\int_0^1 2x^3\,e^{x^2}\,dx = \int_0^1 x^2 e^{x^2}(2x\,dx)$$
$$= \int ue^4\,du$$
$$= \left[(u - 1)e^u\right] \qquad \text{(Formula 82)}$$
$$= \left[\left(x^2 - 1\right)e^{x^2}\right]_0^1$$
$$= 0 - (-1) = 1$$

43. By Formula 89: $n = 4$

$$\int_1^2 x^4 \ln x\,dx = \left[\frac{x^5}{25}(-1 + 5\ln x)\right]_1^2$$
$$= \frac{32}{25}[-1 + 5\ln 2] - \frac{1}{25}[-1 + 0]$$
$$= -\frac{31}{25} + \frac{32}{5}\ln 2 \approx 3.1961$$

44. By Formula 52: $u = 2x, du = 2dx$

$$\int_0^{\pi/2} x \sin 2x\,dx = \frac{1}{4}\int_0^{\pi/2}(2x)\sin 2x\,(2dx)$$
$$= \frac{1}{4}[\sin 2x - 2x\cos 2x]_0^{\pi/2}$$
$$= \frac{1}{4}\left[0 - \pi(-1)\right]$$
$$= \frac{\pi}{4}$$

45. By Formula 23: $u = \sin x, du = \cos x$

$$\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\,dx = \left[\arctan(\sin x)\right]_{-\pi/2}^{\pi/2}$$
$$= \arctan(1) - \arctan(-1) = \frac{\pi}{2}$$

46. By Formula 7: $a = 5, b = 2$

$$\int_0^5 \frac{x^2}{(5 + 2x)^2}\,dx = \frac{1}{8}\left[2x - \frac{25}{5 + 2x} - 10\ln\left|5 + 2x\right|\right]_0^5$$
$$= \frac{1}{8}\left[\left(10 - \frac{25}{15} - 10\ln 15\right) - (-5 - 10\ln 5)\right]$$
$$= \frac{5}{3} - \frac{1}{8}(10)\ln\left(\frac{15}{5}\right)$$
$$= \frac{5}{3} - \frac{5}{4}\ln 3$$

47. By Formulas 54 and 55:

$$\begin{aligned}\int t^3 \cos t\, dt &= t^3 \sin t - 3\int t^2 \sin t\, dt\\ &= t^3 \sin t - 3\left(-t^2 \cos t + 2\int t \cos t\, dt\right)\\ &= t^3 \sin t + 3t^2 \cos t - 6\left(t \sin t - \int \sin t\, dt\right)\\ &= t^3 \sin t + 3t^2 \cos t - 6t \sin t - 6 \cos t + C\end{aligned}$$

So,

$$\begin{aligned}\int_0^{\pi/2} t^3 \cos t\, dt &= \Big[t^3 \sin t + 3t^2 \cos t - 6t \sin t - 6 \cos t\Big]_0^{\pi/2}\\ &= \left(\frac{\pi^3}{8} - 3\pi\right) + 6 = \frac{\pi^3}{8} + 6 - 3\pi \approx 0.4510.\end{aligned}$$

48. By Formula 26: $a = 4$

$$\begin{aligned}\int_0^3 \sqrt{x^2 + 16}\, dx &= \frac{1}{2}\left[x\sqrt{x^2 + 16} + 16 \ln\left|x + \sqrt{x^2 + 16}\right|\right]_0^3\\ &= \frac{1}{2}\Big[\big(3(5) + 16 \ln|3 + 5|\big) - (16 \ln 4)\Big]\\ &= \frac{15}{2} + 8 \ln 8 - 8 \ln 4\\ &= \frac{15}{2} + 8 \ln 2\end{aligned}$$

49. $$\frac{u^2}{(a + bu)^2} = \frac{1}{b^2} - \frac{(2a/b)u + (a^2/b^2)}{(a + bu)^2} = \frac{1}{b^2} + \frac{A}{a + bu} + \frac{B}{(a + bu)^2}$$

$$-\frac{2a}{b}u - \frac{a^2}{b^2} = A(a + bu) + B = (aA + B) + bAu$$

Equating the coefficients of like terms you have $aA + B = -a^2/b^2$ and $bA = -2a/b$. Solving these equations you have $A = -2a/b^2$ and $B = a^2/b^2$.

$$\begin{aligned}\int \frac{u^2}{(a + bu)^2}\, du &= \frac{1}{b^2}\int du - \frac{2a}{b^2}\left(\frac{1}{b}\right)\int \frac{1}{a + bu} b\, du + \frac{a^2}{b^2}\left(\frac{1}{b}\right)\int \frac{1}{(a + bu)^2} b\, du = \frac{1}{b^2}u - \frac{2a}{b^3} \ln|a + bu| - \frac{a^2}{b^3}\left(\frac{1}{a + bu}\right) + C\\ &= \frac{1}{b^3}\left(bu - \frac{a^2}{a + bu} - 2a \ln|a + bu|\right) + C\end{aligned}$$

50. Integration by parts: $w = u^n$, $dw = nu^{n-1}\, du$, $dv = \dfrac{du}{\sqrt{a + bu}}$, $v = \dfrac{2}{b}\sqrt{a + bu}$

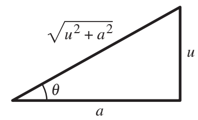
$$\begin{aligned}\int \frac{u^n}{\sqrt{a + bu}}\, du &= \frac{2u^n}{b}\sqrt{a + bu} - \frac{2n}{b}\int u^{n-1}\sqrt{a + bu}\, du\\ &= \frac{2u^n}{b}\sqrt{a + bu} - \frac{2n}{b}\int u^{n-1}\sqrt{a + bu} \cdot \frac{\sqrt{a + bu}}{\sqrt{a + bu}}\, du\\ &= \frac{2u^n}{b}\sqrt{a + bu} - \frac{2n}{b}\int \frac{au^{n-1} + bu^n}{\sqrt{a + bu}}\, du\\ &= \frac{2u^n}{b}\sqrt{a + bu} - \frac{2na}{b}\int \frac{u^{n-1}}{\sqrt{a + bu}}\, du - 2n\int \frac{u^n}{\sqrt{a + bu}}\, du\end{aligned}$$

Therefore, $(2n + 1)\displaystyle\int \frac{u^n}{\sqrt{a + bu}}\, du = \frac{2}{b}\left[u^n\sqrt{a + bu} - na\int \frac{u^{n-1}}{\sqrt{a + bu}}\, du\right]$ and

$$\int \frac{u^n}{\sqrt{a + bu}} = \frac{2}{(2n + 1)b}\left[u^n\sqrt{a + bu} - na\int \frac{u^{n-1}}{\sqrt{a + bu}}\, du\right].$$

51. When you have $u^2 + a^2$:

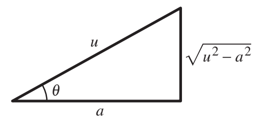
$$u = a\tan\theta$$
$$du = a\sec^2\theta\,d\theta$$
$$u^2 + a^2 = a^2\sec^2\theta$$



$$\int \frac{1}{(u^2 + a^2)^{3/2}}\,du = \int \frac{a\sec^2\theta\,d\theta}{a^3\sec^3\theta} = \frac{1}{a^2}\int \cos\theta\,d\theta = \frac{1}{a^2}\sin\theta + C = \frac{u}{a^2\sqrt{u^2 + a^2}} + C$$

When you have $u^2 - a^2$:

$$u = a\sec\theta$$
$$du = a\sec\theta\tan\theta\,d\theta$$
$$u^2 - a^2 = a^2\tan^2\theta$$



$$\int \frac{1}{(u^2 - a^2)^{3/2}}\,du = \int \frac{a\sec\theta\tan\theta\,d\theta}{a^3\tan^3\theta} = \frac{1}{a^2}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{a^2}\int \csc\theta\cot\theta\,d\theta = -\frac{1}{a^2}\csc\theta + C = \frac{-u}{a^2\sqrt{u^2 - a^2}} + C$$

52. $\int u^n(\cos u)\,du = u^n\sin u - n\int u^{n-1}(\sin u)\,du$

$w = u^n,\ dv = \cos u\,du,\ dw = nu^{n-1}\,du,\ v = \sin u$

53.
$$\int(\arctan u)\,du = u\arctan u - \frac{1}{2}\int \frac{2u}{1 + u^2}\,du$$
$$= u\arctan u - \frac{1}{2}\ln(1 + u^2) + C$$
$$= u\arctan u - \ln\sqrt{1 + u^2} + C$$

$w = \arctan u,\ dv = du,\ dw = \dfrac{du}{1 + u^2},\ v = u$

54.
$$\int(\ln u)^n\,du = u(\ln u)^n - \int n(\ln u)^{n-1}\left(\frac{1}{u}\right)u\,du$$
$$= u(\ln u)^n - n\int(\ln u)^{n-1}\,du$$

$w = (\ln u)^n,\ dv = du,\ dw = n(\ln u)^{n-1}\left(\dfrac{1}{u}\right)du,$

$v = u$

55.
$$\int \frac{1}{2 - 3\sin\theta}\,d\theta = \int\left[\frac{\dfrac{2\,du}{1 + u^2}}{2 - 3\left(\dfrac{2u}{1 + u^2}\right)}\right],\ u = \tan\frac{\theta}{2}$$
$$= \int \frac{2}{2(1 + u^2) - 6u}\,du$$
$$= \int \frac{1}{u^2 - 3u + 1}\,du$$
$$= \int \frac{1}{\left(u - \frac{3}{2}\right)^2 - \frac{5}{4}}\,du$$
$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\left(u - \frac{3}{2}\right) - \frac{\sqrt{5}}{2}}{\left(u - \frac{3}{2}\right) + \frac{\sqrt{5}}{2}}\right| + C$$
$$= \frac{1}{\sqrt{5}}\ln\left|\frac{2u - 3 - \sqrt{5}}{2u - 3 + \sqrt{5}}\right| + C$$
$$= \frac{1}{\sqrt{5}}\ln\left|\frac{2\tan\left(\frac{\theta}{2}\right) - 3 - \sqrt{5}}{2\tan\left(\frac{\theta}{2}\right) - 3 + \sqrt{5}}\right| + C$$

56.
$$\int \frac{\sin\theta}{1 + \cos^2\theta}\,d\theta = -\int \frac{-\sin\theta}{1 + (\cos\theta)^2}\,d\theta$$
$$= -\arctan(\cos\theta) + C$$

57. $\displaystyle\int_0^{\pi/2} \frac{1}{1+\sin\theta+\cos\theta}\,d\theta = \int_0^1 \left[\frac{\frac{2\,du}{1+u^2}}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\right]$

$$= \int_0^1 \frac{1}{1+u}\,du$$

$$= \Big[\ln|1+u|\Big]_0^1$$

$$= \ln 2$$

$u = \tan\dfrac{\theta}{2}$

58. $\displaystyle\int_0^{\pi/2} \frac{1}{3-2\cos\theta}\,d\theta = \int_0^1 \left[\frac{\frac{2u}{1+u^2}}{3-\frac{2(1-u^2)}{1+u^2}}\right]$

$$= 2\int_0^1 \frac{1}{5u^2+1}\,du$$

$$= \left[\frac{2}{\sqrt{5}}\arctan\left(\sqrt{5}\,u\right)\right]_0^1$$

$$= \frac{2}{\sqrt{5}}\arctan\sqrt{5}$$

$u = \tan\dfrac{\theta}{2}$

59. $\displaystyle\int \frac{\sin\theta}{3-2\cos\theta}\,d\theta = \frac{1}{2}\int \frac{2\sin\theta}{3-2\cos\theta}\,d\theta$

$$= \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln(3-2\cos\theta) + C$$

$u = 3 - 2\cos\theta,\ du = 2\sin\theta\,d\theta$

60. $\displaystyle\int \frac{\cos\theta}{1+\cos\theta}\,d\theta = \int \frac{\cos\theta(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}\,d\theta$

$$= \int \frac{\cos\theta - \cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int\left(\csc\theta\cot\theta - \cot^2\theta\right)d\theta$$

$$= \int\left(\csc\theta\cot\theta - \left(\csc^2\theta - 1\right)\right)d\theta$$

$$= -\csc\theta + \cot\theta + \theta + C$$

61. $\displaystyle\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}}\,d\theta = 2\int \sin\sqrt{\theta}\left(\frac{1}{2\sqrt{\theta}}\right)d\theta$

$$= -2\cos\sqrt{\theta} + C$$

$u = \sqrt{\theta},\ du = \dfrac{1}{2\sqrt{\theta}}\,d\theta$

62. $\displaystyle\int \frac{4}{\csc\theta - \cot\theta}\,d\theta = \int \frac{4}{\left(\frac{1}{\sin\theta}\right) - \left(\frac{\cos\theta}{\sin\theta}\right)}\,d\theta$

$$= 4\int \frac{\sin\theta}{1-\cos\theta}\,d\theta$$

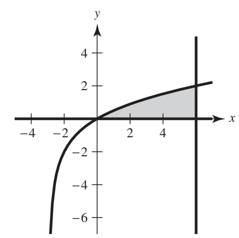
$$= 4\ln|1-\cos\theta| + C$$

$u = 1 - \cos\theta,\ du = \sin\theta\,d\theta$

63. By Formula 21: $a = 3, b = 1$

$$A = \int_0^6 \frac{x}{\sqrt{x+3}}\,dx = \left[\frac{-2(6-x)}{3}\sqrt{x+3}\right]_0^6$$

$$= 4\sqrt{3} \approx 6.928 \text{ square units}$$



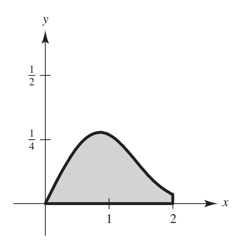
64. $\displaystyle A = \int_0^2 \frac{x}{1+e^{x^2}}\,dx$

$$= \frac{1}{2}\int_0^2 \frac{2x\,dx}{1+e^{x^2}}$$

$$= \frac{1}{2}\left[x^2 - \ln\left(1+e^{x^2}\right)\right]_0^2$$

$$= \frac{1}{2}\left[4 - \ln\left(1+e^4\right)\right] + \frac{1}{2}\ln 2$$

$$\approx 0.337 \text{ square units}$$



65. (a) $n = 1$: $u = \ln x, du = \frac{1}{x}\,dx, dv = x\,dx, v = \frac{x^2}{2}$

$$\int x \ln x\,dx = \frac{x^2}{2}\ln x - \int\left(\frac{x^2}{2}\right)\frac{1}{x}\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

$n = 2$: $u = \ln x, du = \frac{1}{x}\,dx, dv = x^2\,dx, v = \frac{x^3}{3}$

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \int\left(\frac{x^3}{3}\right)\frac{1}{x}\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

$n = 3$: $u = \ln x, du = \frac{1}{x}\,dx, dv = x^3\,dx, v = \frac{x^4}{4}$

$$\int x^3 \ln x\,dx = \frac{x^4}{4}\ln x - \int\left(\frac{x^4}{4}\right)\frac{1}{x}\,dx = \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$$

(b) $\int x^n \ln x\,dx = \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C$

(c) Let $u = \ln x, du = \frac{1}{x}\,dx, dv = x^n\,dx, v = \frac{x^{n+1}}{n+1}$

$$\int x^n \ln x\,dx = (\ln x)\frac{x^{n+1}}{n+1} - \int \frac{x^{n+1}}{n+1}\left(\frac{1}{x}\,dx\right)$$

$$= \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C$$

66. (a) Arctangent formula, Formula 23,

$\int \frac{1}{u^2+1}\,du, u = e^x$

(b) Log Rule: $\int \frac{1}{u}\,du, u = e^x + 1$

(c) Substitution: $u = x^2, du = 2x\,dx$, then Formula 81

(d) Integration by parts

(e) Cannot be integrated.

(f) Formula 16 with $u = e^{2x}$

67. $W = \int_0^5 2000xe^{-x}\,dx$

$$= -2000\int_0^5 -xe^{-x}\,dx$$

$$= 2000\int_0^5 (-x)e^{-x}(-1)\,dx$$

$$= 2000\Big[(-x)e^{-x} - e^{-x}\Big]_0^5$$

$$= 2000\left(-\frac{6}{e^5} + 1\right)$$

$$\approx 1919.145 \text{ ft-lb}$$

68. $W = \int_0^5 \frac{500x}{\sqrt{26 - x^2}}\,dx$

$$= -250\int_0^5 \left(26 - x^2\right)^{-1/2}(-2x)\,dx$$

$$= \Big[-500\sqrt{26 - x^2}\Big]_0^5$$

$$= 500\left(\sqrt{26} - 1\right)$$

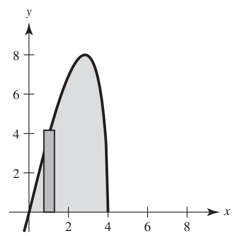
$$\approx 2049.51 \text{ ft-lb}$$

69. $$\frac{1}{2-0}\int_0^2 \frac{5000}{1+e^{4.8-1.9t}}\,dt = \frac{2500}{-1.9}\int_0^2 \frac{-1.9\,dt}{1+e^{4.8-1.9t}}$$
$$= -\frac{2500}{1.9}\Big[(4.8-1.9t) - \ln\big(1+e^{4.8-1.9t}\big)\Big]_0^2$$
$$= -\frac{2500}{1.9}\Big[\big(1-\ln(1+e)\big) - \big(4.8-\ln\big(1+e^{4.8}\big)\big)\Big]$$
$$= \frac{2500}{1.9}\left[3.8 + \ln\left(\frac{1+e}{1+e^{4.8}}\right)\right] \approx 401.4$$

70. (a) The slope of f at $x = -1$ is approximately 0.5 $(f' > 0$ at $x = -1)$.

(b) $f' > 0$ on $(-\infty, 0)$, so f is increasing on $(-\infty, 0)$.

$f' < 0$ on $(0, \infty)$, so f is decreasing on $(0, \infty)$.

71. 

$$V = 2\pi\int_0^4 x\left(x\sqrt{16-x^2}\right)dx$$
$$= 2\pi\int_0^4 x^2\sqrt{16-x^2}\,dx$$

By Formula 38: $(a = 4)$

$$V = 2\pi\left[\frac{1}{8}\left(x\left(2x^2-16\right)\sqrt{16-x^2} + 256\arcsin\left(\frac{x}{4}\right)\right)\right]_0^4$$
$$= 2\pi\left[32\left(\frac{\pi}{2}\right)\right] = 32\pi^2$$

72. (a) $$V = 20(2)\int_0^3 \frac{2}{\sqrt{1+y^2}}\,dy$$
$$= \left[80\ln\left|y+\sqrt{1+y^2}\right|\right]_0^3$$
$$= 80\ln\left(3+\sqrt{10}\right)$$
$$\approx 145.5 \text{ ft}^3$$
$$W = 148\left(80\ln\left(3+\sqrt{10}\right)\right)$$
$$= 11{,}840\ln\left(3+\sqrt{10}\right)$$
$$\approx 21{,}530.4 \text{ lb}$$

(b) By symmetry, $\bar{x} = 0$.

$$M = \rho(2)\int_0^3 \frac{2}{\sqrt{1+y^2}}\,dy$$
$$= \left[4\rho\ln\left|y+\sqrt{1+y^2}\right|\right]_0^3$$
$$= 4\rho\ln\left(3+\sqrt{10}\right)$$
$$M_x = 2\rho\int_0^3 \frac{2y}{\sqrt{1+y^2}}\,dy$$
$$= \left[4\rho\sqrt{1+y^2}\right]_0^3$$
$$= 4\rho\left(\sqrt{10}-1\right)$$
$$\bar{y} = \frac{M_x}{M} = \frac{4\rho\left(\sqrt{10}-1\right)}{4\rho\ln\left(3+\sqrt{10}\right)} \approx 1.19$$

Centroid: $(\bar{x}, \bar{y}) \approx (0, 1.19)$

73. Let $I = \displaystyle\int_0^{\pi/2} \frac{dx}{1+(\tan x)^{\sqrt{2}}}$.

For $x = \dfrac{\pi}{2} - u$, $dx = -du$, and

$$I = \int_{\pi/2}^0 \frac{-du}{1+\left(\tan(\pi/2-u)\right)^{\sqrt{2}}} = \int_0^{\pi/2}\frac{du}{1+(\cot u)^{\sqrt{2}}} = \int_0^{\pi/2}\frac{(\tan u)^{\sqrt{2}}}{(\tan u)^{\sqrt{2}}+1}\,du.$$

$$2I = \int_0^{\pi/2}\frac{dx}{1+(\tan x)^{\sqrt{2}}} + \int_0^{\pi/2}\frac{(\tan x)^{\sqrt{2}}}{(\tan x)^{\sqrt{2}}+1}\,dx = \int_0^{\pi/2} dx = \frac{\pi}{2}$$

So, $I = \dfrac{\pi}{4}$.

Section 8.8 Improper Integrals

1. An integral is improper if one or both of the limits of integration are infinite, or the function has a finite number of infinite discontinuities on the interval of integration.

2. An improper integral converges when the limit of the integral exists.

3. To evaluate the improper integral $\int_a^\infty f(x)\,dx$, find the limit as $b \to \infty$ when f is continuous on $[a^a, \infty)$ or find the limit as $a \to -\infty$ when f is continuous on $(-\infty, b]$.

4. (a) The function has a discontinuity at $x = -2$. So, the integral is improper on $[a, 5]$ for $a \le -2$. The integral is also improper when the lower limit of integration is infinite, that is, when $a = -\infty$.

(b) The function has a discontinuity at $x = \frac{1}{3}$. So, the integral is improper on $[a, 4]$ for $a \le \frac{1}{3}$. The integral is also improper when the lower limit of integration is infinite, that is, when $a = -\infty$.

5. $\int_0^1 \frac{dx}{5x-3}$ is improper because $5x - 3 = 0$ when $x = \frac{3}{5}$, and $0 \le \frac{3}{5} \le 1$.

6. $\int_1^2 \frac{dx}{x^3}$ is not improper because $f(x) = \frac{1}{x^3}$ is continuous on $[1, 2]$.

7. $\int_0^1 \frac{2x-5}{x^2-5x+6}\,dx = \int_0^1 \frac{2x-5}{(x-2)(x-3)}\,dx$ is not improper because $\frac{2x-5}{(x-2)(x-3)}$ is continuous on $[0, 1]$.

8. $\int_1^\infty \ln(x^2)\,dx$ is improper because the upper limit of integration is ∞.

9. $\int_0^2 e^{-x}\,dx$ is not improper because $f(x) = e^{-x}$ is continuous on $[0, 2]$.

10. $\int_0^\infty \cos x\,dx$ is improper because the upper limit of integration is ∞.

11. $\int_{-\infty}^\infty \frac{\sin x}{4+x^2}\,dx$ is improper because the limits of integration are $-\infty$ and ∞.

12. $\int_0^{\pi/4} \csc x\,dx$ is improper because $f(x) = \csc x$ is undefined at $x = 0$.

13. Infinite discontinuity at $x = 0$.

$$\begin{aligned}\int_0^4 \frac{1}{\sqrt{x}}\,dx &= \lim_{b\to 0^+}\int_b^4 \frac{1}{\sqrt{x}}\,dx\\ &= \lim_{b\to 0^+}\Big[2\sqrt{x}\Big]_b^4\\ &= \lim_{b\to 0^+}\left(4 - 2\sqrt{b}\right) = 4\end{aligned}$$

Converges

14. Infinite discontinuity at $x = 3$.

$$\begin{aligned}\int_3^4 \frac{1}{(x-3)^{3/2}}\,dx &= \lim_{b\to 3^+}\int_b^4 (x-3)^{-3/2}\,dx\\ &= \lim_{b\to 3^+}\Big[-2(x-3)^{-1/2}\Big]_b^4\\ &= \lim_{b\to 3^+}\left[-2 + \frac{2}{\sqrt{b-3}}\right] = \infty\end{aligned}$$

Diverges

15. Infinite discontinuity at $x = 1$.

$$\begin{aligned}\int_0^2 \frac{1}{(x-1)^2}\,dx &= \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx\\ &= \lim_{b\to 1^-}\int_0^b \frac{1}{(x-1)^2}\,dx + \lim_{c\to 1^+}\int_c^2 \frac{1}{(x-1)^2}\,dx\\ &= \lim_{b\to 1^-}\left[-\frac{1}{x-1}\right]_0^b + \lim_{c\to 1^+}\left[-\frac{1}{x-1}\right]_c^2\\ &= (\infty - 1) + (-1 + \infty)\end{aligned}$$

Diverges

16. Infinite limit of integration.

$$\begin{aligned}\int_{-\infty}^0 e^{3x}\,dx &= \lim_{b\to -\infty}\int_b^0 e^{3x}\,dx\\ &= \lim_{b\to -\infty}\left[\tfrac{1}{3}e^{3x}\right]_b^0\\ &= \lim_{b\to -\infty}\left[\tfrac{1}{3} - \tfrac{1}{3}e^{3b}\right] = \tfrac{1}{3}\end{aligned}$$

Converges

17. $\int_2^{\infty} \frac{1}{x^3}\,dx = \int_2^{\infty} x^{-3}\,dx$

$= \lim_{b\to\infty} \int_2^b x^{-3}\,dx$

$= \lim_{b\to\infty} \left[\frac{x^{-2}}{-2}\right]_2^b$

$= \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_2^b$

$= \lim_{b\to\infty} \left[-\frac{1}{2b^2} + \frac{1}{8}\right] = \frac{1}{8}$

18. $\int_3^{\infty} \frac{1}{(x-1)^4}\,dx = \lim_{b\to\infty} \int_3^b (x-1)^{-4}\,dx$

$= \lim_{b\to\infty} \left[\frac{-1}{3(x-1)^3}\right]_3^b$

$= \lim_{b\to\infty} \left[\frac{-1}{3(b-1)^3} + \frac{1}{3(8)}\right] = \frac{1}{24}$

19. $\int_1^{\infty} \frac{3}{\sqrt[3]{x}}\,dx = \lim_{b\to\infty} \int_1^b 3x^{-1/3}\,dx$

$= \lim_{b\to\infty} \left[\frac{9}{2}x^{2/3}\right]_1^b = \infty$

Diverges

20. $\int_1^{\infty} \frac{4}{\sqrt[4]{x}}\,dx = \lim_{b\to\infty} \int_1^b 4x^{-1/4}\,dx$

$= \lim_{b\to\infty} \left[\frac{16}{3}x^{3/4}\right]_1^b = \infty$ Diverges

21. $\int_0^{\infty} e^{x/3}\,dx = \lim_{b\to\infty} \int_0^b e^{x/3}\,dx$

$= \lim_{b\to\infty} \left[3e^{x/3}\right]_0^b$

$= \lim_{b\to\infty} \left(3e^{b/3} - 3\right)$

$= \infty$

Diverges

22. $\int_{-\infty}^{0} xe^{-4x}\,dx = \lim_{b\to-\infty} \int_b^0 xe^{-4x}\,dx$

$= \lim_{b\to-\infty} \left[\left(\frac{-x}{4} - \frac{1}{16}\right)e^{-4x}\right]_b^0$ (Integration by parts)

$= \lim_{b\to-\infty} \left[-\frac{1}{16} + \frac{b}{4} + \frac{1}{16}e^{-4b}\right] = -\infty$

Diverges

23. $\int_0^{\infty} x^2e^{-x}\,dx = \lim_{b\to\infty} \int_0^b x^2e^{-x}\,dx$

$= \lim_{b\to\infty} \left[-e^{-x}\left(x^2 + 2x + 2\right)\right]_0^b$

$= \lim_{b\to\infty} \left(-\frac{b^2 + 2b + 2}{e^b} + 2\right) = 2$

Because $\lim_{b\to\infty} \left(-\frac{b^2 + 2b + 2}{e^b}\right) = 0$ by L'Hôpital's Rule.

24. $\int_0^{\infty} e^{-x} \cos x\,dx = \lim_{b\to\infty} \frac{1}{2}\left[e^{-x}(-\cos x + \sin x)\right]_0^b$

$= \frac{1}{2}\left[0 - (-1)\right] = \frac{1}{2}$

25. $\int_4^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{b\to\infty} \int_4^b (\ln x)^{-3}\frac{1}{x}\,dx$

$= \lim_{b\to\infty} \left[-\frac{1}{2}(\ln x)^{-2}\right]_4^b$

$= \lim_{b\to\infty} \left[-\frac{1}{2}(\ln b)^{-2} + \frac{1}{2}(\ln 4)^{-2}\right]$

$= \frac{1}{2}\frac{1}{(2\ln 2)^2} = \frac{1}{2(\ln 4)^2}$

26. $\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x}\,dx$

$= \lim_{b\to\infty} \left[\frac{(\ln x)^2}{2}\right]_1^b = \infty$

Diverges

27. $$\int_{-\infty}^{\infty} \frac{4}{16 + x^2}\,dx = \int_{-\infty}^{0} \frac{4}{16 + x^2}\,dx + \int_{0}^{\infty} \frac{4}{16 + x^2}\,dx$$
$$= \lim_{b\to-\infty} \int_{b}^{0} \frac{4}{16 + x^2}\,dx + \lim_{c\to\infty} \int_{0}^{c} \frac{4}{16 + x^2}\,dx$$
$$= \lim_{b\to-\infty}\left[\arctan\left(\frac{x}{4}\right)\right]_b^0 + \lim_{c\to\infty}\left[\arctan\left(\frac{x}{4}\right)\right]_0^c$$
$$= \lim_{b\to-\infty}\left[0 - \arctan\left(\frac{b}{4}\right)\right] + \lim_{c\to\infty}\left[\arctan\left(\frac{c}{4}\right) - 0\right]$$
$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$$

28. $$\int_0^{\infty} \frac{x^3}{\left(x^2 + 1\right)^2}\,dx = \lim_{b\to\infty} \int_0^b \frac{x}{x^2 + 1}\,dx - \lim_{b\to\infty} \int_0^b \frac{x}{\left(x^2 + 1\right)^2}\,dx = \lim_{b\to\infty}\left[\frac{1}{2}\ln\left(x^2 + 1\right) + \frac{1}{2\left(x^2 + 1\right)}\right]_0^b = \infty - \frac{1}{2}$$

Diverges

29. $$\int_0^{\infty} \frac{1}{e^x + e^{-x}}\,dx = \lim_{b\to\infty} \int_0^b \frac{e^x}{1 + e^{2x}}\,dx$$
$$= \lim_{b\to\infty}\left[\arctan\left(e^x\right)\right]_0^b$$
$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

30. $$\int_0^{\infty} \frac{e^x}{1 + e^x}\,dx = \lim_{b\to\infty}\left[\ln\left(1 + e^x\right)\right]_0^b = \infty - \ln 2$$

Diverges

31. $$\int_0^{\infty} \cos \pi x\,dx = \lim_{b\to\infty}\left[\frac{1}{\pi}\sin \pi x\right]_0^b$$

Diverges because $\sin \pi b$ does not approach a limit as $b \to \infty$.

32. $$\int_0^{\infty} \sin\frac{x}{2}\,dx = \lim_{b\to\infty}\left[-2\cos\frac{x}{2}\right]_0^b$$

Diverges because $\cos\frac{x}{2}$ does not approach a limit as $x \to \infty$.

33. $$\int_0^1 \frac{1}{x^2}\,dx = \lim_{b\to0^+}\left[\frac{-1}{x}\right]_b^1 = \lim_{b\to0^+}\left(-1 + \frac{1}{b}\right) = -1 + \infty$$

Diverges

34. $$\int_0^5 \frac{10}{x}\,dx = \lim_{b\to0^+} \int_b^5 \frac{10}{x}\,dx$$
$$= \lim_{b\to0^+}\left[10\ln x\right]_b^5$$
$$= \lim_{b\to0^+}\left(10\ln 5 - 10\ln b\right) = \infty$$

Diverges

35. $$\int_0^2 \frac{1}{\sqrt[3]{x - 1}}\,dx = \int_0^1 \frac{1}{\sqrt[3]{x - 1}}\,dx + \int_1^2 \frac{1}{\sqrt[3]{x - 1}}\,dx = \lim_{b\to1^-}\left[\frac{3}{2}(x - 1)^{2/3}\right]_0^b + \lim_{c\to1^+}\left[\frac{3}{2}(x - 1)^{2/3}\right]_c^2 = \frac{-3}{2} + \frac{3}{2} = 0$$

36. $$\int_0^8 \frac{3}{\sqrt{8 - x}}\,dx = \lim_{b\to8^-} 3\int_0^b (8 - x)^{-1/2}\,dx$$
$$= \lim_{b\to8^-}\left[-6\sqrt{8 - x}\right]_0^b$$
$$= \lim_{b\to8^-}\left(-6\sqrt{8 - b} + 6\sqrt{8}\right)$$
$$= 12\sqrt{2}$$

37. $$\int_0^1 x\ln x\,dx = \lim_{b\to0^+}\left[\frac{x^2}{2}\ln|x| - \frac{x^2}{4}\right]_b^1$$
$$= \lim_{b\to0^+}\left(\frac{-1}{4} - \frac{b^2\ln b}{2} + \frac{b^2}{4}\right) = \frac{-1}{4}$$

because $\lim_{b\to0^+}\left(b^2\ln b\right) = 0$ by L'Hôpital's Rule.

38. $$\int_0^e \ln x^2\,dx = \lim_{b\to0^+} \int_0^e 2\ln x\,dx$$
$$= \lim_{b\to0^+}\left[2x\ln x - 2x\right]_b^e$$
$$= \lim_{b\to0^+}\left[(2e - 2e) - (2b\ln b - 2b)\right]$$
$$= 0$$

39. $$\int_0^{\pi/2} \tan\theta\,d\theta = \lim_{b\to(\pi/2)^-}\left[\ln\left|\sec\theta\right|\right]_0^b = \infty$$

Diverges

40. $\int_0^{\pi/2} \sec\theta\, d\theta = \lim_{b\to(\pi/2)^-}\Big[\ln|\sec\theta + \tan\theta|\Big]_0^b = \infty$

Diverges

41. $\int_2^4 \frac{2}{x\sqrt{x^2-4}}\,dx = \lim_{b\to 2^+}\int_b^4 \frac{2}{x\sqrt{x^2-4}}\,dx$

$= \lim_{b\to 2^+}\left[\operatorname{arcsec}\left|\frac{x}{2}\right|\right]_b^4$

$= \lim_{b\to 2^+}\left(\operatorname{arcsec} 2 - \operatorname{arcsec}\left(\frac{b}{2}\right)\right)$

$= \frac{\pi}{3} - 0 = \frac{\pi}{3}$

42. $\int_3^6 \frac{1}{\sqrt{36-x^2}}\,dx = \lim_{b\to 6^-}\int_3^b \frac{1}{\sqrt{36-x^2}}\,dx$

$= \lim_{b\to 6^-}\left[\arcsin\frac{x}{6}\right]_3^b$

$= \lim_{b\to 6^-}\left[\arcsin\frac{b}{6} - \arcsin\frac{1}{2}\right]$

$= \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

43. $\int_3^5 \frac{1}{\sqrt{x^2-9}}\,dx = \lim_{b\to 3^+}\left[\ln\left|x+\sqrt{x^2-9}\right|\right]_b^5$

$= \lim_{b\to 3^+}\left[\ln 9 - \ln\left(b+\sqrt{b^2-9}\right)\right]$

$= \ln 9 - \ln 3$

$= \ln\frac{9}{3} = \ln 3$

44. $\int_0^5 \frac{1}{25-x^2}\,dx = \lim_{b\to 5^-}\int_0^b \frac{1}{25-x^2}\,dx$

$= \lim_{b\to 5^-}\int_0^b \frac{1}{10}\left(\frac{1}{x+5} - \frac{1}{x-5}\right)dx$ (partial fractions)

$= \lim_{b\to 5^-}\left[\frac{1}{10}\ln\left|\frac{x+5}{x-5}\right|\right]_0^b$

$= \infty - 0$ Diverges

45. $\int_3^\infty \frac{1}{x\sqrt{x^2-9}}\,dx = \lim_{b\to 3^+}\int_b^5 \frac{1}{x\sqrt{x^2-9}}\,dx + \lim_{c\to\infty}\int_5^\infty \frac{1}{x\sqrt{x^2-9}}\,dx$

$= \lim_{b\to 3^+}\left[\frac{1}{3}\operatorname{arcsec}\frac{x}{3}\right]_b^5 + \lim_{c\to\infty}\left[\frac{1}{3}\operatorname{arcsec}\left(\frac{x}{3}\right)\right]_5^\infty$

$= \lim_{b\to 3^+}\left[\frac{1}{3}\operatorname{arcsec}\left(\frac{5}{3}\right) - \frac{1}{3}\operatorname{arcsec}\left(\frac{b}{3}\right)\right] + \lim_{c\to\infty}\left[\frac{1}{3}\operatorname{arcsec}\left(\frac{c}{3}\right) - \frac{1}{3}\operatorname{arcsec}\left(\frac{5}{3}\right)\right] = -0 + \frac{1}{3}\left(\frac{\pi}{2}\right) = \frac{\pi}{6}$

46. $\int_4^\infty \frac{\sqrt{x^2-16}}{x^2}\,dx = \lim_{b\to\infty}\int_4^b \frac{\sqrt{x^2-16}}{x^2}\,dx$

$= \lim_{b\to\infty}\left[\frac{-\sqrt{x^2-16}}{x} + \ln\left|x+\sqrt{x^2-16}\right|\right]_4^b$ (Formula 30)

$= \lim_{b\to\infty}\left[-\frac{\sqrt{b^2-16}}{b} + \ln\left|b+\sqrt{b^2-16}\right| - \ln 4\right] = \infty$

Diverges

47. $\int_0^{\infty} \frac{4}{\sqrt{x}(x+6)}\,dx = \int_0^1 \frac{4}{\sqrt{x}(x+6)}\,dx + \int_1^{\infty} \frac{4}{\sqrt{x}(x+6)}\,dx$

Let $u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$.

$$\int \frac{4}{\sqrt{x}(x+6)}\,dx = \int \frac{4(2u\,du)}{u(u^2+6)} = 8\int \frac{du}{u^2+6} = \frac{8}{\sqrt{6}}\arctan\left(\frac{u}{\sqrt{6}}\right) + C = \frac{8}{\sqrt{6}}\arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$

So, $\int_0^{\infty} \frac{4}{\sqrt{x}(x+6)}\,dx = \lim_{b\to 0^+}\left[\frac{8}{\sqrt{6}}\arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right)\right]_b^1 + \lim_{c\to\infty}\left[\frac{8}{\sqrt{6}}\arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right)\right]_1^c$

$$= \left[\frac{8}{\sqrt{6}}\arctan\left(\frac{1}{\sqrt{6}}\right) - \frac{8}{\sqrt{6}}(0)\right] + \left[\frac{8}{\sqrt{6}}\left(\frac{\pi}{2}\right) - \frac{8}{\sqrt{6}}\arctan\left(\frac{1}{\sqrt{6}}\right)\right] = \frac{8\pi}{2\sqrt{6}} = \frac{2\pi\sqrt{6}}{3}.$$

48. $\int \frac{1}{x\ln x}\,dx = \ln\left|\ln|x|\right| + C$

So,

$$\int_1^{\infty} \frac{1}{x\ln x}\,dx = \int_1^e \frac{1}{x\ln x}\,dx + \int_e^{\infty} \frac{1}{x\ln x}\,dx = \lim_{b\to 1^+}\left[\ln(\ln x)\right]_1^e + \lim_{c\to\infty}\left[\ln(\ln x)\right]_e^{\infty}.$$

Diverges

49. If $p = 1$, $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln x\right]_1^b$

$$= \lim_{b\to\infty}(\ln b) = \infty.$$

Diverges. For $p \neq 1$,

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^b = \lim_{b\to\infty}\left(\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\right).$$

This converges to $\frac{1}{p-1}$ if $1 - p < 0$ or $p > 1$.

50. If $p = 1$, $\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0^+}\left[\ln x\right]_a^1 = \lim_{a\to 0^+} -\ln a = \infty.$

Diverges. For $p \neq 1$,

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+}\left[\frac{x^{1-p}}{1-p}\right]_a^1 = \lim_{a\to 0^+}\left(\frac{1}{1-p} - \frac{a^{1-p}}{1-p}\right).$$

This converges to $\frac{1}{1-p}$ if $1 - p > 0$ or $p < 1$.

51. For $n = 1$:

$$\int_0^{\infty} xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx$$
$$= \lim_{b\to\infty}\left[-e^{-x}x - e^{-x}\right]_0^b \qquad (\text{Parts: } u = x,\ dv = e^{-x}\,dx)$$
$$= \lim_{b\to\infty}\left(-e^{-b}b - e^{-b} + 1\right)$$
$$= \lim_{b\to\infty}\left(\frac{-b}{e^b} - \frac{1}{e^b} + 1\right) = 1 \qquad (\text{L'Hôpital's Rule})$$

Assume that $\int_0^{\infty} x^n e^{-x}\,dx$ converges. Then for $n + 1$ you have

$$\int x^{n+1}e^{-x}\,dx = -x^{n+1}e^{-x} + (n+1)\int x^n e^{-x}\,dx$$

by parts $\left(u = x^{n+1},\ du = (n+1)x^n\,dx,\ dv = e^{-x}\,dx,\ v = -e^{-x}\right)$.

So,

$$\int_0^{\infty} x^{n+1}e^{-x}\,dx = \lim_{b\to\infty}\left[-x^{n+1}e^{-x}\right]_0^b + (n+1)\int_0^{\infty} x^n e^{-x}\,dx = 0 + (n+1)\int_0^{\infty} x^n e^{-x}\,dx, \text{ which converges.}$$

52. (a) $\int_1^{\infty} e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-b}\right]_1^b = 1$

Because $e^{-x^2} \le e^{-x}$ on $[1, \infty)$ and

$\int_1^{\infty} e^{-x}\,dx$

converges, then so does

$\int_1^{\infty} e^{-x^2}\,dx.$

(b) $\int_1^{\infty} \frac{1}{x^5}\,dx$ converges (see Exercise 49).

Because $\frac{1}{x^5 + 1} < \frac{1}{x^5}$ on $[1, \infty)$, then $\int_1^{\infty} \frac{1}{x^5 + 1}\,dx$ also converges.

53. $\int_0^1 \frac{1}{\sqrt[6]{x}}\,dx = \int_0^1 \frac{1}{x^{1/6}}\,dx$ converges by Exercise 50. $\left(p = \frac{1}{6}\right)$

54. $\int_0^1 \frac{1}{x^9}\,dx$ diverges by Exercise 50. $(p = 9)$.

55. $\int_1^{\infty} \frac{1}{x^5}\,dx$ converges by Exercise 49. $(p = 5)$

56. $\int_0^{\infty} x^4 e^{-x}\,dx$ converges by Exercise 51. $(n = 4)$

57. Because $\frac{1}{x^2 + 5} \le \frac{1}{x^2}$ on $[1, \infty)$ and $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges by Exercise 49, $\int_1^{\infty} \frac{1}{x^2 + 5}\,dx$ converges.

58. Because $\frac{1}{\sqrt{x - 1}} \ge \frac{1}{x}$ on $[2, \infty)$ and $\int_2^{\infty} \frac{1}{x}\,dx$ diverges by Exercise 55, $\int_2^{\infty} \frac{1}{\sqrt{x - 1}}\,dx$ diverges.

59. $\int_1^{\infty} \frac{2}{x^2}\,dx$ converges, and $\frac{1 - \sin x}{x^2} \le \frac{2}{x^2}$ on $[1, \infty)$, so $\int_1^{\infty} \frac{1 - \sin x}{x^2}\,dx$ converges.

60. $\int_0^{\infty} \frac{1}{e^x}\,dx = \int_0^{\infty} e^{-x}\,dx$ converges, and $\frac{1}{e^x} \ge \frac{1}{e^x + x}$ on $[0, \infty)$, so $\int_0^{\infty} \frac{1}{e^x + x}\,dx$ converges.

61. $\int_{-1}^{1} \frac{1}{x^3}\,dx = \int_{-1}^{0} \frac{1}{x^3}\,dx + \int_0^1 \frac{1}{x^3}\,dx$

These two integrals diverge by Exercise 50.

62. $\frac{10}{x^2 - 2x} = \frac{10}{x(x - 2)} \Rightarrow x = 0, 2.$

You must analyze three improper integrals, and each must converge in order for the original integral to converge.

$$\int_0^3 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

63.
$$\begin{aligned}
A &= \int_{-\infty}^{-1} -\frac{7}{(x - 1)^3}\,dx \\
&= \lim_{b\to-\infty}\int_b^{-1} -7(x - 1)^{-3}\,dx \\
&= \lim_{b\to-\infty}\left[\frac{7}{2}(x - 1)^{-2}\right]_b^{-1} \\
&= \lim_{b\to-\infty}\left[\frac{7}{2(-2)^2} - \frac{7}{2(b - 1)^2}\right] \\
&= \frac{7}{8} - 0 = \frac{7}{8}
\end{aligned}$$

64.
$$\begin{aligned}
A &= \int_0^1 -\ln x\,dx \\
&= -\lim_{b\to0^+}\int_b^1 \ln x\,dx \\
&= -\lim_{b\to0^+}\left[x\ln x - x\right]_b^1 \\
&= -\lim_{b\to0^+}\left[(0 - 1) - b\ln b + b\right] \\
&= 1
\end{aligned}$$

Note: $\lim_{b\to0^+} b\ln b = \lim_{b\to0^+}\frac{\ln b}{1/b} = \lim_{b\to0^+}\frac{1/b}{-1/b^2} = 0$

65. $A = \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$

$= \lim_{b\to-\infty} \int_b^0 \frac{1}{x^2+1}\,dx + \lim_{b\to\infty}\int_0^b \frac{1}{x^2+1}\,dx$

$= \lim_{b\to-\infty} \left[\arctan(x)\right]_b^0 + \lim_{b\to\infty}\left[\arctan(x)\right]_0^b$

$= \lim_{b\to-\infty} \left[0 - \arctan(b)\right] + \lim_{b\to\infty}\left[\arctan(b) - 0\right]$

$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$

66. $A = \int_{-\infty}^{\infty} \frac{8}{x^2+4}\,dx$

$= \lim_{b\to-\infty} \int_b^0 \frac{8}{x^2+4}\,dx + \lim_{b\to\infty}\int_0^b \frac{8}{x^2+4}\,dx$

$= \lim_{b\to-\infty} \left[4\arctan\left(\frac{x}{2}\right)\right]_b^0 + \lim_{b\to\infty}\left[4\arctan\left(\frac{x}{2}\right)\right]_0^b$

$= \lim_{b\to-\infty} \left[0 - 4\arctan\left(\frac{b}{2}\right)\right] + \lim_{b\to\infty}\left[4\arctan\left(\frac{b}{2}\right) - 0\right]$

$= -4\left(\frac{-\pi}{2}\right) + 4\left(\frac{\pi}{2}\right) = 4\pi$

67. (a) $A = \int_0^{\infty} e^{-x}\,dx$

$= \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = 0 - (-1) = 1$

(b) **Disk:**

$V = \pi\int_0^{\infty}\left(e^{-x}\right)^2\,dx$

$= \lim_{b\to\infty}\pi\left[-\frac{1}{2}e^{-2x}\right]_0^b = \frac{\pi}{2}$

(c) **Shell:**

$V = 2\pi\int_0^{\infty} xe^{-x}\,dx$

$= \lim_{b\to\infty} 2\pi\left[-e^{-x}(x+1)\right]_0^b = 2\pi$

68. (a) $A = \int_1^{\infty}\frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{\infty} = 1$

(b) **Disk:**

$V = \pi\int_1^{\infty}\frac{1}{x^4}\,dx = \lim_{b\to\infty}\left[-\frac{\pi}{3x^3}\right]_1^b = \frac{\pi}{3}$

(c) **Shell:**

$V = 2\pi\int_1^{\infty} x\left(\frac{1}{x^2}\right)dx = \lim_{b\to\infty}\left[2\pi(\ln x)\right]_1^b = \infty$

Diverges

69. $y = \sqrt{16 - x^2}, \quad 0 \le x \le 4$

$y' = \frac{-x}{\sqrt{16-x^2}}$

$s = \int_0^4 \sqrt{1 + \frac{x^2}{16-x^2}}\,dx = \int_0^4 \frac{4}{\sqrt{16-x^2}}\,dx = \lim_{t\to4^-}\int_0^t \frac{4}{\sqrt{16-x^2}}\,dx = \lim_{t\to4^-}\left[4\arcsin\left(\frac{x}{4}\right)\right]_0^t = \lim_{t\to4^-} 4\arcsin\left(\frac{t}{4}\right) = 2\pi$

70. $y = 2e^{-x}$

$y' = -2e^{-x}$

$S = 2\pi\int_0^{\infty}\left(2e^{-x}\right)\sqrt{1+4e^{-2x}}\,dx$

Let $u = e^{-x}$, $du = -e^{-x}\,dx$.

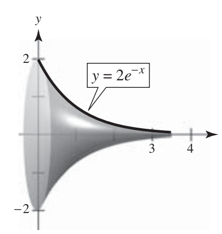
$\int e^{-x}\sqrt{1+4e^{-2x}}\,dx = -\int\sqrt{1+4u^2}\,du$

$= -\frac{1}{4}\left[2u\sqrt{4u^2+1} + \ln\left|2u + \sqrt{4u^2+1}\right|\right] + C$

$= -\frac{1}{4}\left[2e^{-x}\sqrt{4e^{-2x}+1} + \ln\left|2e^{-x} + \sqrt{4e^{-2x}+1}\right|\right] + C$

$S = 4\pi\lim_{b\to\infty}\int_0^b\left(e^{-x}\right)\sqrt{1+4e^{-2x}}\,dx$

$= -\pi\lim_{b\to\infty}\left[2e^{-x}\sqrt{4e^{-2x}+1} + \ln\left|2e^{-x} + \sqrt{4e^{-2x}+1}\right|\right]_0^b = \pi\left[2\sqrt{5} + \ln\left(2+\sqrt{5}\right)\right] \approx 18.5849$



71. (a) $F(x) = \dfrac{K}{x^2}, 5 = \dfrac{K}{(4000)^2}, K = 80{,}000{,}000$

$$W = \int_{4000}^{\infty} \frac{80{,}000{,}000}{x^2}\,dx = \lim_{b\to\infty}\left[\frac{-80{,}000{,}000}{x}\right]_{4000}^{b} = 20{,}000 \text{ mi-ton}$$

(b) $\dfrac{W}{2} = 10{,}000 = \left[\dfrac{-80{,}000{,}000}{x}\right]_{4000}^{b} = \dfrac{-80{,}000{,}000}{b} + 20{,}000$

$$\frac{80{,}000{,}000}{b} = 10{,}000$$

$$b = 8000$$

Therefore, the rocket has traveled 4000 miles above Earth's surface.

72. (a) $F(x) = \dfrac{k}{x^2}, 10 = \dfrac{k}{4000^2}, k = 10(4000^2)$

$$W = \int_{4000}^{\infty} \frac{10(4000^2)}{x^2}\,dx = \lim_{b\to\infty}\left[\frac{-10(4000^2)}{x}\right]_{4000}^{b}$$

$$= \frac{10(4000^2)}{4000} = 40{,}000 \text{ mi-ton}$$

(b) $\dfrac{W}{2} = 20{,}000 = \left[\dfrac{-10(4000^2)}{x}\right]_{4000}^{b}$

$$= \frac{-10(4000^2)}{b} + 40{,}000$$

$$\frac{10(4000^2)}{b} = 20{,}000$$

$$b = 8000$$

Therefore, the rocket has traveled 4000 miles above Earth's surface.

73. (a) $\displaystyle\int_{-\infty}^{\infty} f(t)\,dt = \int_0^{\infty} \frac{1}{9}e^{-t/9}\,dt$

$$= \lim_{b\to\infty}\left[-e^{-t/9}\right]_0^b = 1$$

(b) $\displaystyle P(0 \le x \le 6) = \int_0^6 \frac{1}{9}e^{-t/9}\,dt$

$$= \left[-e^{-t/9}\right]_0^6$$

$$= -e^{-2/3} + 1$$

$$\approx 0.4866 = 48.66\%$$

74. (a) $\displaystyle\int_{-\infty}^{\infty} f(t)\,dt = \int_0^{\infty} \frac{5}{6}e^{-5t/6}\,dt$

$$= \lim_{b\to\infty}\left[-e^{-5t/6}\right]_0^b = 1$$

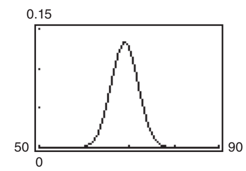
(b) $\displaystyle P(0 \le x \le 6) = \int_0^6 \frac{5}{6}e^{-5t/6}\,dt$

$$= \left[-e^{-5t/6}\right]_0^6$$

$$= 1 - e^{-5}$$

$$\approx 0.9933 = 99.33\%$$

75. (a)



Using a graphing utility, the area under the curve is approximately 1.

(b) $\displaystyle P(72 \le x \le \infty) = \int_{72}^{\infty} f(x)\,dx \approx 0.1587$

(c) $\displaystyle 0.5 - P(69 \le x \le 72) = 0.5 - \int_{69}^{72} f(x)\,dx$

$$= 0.5 - 0.3413 = 0.1587$$

The answers are the same by symmetry:

$$0.5 = \int_{69}^{\infty} f(x)\,dx = \int_{69}^{72} f(x)\,dx + \int_{72}^{\infty} f(x)\,dx.$$

76. (a) The area under the curve is greater on the interval $26 \le x \le 28$ than on the interval $22 \le x \le 24$. So, the probability is greater for choosing a car getting between 26 and 28 miles per gallon.

(b) The area under the curve is greater on the interval $x \ge 30$ than on the interval $20 \le x \le 22$. So, the probability is greater for choosing a car getting at least 30 miles per gallon.

77. (a) $C = 700{,}000 + \int_0^5 25{,}000e^{-0.06t}\,dt$

$$= 700{,}000 - \left[\frac{25{,}000}{0.06}e^{-0.06t}\right]_0^5 \approx \$807{,}992.41$$

(b) $C = 700{,}000 + \int_0^{10} 25{,}000e^{-0.06t}\,dt = 700{,}000 - \left[\frac{25{,}000}{0.06}e^{-0.06t}\right]_0^{10} \approx \$887{,}995.15$

(c) $C = 700{,}000 + \int_0^{\infty} 25{,}000e^{-0.06t}\,dt$

$$= 700{,}000 - \lim_{b\to\infty}\left[\frac{25{,}000}{0.06}e^{-0.06t}\right]_0^b$$

$$\approx \$1{,}116{,}666.67$$

78. (a) $C = 700{,}000 + \int_0^5 25{,}000(1 + 0.08t)\,e^{-0.06t}\,dt$

$$= 700{,}000 + 25{,}000\left[-\frac{1}{0.06}e^{-0.06t} - 0.08\left(\frac{t}{0.06}e^{-0.06t} + \frac{1}{(0.06)^2}e^{-0.06t}\right)\right]_0^5$$

$$\approx \$828{,}512.58$$

(b) $C = 700{,}000 + \int_0^{10} 25{,}000(1 + 0.08t)e^{-0.06t}\,dt$

$$= 700{,}00 + 25{,}000\left[-\frac{1}{0.06}e^{-0.06t} - 0.08\left(\frac{t}{0.06}e^{-0.06t} + \frac{1}{(0.06)^2}e^{-0.06t}\right)\right]_{\infty}^{10}$$

$$\approx \$955{,}718.14$$

(c) $C = 700{,}000 + \int_0^{\infty} 25{,}000(1 + 0.08t)e^{-0.06t}\,dt$

$$= 700{,}000 + 25{,}000\lim_{b\to\infty}\left[-\frac{1}{0.06}e^{-0.06t} - 0.08\left(\frac{t}{0.06}e^{-0.06t} + \frac{1}{(0.06)^2}e^{-0.06t}\right)\right]_0^b$$

$$\approx \$1{,}672{,}222.22$$

79. Let $K = \frac{2\pi NI\,r}{k}$. Then

$$P = K\int_c^{\infty}\frac{1}{(r^2 + x^2)^{3/2}}\,dx.$$

Let $x = r\tan\theta$, $dx = r\sec^2\theta\,d\theta$, $\sqrt{r^2 + x^2} = r\sec\theta$.

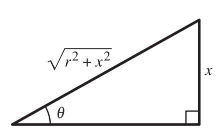
$$\int\frac{1}{(r^2 + x^2)^{3/2}}\,dx = \int\frac{r\sec^2\theta\,d\theta}{r^3\sec^3\theta} = \frac{1}{r^2}\int\cos\theta\,d\theta$$

$$= \frac{1}{r^2}\sin\theta + C = \frac{1}{r^2}\frac{x}{\sqrt{r^2 + x^2}} + C$$

So,

$$P = K\frac{1}{r^2}\lim_{b\to\infty}\left[\frac{x}{\sqrt{r^2 + x^2}}\right]_c^b = \frac{K}{r^2}\left[1 - \frac{c}{\sqrt{r^2 + c^2}}\right]$$

$$= \frac{K\left(\sqrt{r^2 + c^2} - c\right)}{r^2\sqrt{r^2 + c^2}} = \frac{2\pi NI\left(\sqrt{r^2 + c^2} - c\right)}{kr\sqrt{r^2 + c^2}}.$$



80. $F = \int_0^\infty \frac{GM\delta}{(a + x)^2}\,dx$

$= \lim_{b\to\infty}\left[\frac{-GM\delta}{a + x}\right]_0^b$

$= \frac{GM\delta}{a}$

81. False. $f(x) = 1/(x + 1)$ is continuous on $[0, \infty)$, $\lim_{x\to\infty} 1/(x + 1) = 0$, but

$\int_0^\infty \frac{1}{x + 1}\,dx = \lim_{b\to\infty}\left[\ln|x + 1|\right]_0^b = \infty.$

Diverges

82. False. This is equivalent to Exercise 81.

83. True

84. True

85. True

86. False. For example, let $f(x) = \frac{1}{x}$.

Then $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{1}{x} = 0 = L.$

However, $\int_0^\infty f(x)\,dx = \int_0^\infty \frac{1}{x}\,dx$ diverges.

87. (a) $\int_{-\infty}^\infty \sin x\,dx = \int_{-\infty}^0 \sin x\,dx + \int_0^\infty \sin x\,dx$

$= \lim_{b\to-\infty}\int_b^0 \sin x\,dx + \lim_{c\to\infty}\int_0^c \sin x\,dx$

$= \lim_{b\to-\infty}\left[-\cos x\right]_b^0 + \lim_{c\to\infty}\left[-\cos x\right]_0^c$

Because $\lim_{b\to-\infty}[-\cos b]$ diverges, as does $\lim_{c\to\infty}[-\cos c]$,

$\int_{-\infty}^\infty \sin x\,dx$ diverges.

(b) $\lim_{a\to\infty}\int_{-a}^a \sin x\,dx = \lim_{a\to\infty}\left[-\cos x\right]_{-a}^a$

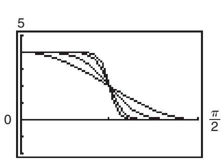
$= \lim_{a\to\infty}\left[-\cos(a) + \cos(-a)\right] = 0$

(c) The definition of $\int_{-\infty}^\infty f(x)\,dx$ is not

$\lim_{a\to\infty}\int_{-a}^a f(x)\,dx.$

88. (a) Yes, the integrand is not defined at $x = \pi/2$.

(b)



(c) As $n \to \infty$, the integral approaches $4(\pi/4) = \pi$.

(d) $I_n = \int_0^{\pi/2}\frac{4}{1 + (\tan x)^n}\,dx$

$I_2 \approx 3.14159$

$I_4 \approx 3.14159$

$I_8 \approx 3.14159$

$I_{12} \approx 3.14159$

89. You know that $\int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx$

Therefore,

$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$

$\le \lim_{b\to\infty}\int_a^b |f(x)|\,dx$, which coverges.

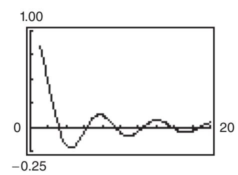
So, $\int_a^\infty f(x)\,dx$ converges.

90. (a) $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln|x|\right]_1^b = \infty$

$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = 1$

$\int_1^\infty \frac{1}{x^n}\,dx$ will converge if $n > 1$ and will diverge if $n \le 1$.

(b) It would appear to converge.



(c) Let $dv = \sin x\,dx \Rightarrow v = -\cos x$

$u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2}\,dx.$

$\int_1^\infty \frac{\sin x}{x}\,dx = \lim_{b\to 0}\left[-\frac{\cos x}{x}\right]_1^b - \int_1^\infty \frac{\cos x}{x^2}\,dx$

$= \cos 1 - \int_1^\infty \frac{\cos x}{x^2}\,dx$

Because $\int_1^\infty \frac{\cos x}{x^2}\,dx \le \int_1^\infty \frac{|\cos x|}{x^2}\,dx \le \int_1^\infty \frac{1}{x^2}\,dx$,

and $\int_1^\infty \frac{1}{x^2}\,dx$ converges, then Exercise 89 implies that $\int_1^\infty \frac{\cos x}{x^2}\,dx$ converges. Finally, you see that $\int_1^\infty \frac{\sin x}{x}\,dx$ converges.

91. $f(t) = 1$

$F(s) = \int_0^\infty e^{-st}\,dt = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^b = \frac{1}{s},\ s > 0$

92. $f(t) = t$

$$F(s) = \int_0^\infty te^{-st}\,dt = \lim_{b\to\infty}\left[\frac{1}{s^2}(-st - 1)e^{-st}\right]_0^b$$
$$= \frac{1}{s^2},\ s > 0$$

93. $f(t) = t^2$

$$F(s) = \int_0^\infty t^2e^{-st}\,dt = \lim_{b\to\infty}\left[\frac{1}{s^3}\left(-s^2t^2 - 2st - 2\right)e^{-st}\right]_0^b$$
$$= \frac{2}{s^3},\ s > 0$$

94. $f(t) = e^{at}$

$$F(s) = \int_0^\infty e^{at}e^{-st}\,dt = \int_0^\infty e^{t(a-s)}\,dt$$
$$= \lim_{b\to\infty}\left[\frac{1}{a-s}e^{t(a-s)}\right]_0^b$$
$$= 0 - \frac{1}{a-s} = \frac{1}{s-a},\ s > a$$

95. $f(t) = \cos at$

$$F(s) = \int_0^\infty e^{-st}\cos at\,dt$$
$$= \lim_{b\to\infty}\left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^b$$
$$= 0 + \frac{s}{s^2+a^2} = \frac{s}{s^2+a^2},\ s > 0$$

96. $f(t) = \sin at$

$$F(s) = \int_0^\infty e^{-st}\sin at\,dt$$
$$= \lim_{b\to\infty}\left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^b$$
$$= 0 + \frac{a}{s^2+a^2} = \frac{a}{s^2+a^2},\ s > 0$$

97. $f(t) = \cosh at$

$$F(s) = \int_0^\infty e^{-st}\cosh at\,dt = \int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right)dt = \frac{1}{2}\int_0^\infty\left[e^{t(-s+a)} + e^{t(-s-a)}\right]dt$$
$$= \lim_{b\to\infty}\frac{1}{2}\left[\frac{1}{(-s+a)}e^{t(-s+a)} + \frac{1}{(-s-a)}e^{t(-s-a)}\right]_0^b = 0 - \frac{1}{2}\left[\frac{1}{(-s+a)} + \frac{1}{(-s-a)}\right]$$
$$= \frac{-1}{2}\left[\frac{1}{(-s+a)} + \frac{1}{(-s-a)}\right] = \frac{s}{s^2-a^2},\ s > |a|$$

98. $f(t) = \sinh at$

$$F(s) = \int_0^\infty e^{-st}\sinh at\,dt = \int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right)dt = \frac{1}{2}\int_0^\infty\left[e^{t(-s+a)} - e^{t(-s-a)}\right]dt$$
$$= \lim_{b\to\infty}\frac{1}{2}\left[\frac{1}{(-s+a)}e^{t(-s+a)} - \frac{1}{(-s-a)}e^{t(-s-a)}\right]_0^b = 0 - \frac{1}{2}\left[\frac{1}{(-s+a)} - \frac{1}{(-s-a)}\right]$$
$$= \frac{-1}{2}\left[\frac{1}{(-s+a)} - \frac{1}{(-s-a)}\right] = \frac{a}{s^2-a^2},\ s > |a|$$

99. $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx$

(a) $\Gamma(1) = \int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = 1$

$\Gamma(2) = \int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}(x+1)\right]_0^b = 1$

$\Gamma(3) = \int_0^\infty x^2e^{-x}\,dx = \lim_{b\to\infty}\left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^b = 2$

(b) $\Gamma(n+1) = \int_0^\infty x^ne^{-x}\,dx = \lim_{b\to\infty}\left[-x^ne^{-x}\right]_0^b + \lim_{b\to\infty}n\int_0^b x^{n-1}e^{-x}\,dx = 0 + n\Gamma(n) \qquad \left(u = x^n,\ dv = e^{-x}\,dx\right)$

(c) $\Gamma(n) = (n-1)!$

100. For $n = 1$,

$$I_1 = \int_0^\infty \frac{x}{(x^2+1)^4}\,dx = \lim_{b\to\infty} \frac{1}{2}\int_0^b (x^2+1)^{-4}(2x\,dx) = \lim_{b\to\infty}\left[-\frac{1}{6}\cdot\frac{1}{(x^2+1)^3}\right]_0^b = \frac{1}{6}.$$

For $n > 1$,

$$I_n = \int_0^\infty \frac{x^{2n-1}}{(x^2+1)^{n+3}}\,dx = \lim_{b\to\infty}\left[\frac{-x^{2n-2}}{2(n+2)(x^2+1)^{n+2}}\right]_0^b + \frac{n-1}{n+2}\int_0^\infty \frac{x^{2n-3}}{(x^2+1)^{n+2}}\,dx = 0 + \frac{n-1}{n+2}(I_{n-1})$$

$$\left(\text{Parts: } u = x^{2n-2},\ du = (2n-2)x^{2n-3}\,dx,\ dv = \frac{x}{(x^2+1)^{n+3}}\,dx,\ v = \frac{-1}{2(n+2)(x^2+1)^{n+2}}\right)$$

(a) $\displaystyle\int_0^\infty \frac{x}{(x^2+1)^4}\,dx = \lim_{b\to\infty}\left[-\frac{1}{6(x^2+1)^3}\right]_0^b = \frac{1}{6}$

(b) $\displaystyle\int_0^\infty \frac{x^3}{(x^2+1)^5}\,dx = \frac{1}{4}\int_0^\infty \frac{x}{(x^2+1)^4}\,dx = \frac{1}{4}\left(\frac{1}{6}\right) = \frac{1}{24}$

(c) $\displaystyle\int_0^\infty \frac{x^5}{(x^2+1)^6} = \frac{2}{5}\int_0^\infty \frac{x^3}{(x^2+1)^5}\,dx = \frac{2}{5}\left(\frac{1}{24}\right) = \frac{1}{60}$

101. $$\int_0^\infty \left(\frac{1}{\sqrt{x^2+1}} - \frac{c}{x+1}\right)dx = \lim_{b\to\infty}\int_0^b \left(\frac{1}{\sqrt{x^2+1}} - \frac{c}{x+1}\right)dx$$
$$= \lim_{b\to\infty}\left[\ln\left|x+\sqrt{x^2+1}\right| - c\ln|x+1|\right]_0^b$$
$$= \lim_{b\to\infty}\left[\ln\left(b+\sqrt{b^2+1}\right) - \ln(b+1)^c\right] = \lim_{b\to\infty}\ln\left[\frac{b+\sqrt{b^2+1}}{(b+1)^c}\right]$$

This limit exists for $c = 1$, and you have

$$\lim_{b\to\infty}\ln\left[\frac{b+\sqrt{b^2+1}}{(b+1)}\right] = \ln 2.$$

102. $$\int_1^\infty \left(\frac{cx}{x^2+2} - \frac{1}{3x}\right)dx = \lim_{b\to\infty}\int_1^b \left(\frac{cx}{x^2+2} - \frac{1}{3x}\right)dx$$
$$= \lim_{b\to\infty}\left[\frac{c}{2}\ln(x^2+2) - \frac{1}{3}\ln|x|\right]_1^b$$
$$= \lim_{b\to\infty}\ln\left[\frac{(x^2+2)^{c/2}}{x^{1/3}}\right]_1^b$$
$$= \lim_{b\to\infty}\left[\ln\frac{(b^2+2)^{c/2}}{b^{1/3}} - \ln 3^{c/2}\right]$$

This limit exists if $c = 1/3$, and you have

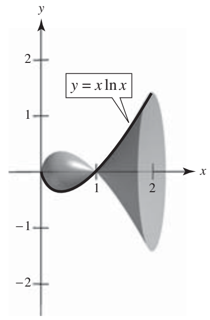
$$\lim_{b\to\infty}\left[\ln\frac{(b^2+2)^{1/6}}{b^{1/3}} - \ln 3^{1/6}\right] = -\ln 3^{1/6} = \frac{-\ln 3}{6}.$$

103. $f(x) = \begin{cases} x \ln x, & 0 < x \le 2 \\ 0, & x = 0 \end{cases}$

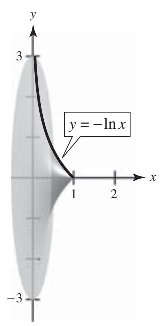
$V = \pi \int_0^2 (x \ln x)^2 \, dx$

Let $u = \ln x, e^u = x, e^u \, du = dx.$

$$\begin{aligned} V &= \pi \int_{-\infty}^{\ln 2} e^{2u} u^2 (e^4 \, du) \\ &= \pi \int_{-\infty}^{\ln 2} e^{3u} u^2 \, du \\ &= \lim_{b \to -\infty} \left[\pi \left[\frac{u^2}{3} - \frac{2u}{9} + \frac{2}{27} \right] e^{3u} \right]_b^{\ln 2} \\ &= 8\pi \left[\frac{(\ln 2)^2}{3} - \frac{2 \ln 2}{9} + \frac{2}{27} \right] \approx 2.0155 \end{aligned}$$



104. $$\begin{aligned} V &= \pi \int_0^1 (-\ln x)^2 \, dx \\ &= \lim_{b \to 0^+} \pi \int_b^1 (\ln x)^2 \, dx \\ &= \lim_{b \to 0^+} \pi x \left[(\ln x)^2 - 2 \ln x + 2 \right]_b^1 \\ &= \lim_{b \to 0^+} \pi \left[2 - b(\ln b)^2 - 2b \ln b - 2b \right] \\ &= 2\pi \end{aligned}$$



105. $u = \sqrt{x}, u^2 = x, 2u \, du = dx$

$$\int_0^1 \frac{\sin x}{\sqrt{x}} \, dx = \int_0^1 \frac{\sin(u^2)}{u} (2u \, du) = \int_0^1 2 \sin(u^2) \, du$$

Trapezoidal Rule $(n = 5)$: 0.6278

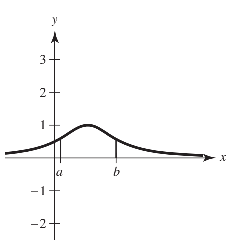
106. $u = \sqrt{1 - x}, 1 - x = u^2, 2u \, du = -dx$

$$\begin{aligned} \int_0^1 \frac{\cos x}{\sqrt{1 - x}} \, dx &= \int_1^0 \frac{\cos(1 - u^2)}{u} (-2u \, du) \\ &= \int_0^1 2 \cos(1 - u^2) \, du \end{aligned}$$

Trapezoidal Rule $(n = 5)$: 1.4997

107. Assume $a < b$. The proof is similar if $a > b$.

$$\begin{aligned} \int_{-\infty}^a f(x) \, dx + \int_a^\infty f(x) \, dx &= \lim_{c \to -\infty} \int_c^a f(x) \, dx + \lim_{d \to \infty} \int_a^d f(x) \, dx \\ &= \lim_{c \to -\infty} \int_c^a f(x) \, dx + \lim_{d \to \infty} \left[\int_a^b f(x) \, dx + \int_b^d f(x) \, dx \right] \\ &= \lim_{c \to -\infty} \int_c^a f(x) \, dx + \int_a^b f(x) \, dx + \lim_{d \to \infty} \int_b^d f(x) \, dx \\ &= \lim_{c \to -\infty} \left[\int_c^a f(x) \, dx + \int_a^b f(x) \, dx \right] + \lim_{d \to \infty} \int_b^d f(x) \, dx \\ &= \lim_{c \to -\infty} \int_c^b f(x) \, dx + \lim_{d \to \infty} \int_b^d f(x) \, dx \\ &= \int_{-\infty}^b f(x) \, dx + \int_b^\infty f(x) \, dx \end{aligned}$$



Review Exercises for Chapter 8

1. $\displaystyle\int x^2\sqrt{x^3-27}\,dx = \frac{1}{3}\int\left(x^3-27\right)^{1/2}3x^2\,dx$

$\displaystyle= \frac{1}{3}\frac{\left(x^3-27\right)^{3/2}}{(3/2)} + C$

$\displaystyle= \frac{2}{9}\left(x^3-27\right)^{3/2} + C$

2. $\displaystyle\int xe^{5-x^2}\,dx = -\frac{1}{2}\int e^{5-x^2}(-2x\,dx)$

$\displaystyle= -\frac{1}{2}e^{5-x^2} + C$

3. $\displaystyle\int \csc^2\left(\frac{x+8}{4}\right)dx = 4\int\csc^2\left(\frac{x+8}{4}\right)\frac{1}{4}\,dx$

$\displaystyle= -4\cot\left(\frac{x+8}{4}\right) + C$

4. $\displaystyle\int\frac{x}{\sqrt[3]{4-x^2}}\,dx = -\frac{1}{2}\int\left(4-x^2\right)^{-1/3}(-2x)\,dx$

$\displaystyle= -\frac{1}{2}\frac{\left(4-x^2\right)^{2/3}}{(2/3)} + C$

$\displaystyle= -\frac{3}{4}\left(4-x^2\right)^{2/3} + C$

5. Let $u = \ln(2x)$, $du = \dfrac{1}{x}\,dx$.

$\displaystyle\int_1^e\frac{\ln(2x)}{x}\,dx = \int_{\ln 2}^{1+\ln 2}u\,du$

$\displaystyle= \left[\frac{u^2}{2}\right]_{\ln 2}^{1+\ln 2}$

$\displaystyle= \frac{1}{2}\left[1 + 2\ln 2 + (\ln 2)^2 - (\ln 2)^2\right]$

$\displaystyle= \frac{1}{2} + \ln 2 \approx 1.1931$

6. Let $u = 2x-3$, $du = 2\,dx$, $x = \dfrac{1}{2}(u+3)$.

$\displaystyle\int_{3/2}^2 2x\sqrt{2x-3}\,dx = \int_0^1(u+3)u^{1/2}\left(\frac{1}{2}\right)du$

$\displaystyle= \frac{1}{2}\int_0^1\left(u^{3/2}+3u^{1/2}\right)du$

$\displaystyle= \frac{1}{2}\left[\frac{2}{5}u^{5/2} + 2u^{3/2}\right]_0^1$

$\displaystyle= \frac{1}{2}\left(\frac{2}{5}+2\right) = \frac{6}{5}$

7. $\displaystyle\int\frac{100}{\sqrt{100-x^2}}\,dx = 100\arcsin\left(\frac{x}{10}\right) + C$

8. $\displaystyle\int\frac{2x}{x-3}\,dx = \int\left(2+\frac{6}{x-3}\right)dx$

$\displaystyle= 2x + 6\ln\left|x-3\right| + C$

9. Let $u = x$, $du = dx$, $dv = e^{1-x}\,dx$, $v = -e^{1-x}$.

$\displaystyle\int xe^{1-x}\,dx = -xe^{1-x} + \int e^{1-x}\,dx$

$\displaystyle= -xe^{1-x} - e^{1-x} + C$

10. Let $u = x^2$, $du = 2x\,dx$, $dv = e^{x/2}$, $v = 2e^{x/2}$

$\displaystyle\int x^2e^{x/2}\,dx = 2x^2e^{x/2} - \int 2e^{x/2}(2x)\,dx$

$\displaystyle= 2x^2e^{x/2} - 4\int xe^{x/2}\,dx$

Use integration by parts again with $u = x$, $du = dx$, $dv = e^{x/2}$, $v = 2e^{x/2}$.

$\displaystyle= 2x^2e^{x/2} - 4\left(2xe^{x/2} - \int 2e^{x/2}\,dx\right)$

$\displaystyle= 2x^2e^{x/2} - 8xe^{x/2} + 16e^{x/2} + C$

$\displaystyle= e^{x/2}\left(2x^2 + 8x + 16\right) + C$

11. $\displaystyle\int e^{2x}\sin 3x\,dx = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\,dx$

$\displaystyle= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\left(\frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x\,dx\right)$

$\displaystyle\frac{13}{9}\int e^{2x}\sin 3x\,dx = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x$

$\displaystyle\int e^{2x}\sin 3x\,dx = \frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + C$

(1) $dv = \sin 3x\,dx \Rightarrow v = -\dfrac{1}{3}\cos 3x$
$u = e^{2x} \Rightarrow du = 2e^{2x}\,dx$

(2) $dv = \cos 3x\,dx \Rightarrow v = \dfrac{1}{3}\sin 3x$
$u = e^{2x} \Rightarrow du = 2e^{2x}\,dx$

12. $\int x\sqrt{x-1}\,dx = \frac{2}{3}x(x-1)^{3/2} - \int \frac{2}{3}(x-1)^{3/2}\,dx$

$= \frac{2}{3}x(x-1)^{3/2} - \frac{4}{15}(x-1)^{5/2} + C$

$= \frac{2}{15}(x-1)^{3/2}(5x - 2(x-1)) + C$

$= \frac{2}{15}(x-1)^{3/2}(3x+2) + C$

$dv = (x-1)^{1/2}\,dx \Rightarrow v = \frac{2}{3}(x-1)^{3/2}$

$h = x \Rightarrow du = dx$

13. Let $u = x, du = dx, dv = \sec^2 x\,dx, v = \tan x$.

$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx$

$= x\tan x + \ln|\cos x| + C$

14. $\int \ln\sqrt{x^2-4}\,dx = \frac{1}{2}\int \ln(x^2-4)\,dx$

$= \frac{1}{2}\left[x\ln(x^2-4) - \int \frac{2x^2}{x^2-4}\,dx\right]$

$= \frac{1}{2}x\ln(x^2-4) - \int\left(1 + \frac{4}{x^2-4}\right)dx$

$= \frac{1}{2}x\ln(x^2-4) - x - \ln\left|\frac{x-2}{x+2}\right| + C$

$dv = dx \quad \Rightarrow \quad v = x$

$u = \ln(x^2-4) \quad \Rightarrow \quad du = \frac{2x}{x^2-4}\,dx$

15. $\int x\arcsin 2x\,dx = \frac{x^2}{2}\arcsin 2x - \int \frac{x^2}{\sqrt{1-4x^2}}\,dx$

$= \frac{x^2}{2}\arcsin 2x - \frac{1}{4}\int \frac{(2x)^2}{\sqrt{1-(2x)^2}}\,dx$

$= \frac{x^2}{2}\arcsin 2x - \frac{1}{4}\left(\frac{1}{2}\right)\left[-(2x)\sqrt{1-4x^2} + \arcsin 2x\right] + C$ (by Formula 43 of Integration Tables)

$= \frac{1}{8}\left[(4x^2-1)\arcsin 2x + 2x\sqrt{1-4x^2}\right] + C$

$dv = x\,dx \quad \Rightarrow \quad v = \frac{x^2}{2}$

$u = \arcsin 2x \quad \Rightarrow \quad du = \frac{2}{\sqrt{1-4x^2}}\,dx$

16. $\int \arctan 2x\,dx = x\arctan 2x - \int \frac{2x}{1+4x^2}\,dx$

$= x\arctan 2x - \frac{1}{4}\ln(1+4x^2) + C$

$dv = dx \quad \Rightarrow \quad v = x$

$u = \arctan 2x \quad \Rightarrow \quad du = \frac{2}{1+4x^2}\,dx$

17. $\int \sin x\cos^4 x\,dx = -\frac{\cos^5 x}{5} + C \qquad (u = \cos x, du = -\sin x\,dx)$

18. $\int \sin^2 x\cos^3 x\,dx = \int \sin^2 x\cos^2 x\cos x\,dx$

$= \int \sin^2 x(1-\sin^2 x)\cos x\,dx$

$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

19. $\int \cos^3(\pi x - 1)\,dx = \int \left[1 - \sin^2(\pi x - 1)\right]\cos(\pi x - 1)\,dx$

$$= \frac{1}{\pi}\left[\sin(\pi x - 1) - \frac{1}{3}\sin^3(\pi x - 1)\right] + C$$

$$= \frac{1}{3\pi}\sin(\pi x - 1)\left[3 - \sin^2(\pi x - 1)\right] + C$$

$$= \frac{1}{3\pi}\sin(\pi x - 1)\left[3 - \left(1 - \cos^2(\pi x - 1)\right)\right] + C$$

$$= \frac{1}{3\pi}\sin(\pi x - 1)\left[2 + \cos^2(\pi x - 1)\right] + C$$

20. $\int \sin^2\frac{\pi x}{2}\,dx = \int \frac{1}{2}(1 - \cos \pi x)\,dx = \frac{1}{2}\left(x - \frac{1}{\pi}\sin \pi x\right) + C = \frac{1}{2\pi}(\pi x - \sin \pi x) + C$

21. $\int \sec^4\left(\frac{x}{2}\right)dx = \int \left[\tan^2\left(\frac{x}{2}\right) + 1\right]\sec^2\left(\frac{x}{2}\right)dx$

$$= \int \tan^2\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx + \int \sec^2\left(\frac{x}{2}\right)dx$$

$$= \frac{2}{3}\tan^3\left(\frac{x}{2}\right) + 2\tan\left(\frac{x}{2}\right) + C = \frac{2}{3}\left[\tan^3\left(\frac{x}{2}\right) + 3\tan\left(\frac{x}{2}\right)\right] + C$$

22. $\int \tan\theta\sec^4\theta\,d\theta = \int \left(\tan^3\theta + \tan\theta\right)\sec^2\theta\,d\theta + \frac{1}{4}\tan^4\theta + \frac{1}{2}\tan^2\theta + C_1$

or

$\int \tan\theta\sec^4\theta\,d\theta = \int \sec^3\theta(\sec\theta\tan\theta)\,d\theta + \frac{1}{4}\sec^4\theta + C_2$

23. Let $u = x^2$, $du = 2x\,dx$.

$$\int x\tan^4 x^2\,dx = \frac{1}{2}\int \tan^4 u\,du$$

$$= \frac{1}{2}\int \tan^2 u\left(\sec^2 u - 1\right)du$$

$$= \frac{1}{2}\int \tan^2 u\sec^2 u\,du - \frac{1}{2}\int \tan^2 u\,du$$

$$= \frac{1}{2}\int \tan^2 u\sec^2 u\,du - \frac{1}{2}\int \left(\sec^2 u - 1\right)du$$

$$= \frac{1}{2}\cdot\frac{\tan^3 u}{3} - \frac{1}{2}\tan u + \frac{1}{2}u + C$$

$$= \frac{1}{6}\tan^3 x^2 - \frac{1}{2}\tan x^2 + \frac{x^2}{2} + C$$

24. $\int \frac{\tan^2 x}{\sec^3 x}\,dx = \int \frac{\sin^2 x}{\cos^2 x}\left(\cos^3 x\right)dx$

$$= \int \sin^2 x\cos x\,dx$$

$$= \frac{1}{3}\sin^3 x + C$$

25. $\int \frac{1}{1 - \sin\theta}\,d\theta = \int \frac{1}{1 - \sin\theta}\cdot\frac{1 + \sin\theta}{1 + \sin\theta}\,d\theta = \int \frac{1 + \sin\theta}{\cos^2\theta}\,d\theta = \int \left(\sec^2\theta + \sec\theta\tan\theta\right)d\theta = \tan\theta + \sec\theta + C$

26. $\int \cos 2\theta(\sin\theta + \cos\theta)^2\,d\theta = \int \left(\cos^2\theta - \sin^2\theta\right)(\sin\theta + \cos\theta)^2\,d\theta$

$$= \int (\sin\theta + \cos\theta)^3(\cos\theta - \sin\theta)\,d\theta = \frac{1}{4}(\sin\theta + \cos\theta)^4 + C$$

27. $A = \int_{\pi/4}^{3\pi/4} \sin^4 x\,dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \int \left(\frac{1}{4}\cos^2 2x - \frac{1}{2}\cos 2x + \frac{1}{4}\right) dx$

$$= \left[\frac{1}{32}\sin 4x - \frac{1}{4}\sin 2x + \frac{3x}{8}\right]_{\pi/4}^{3\pi/4}$$

$$= \left(0 + \frac{1}{4} + \frac{9\pi}{32}\right) - \left(0 - \frac{1}{4} + \frac{3\pi}{32}\right)$$

$$\approx 1.0890$$

28. $A = \int_0^{\pi/4} \sin 3x \cos 2x\,dx$

$$= \frac{1}{2}\int_0^{\pi/4} [\sin x + \sin 5x]\,dx$$

$$= \frac{1}{2}\left[-\cos x - \frac{1}{5}\cos 5x\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[-\frac{\sqrt{2}}{2} - \frac{1}{5}\left(-\frac{\sqrt{2}}{2}\right) + 1 + \frac{1}{5}\right]$$

$$= \frac{3}{5} - \frac{\sqrt{2}}{5} \approx 0.317$$

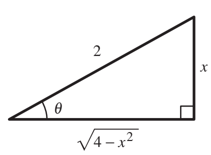
29. $x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta,\ \sqrt{4-x^2} = 2\cos\theta$

$$\int \frac{-12}{x^2\sqrt{4-x^2}}\,dx = \int \frac{-24\cos\theta\,d\theta}{(4\sin^2\theta)(2\cos\theta)}$$

$$= -3\int \csc^2\theta\,d\theta$$

$$= 3\cot\theta + C$$

$$= \frac{3\sqrt{4-x^2}}{x} + C$$



30. $x = 3\sec\theta,\ dx = 3\sec\theta\tan\theta\,d\theta,\ \sqrt{x^2-9} = 3\tan\theta$

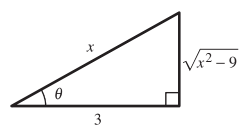
$$\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{3\tan\theta}{3\sec\theta}(3\sec\theta\tan\theta\,d\theta)$$

$$= 3\int \tan^2\theta\,d\theta$$

$$= 3\int (\sec^2\theta - 1)\,d\theta$$

$$= 3(\tan\theta - \theta) + C$$

$$= \sqrt{x^2-9} - 3\,\text{arcsec}\left(\frac{x}{3}\right) + C$$



31. $x = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta,\ 4 + x^2 = 4\sec^2\theta$

$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{8\tan^3\theta}{2\sec\theta}\,2\sec^2\theta\,d\theta$$

$$= 8\int \tan^3\theta\sec\theta\,d\theta$$

$$= 8\int (\sec^2\theta - 1)\tan\theta\sec\theta\,d\theta$$

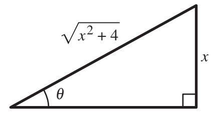
$$= 8\left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C$$

$$= 8\left[\frac{(x^2+4)^{3/2}}{24} - \frac{\sqrt{x^2+4}}{2}\right] + C$$

$$= \sqrt{x^2+4}\left[\frac{1}{3}(x^2+4) - 4\right] + C$$

$$= \frac{1}{3}x^2\sqrt{x^2+4} - \frac{8}{3}\sqrt{x^2+4} + C$$

$$= \frac{1}{3}(x^2+4)^{1/2}(x^2-8) + C$$



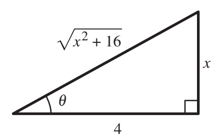
32. $\int \sqrt{25 - 9x^2}\, dx = \frac{1}{3}\int \sqrt{5^2 - (3x)^2}\,(3)\, dx$

$$= \frac{1}{3}\frac{1}{2}\left[25 \arcsin\left(\frac{3x}{5}\right) + 3x\sqrt{25 - 9x^2}\right] + C = \frac{25}{6}\arcsin\left(\frac{3x}{5}\right) + \frac{x}{2}\sqrt{25 - 9x^2} + C$$

(Theorem 8.2)

33. $x = 4 \tan\theta,\ dx = 4\sec^2\theta\, d\theta,\ \sqrt{16 + x^2} = 4\sec\theta$

$$\int \frac{6x^3}{\sqrt{16 + x^2}}\, dx = \int \frac{6(4\tan\theta)^3}{4\sec\theta}\, 4\sec^2\theta\, d\theta$$
$$= 384\int \tan^3\theta \sec\theta\, d\theta$$
$$= 384\int \left(\sec^2\theta - 1\right)\sec\theta\tan\theta\, d\theta$$
$$= 384\left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C$$
$$= \frac{384}{3}\cdot\frac{\left(16 + x^2\right)^{3/2}}{64} - \frac{384\sqrt{16 + x^2}}{4} + C$$
$$= 2\sqrt{x^2 + 16}\left(16 + x^2 - 48\right) + C$$
$$= 2\sqrt{x^2 + 16}\left(x^2 - 32\right) + C$$

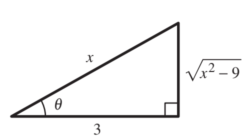


$$\int_0^1 \frac{6x^3}{\sqrt{16 + x^2}}\, dx = \left[2\sqrt{x^2 + 16}\left(x^2 - 32\right)\right]_0^1$$
$$= 2\sqrt{17}(-31) - 2(4)(-32)$$
$$= 256 - 62\sqrt{17}$$

34. $x = 3\sec\theta,\ dx = 3\sec\theta\tan\theta\, d\theta,\ \sqrt{x^2 - 9} = 3\tan\theta$

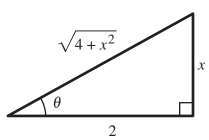
$$\int x^3\sqrt{x^2 - 9}\, dx = \int 27\sec^3\theta\,(3\tan\theta)\,3\sec\theta\tan\theta\, d\theta$$
$$= 243\int \sec^4\theta\tan^2\theta\, d\theta$$
$$= 243\int \left(1 + \tan^2\theta\right)\tan^2\theta\sec^2\theta\, d\theta$$
$$= 243\left[\frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5}\right] + C$$
$$= 243\left[\frac{\left(x^2 - 9\right)^{3/2}}{81} + \frac{\left(x^2 - 9\right)^{5/2}}{1215}\right] + C$$

$$\int_3^4 x^3\sqrt{x^2 - 9}\, dx = 243\left[\frac{\left(x^2 - 9\right)^{3/2}}{81} + \frac{\left(x^2 - 9\right)^{5/2}}{1215}\right]_3^4$$
$$= 243\left[\frac{7^{3/2}}{81} + \frac{7^{5/2}}{1215}\right]$$
$$= 243\left[\frac{7\sqrt{7}}{81} + \frac{49\sqrt{7}}{1215}\right]$$
$$= \frac{154}{5}\sqrt{7}$$



35. (a) Let $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$.

$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{8\tan^3\theta}{2\sec\theta}2\sec^2\theta\,d\theta$$
$$= 8\int \tan^3\theta\sec\theta\,d\theta$$
$$= 8\int \frac{\sin^3\theta}{\cos^4\theta}\,d\theta$$
$$= 8\int (1-\cos^2\theta)\cos^{-4}\theta\sin\theta\,d\theta$$
$$= 8\int (\cos^{-4}\theta - \cos^{-2}\theta)\sin\theta\,d\theta$$
$$= 8\left[\frac{\cos^{-3}\theta}{3} - \frac{\cos^{-1}\theta}{-1}\right] + C$$
$$= \frac{8}{3}\sec\theta(\sec^2\theta - 3) + C$$
$$= \frac{8}{3}\left(\frac{\sqrt{4+x^2}}{2}\right)\left(\frac{4+x^2}{4} - 3\right) + C$$
$$= \frac{1}{3}\sqrt{4+x^2}(x^2-8) + C$$



(b)
$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{x^2}{\sqrt{4+x^2}}x\,dx$$
$$= \int \frac{(u^2-4)u\,du}{u}$$
$$= \int (u^2-4)\,du$$
$$= \frac{1}{3}u^3 - 4u + C$$
$$= \frac{u}{3}(u^2-12) + C$$
$$= \frac{\sqrt{4+x^2}}{3}(x^2-8) + C$$

$u^2 = 4 + x^2$, $2u\,du = 2x\,dx$

(c)
$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = x^2\sqrt{4+x^2} - \int 2x\sqrt{4+x^2}\,dx$$
$$= x^2\sqrt{4+x^2} - \frac{2}{3}(4+x^2)^{3/2} + C = \frac{\sqrt{4+x^2}}{3}(x^2-8) + C$$

$dv = \dfrac{x}{\sqrt{4+x^2}}\,dx \Rightarrow v = \sqrt{4+x^2}$

$u = x^2 \Rightarrow du = 2x\,dx$

36. (a)
$$\int x\sqrt{4+x}\,dx = 64\int \tan^3\theta\sec^3\theta\,d\theta$$
$$= 64\int (\sec^4\theta - \sec^2\theta)\sec\theta\tan\theta\,d\theta$$
$$= \frac{64\sec^3\theta}{15}(3\sec^3\theta - 5) + C$$
$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$x = 4\tan^2\theta$, $dx = 8\tan\theta\sec^2\theta\,d\theta$,

$\sqrt{4+x} = 2\sec\theta$

(b)
$$\int x\sqrt{4+x}\,dx = 2\int (u^4 - 4u^2)\,du$$
$$= \frac{2u^3}{15}(3u^2-20) + C$$
$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$u^2 = 4 + x$, $dx = 2u\,du$

(c)
$$\int x\sqrt{4+x}\,dx = \int (u^{3/2} - 4u^{1/2})\,du$$
$$= \frac{2u^{3/2}}{15}(3u-20) + C$$
$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$u = 4 + x$, $du = dx$

(d)
$$\int x\sqrt{4+x}\,dx = \frac{2x}{3}(4+x)^{3/2} - \frac{2}{3}\int (4+x)^{3/2}\,dx$$
$$= \frac{2x}{3}(4+x)^{3/2} - \frac{4}{15}(4+x)^{5/2} + C$$
$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$dv = \sqrt{4+x}\,dx \Rightarrow v = \dfrac{2}{3}(4+x)^{3/2}$

$u = x \Rightarrow du = dx$

37. $\dfrac{x-8}{x^2-x-6} = \dfrac{x-8}{(x-3)(x+2)} = \dfrac{A}{x-3} + \dfrac{B}{x+2}$

$x - 8 = A(x+2) + B(x-3)$

When $x = -2, -10 = -5B \Rightarrow B = 2.$

When $x = 3, -5 = 5A \Rightarrow A = -1.$

$$\int \frac{x-8}{x^2-x-6}\,dx = \int\left(\frac{-1}{x-3} + \frac{2}{x+2}\right)dx$$
$$= -\ln|x-3| + 2\ln|x+2| + C$$
$$= \ln\left|\frac{(x+2)^2}{x-3}\right| + C$$

38. $\dfrac{5x-2}{x^2-x} = \dfrac{5x-2}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1}$

$5x - 2 = A(x-1) + Bx$

When $x = 1, \quad 3 = B.$

When $x = 0, \quad -2 = -A \Rightarrow A = 2.$

$$\int \frac{5x-2}{x^2-x}\,dx = \int\left(\frac{2}{x} + \frac{3}{x-1}\right)dx$$
$$= 2\ln|x| + 3\ln|x-1| + C$$

39. $\dfrac{x^2+2x}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$

$x^2 + 2x = A(x^2+1) + (Bx+C)(x-1)$

When $x = 1, \quad 3 = 2A \Rightarrow A = \dfrac{3}{2}.$

When $x = 0, \quad 0 = A - C \Rightarrow C = \dfrac{3}{2}.$

When $x = 2, \quad 8 = 5A + 2B + C \Rightarrow B = -\dfrac{1}{2}.$

$$\int \frac{x^2+2x}{x^3-x^2+x-1}\,dx = \frac{3}{2}\int\frac{1}{x-1}\,dx - \frac{1}{2}\int\frac{x-3}{x^2+1}\,dx$$
$$= \frac{3}{2}\int\frac{1}{x-1}\,dx - \frac{1}{4}\int\frac{2x}{x^2+1}\,dx + \frac{3}{2}\int\frac{1}{x^2+1}\,dx$$
$$= \frac{3}{2}\ln|x-1| - \frac{1}{4}\ln|x^2+1| + \frac{3}{2}\arctan x + C$$
$$= \frac{1}{4}\left[6\ln|x-1| - \ln(x^2+1) + 6\arctan x\right] + C$$

40. $\dfrac{4x-2}{3(x-1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2}$

$4x - 2 = 3A(x-1) + 3B$

When $x = 1, \quad 2 = 3B \Rightarrow B = \dfrac{2}{3}.$

When $x = 2, \quad 6 = 3A + 3B \Rightarrow A = \dfrac{4}{3}.$

$$\int\frac{4x-2}{3(x-1)^2}\,dx = \frac{4}{3}\int\frac{1}{x-1}\,dx + \frac{2}{3}\int\frac{1}{(x-1)^2}\,dx = \frac{4}{3}\ln|x-1| - \frac{2}{3(x-1)} + C = \frac{2}{3}\left(2\ln|x-1| - \frac{1}{x-1}\right) + C$$

41. $\dfrac{x^2}{x^2 - 2x + 1} = 1 + \dfrac{2x - 1}{x^2 - 2x + 1} = 1 + \dfrac{A}{x - 1} + \dfrac{B}{(x - 1)^2}$

$2x - 1 = A(x - 1) + B$

When $x = 1, B = 1$.

When $x = 0, -1 = -A + B = -A + 1 \Rightarrow A = 2$.

$$\int \frac{x^2}{x^2 - 2x + 1}\,dx = \int \left(1 + \frac{2}{x - 1} + \frac{1}{(x - 1)^2}\right) dx$$
$$= x + 2\ln|x - 1| + \frac{1}{1 - x} + C$$

42. $\dfrac{x^3 + 4}{x^2 - 4x} = x + 4 + \dfrac{16x + 4}{x^2 - 4x}$

$\dfrac{16x + 4}{x^2 - 4x} = \dfrac{A}{x} + \dfrac{B}{x - 4}$

$16x + 4 = A(x - 4) + Bx$

When $x = 4, 68 = 4B \Rightarrow B = 17$.

When $x = 0, 4 = -4A \Rightarrow A = -1$.

$$\int \frac{x^3 + 4}{x^2 - 4x}\,dx = \int \left(x + 4 - \frac{1}{x} + \frac{17}{x - 4}\right) dx$$
$$= \frac{x^2}{2} + 4x - \ln|x| + 17\ln|x - 4| + C$$

43. Let $u = e^x, du = e^x\,dx$.

$$\frac{4}{(e^{2x} - 1)(e^x + 3)} = \frac{4}{(u^2 - 1)(u + 3)} = \frac{A}{u - 1} + \frac{B}{u + 1} + \frac{C}{u + 3}$$
$$4 = A(u + 1)(u + 3) + B(u - 1)(u + 3) + C(u - 1)(u + 1)$$

When $u = 1, 4 = 8A \Rightarrow A = \frac{1}{2}$.

When $u = -1, 4 = -4B \Rightarrow B = -1$.

When $u = -3, 4 = 8C \Rightarrow C = \frac{1}{2}$.

$$\int \frac{4e^x}{(e^{2x} - 1)(e^x + 3)}\,dx = \int \frac{4}{(u^2 - 1)(u + 3)}\,du$$
$$= \int \left(\frac{1/2}{u - 1} - \frac{1}{u + 1} + \frac{1/2}{u + 3}\right) du$$
$$= \frac{1}{2}\ln|u - 1| - \ln|u + 1| + \frac{1}{2}\ln|u + 3| + C$$
$$= \frac{1}{2}\ln|e^x - 1| - \ln|e^x + 1| + \frac{1}{2}\ln|e^x + 3| + C$$

44. $u = \tan\theta,\ du = \sec^2\theta\,d\theta$

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$1 = A(u-1) + Bu$$

When $u = 0$, $1 = -A \Rightarrow A = -1$.

When $u = 1$, $1 = B$.

$$\int \frac{\sec^2\theta}{\tan\theta(\tan\theta - 1)}\,d\theta = \int \frac{1}{u(u-1)}\,du = \int \frac{1}{u-1}\,du - \int \frac{1}{u}\,du$$

$$= \ln|u-1| - \ln|u| + C = \ln\left|\frac{\tan\theta - 1}{\tan\theta}\right| + C = \ln|1 - \cot\theta| + C$$

45. Trapezoidal Rule $(n = 4)$: $\int_2^3 \frac{2}{1+x^2}\,dx$

$$\approx \frac{1}{8}\left[\frac{2}{1+2^2} + 2\left(\frac{2}{1+(9/4)^2}\right) + 2\left(\frac{2}{1+(5/2)^2}\right) + 2\left(\frac{2}{1+(11/4)^2}\right) + \frac{2}{1+3^2}\right] \approx 0.2848$$

Simpson's Rule $(n = 4)$: $\int_2^3 \frac{2}{1+x^2}\,dx$

$$\approx \frac{1}{12}\left[\frac{2}{1+2^2} + 4\left(\frac{2}{1+(9/4)^2}\right) + 2\left(\frac{2}{1+(5/2)^2}\right) + 4\left(\frac{2}{1+(11/4)^2}\right) + \frac{2}{1+3^2}\right] \approx 0.2838$$

Graphing utility: 0.2838

46. Trapezoidal Rule $(n = 4)$: $\int_0^1 \frac{x^{3/2}}{3-x^2}\,dx \approx \frac{1}{8}\left[0 + \frac{2(1/4)^{3/2}}{3-(1/4)^2} + \frac{2(1/2)^{3/2}}{3-(1/2)^2} + \frac{2(3/4)^{3/2}}{3-(3/4)^2} + \frac{1}{2}\right] \approx 0.1719$

Simpson's Rule $(n = 4)$: $\int_0^1 \frac{x^{3/2}}{3-x^2}\,dx \approx \frac{1}{12}\left[0 + \frac{4(1/4)^{3/2}}{3-(1/4)^2} + \frac{2(1/2)^{3/2}}{3-(1/2)^2} + \frac{4(3/4)^{3/2}}{3-(3/4)^2} + \frac{1}{2}\right] \approx 0.1661$

Graphing utility: 0.1657

47. Trapezoidal Rule $(n = 4)$: $\int_0^{\pi/2} \sqrt{x}\cos x\,dx \approx \frac{\pi}{16}\left[0 + \frac{\sqrt{2\pi}}{2}\sqrt{\frac{\sqrt{2}}{4} + \frac{1}{2}} + \frac{\sqrt{2\pi}}{2} + \frac{\sqrt{6\pi}}{2}\sqrt{-\frac{\sqrt{2}}{4} + \frac{1}{2}} + 0\right] \approx 0.6366$

Simpson's Rule $(n = 4)$: $\int_0^{\pi/2} \sqrt{x}\cos x\,dx \approx \frac{\pi}{24}\left[0 + \sqrt{2\pi}\sqrt{\frac{\sqrt{2}}{4} + \frac{1}{2}} + \frac{\sqrt{2\pi}}{2} + \sqrt{6\pi}\sqrt{-\frac{\sqrt{2}}{4} + \frac{1}{2}} + 0\right] \approx 0.6845$

Graphing utility: 0.7041

48. Trapezoidal Rule $(n = 4)$: $\int_0^{\pi} \sqrt{1+\sin^2 x}\,dx \approx \frac{\pi}{8}\left[1 + \sqrt{6} + 2\sqrt{2} + \sqrt{6} + 1\right] \approx 3.8199$

Simpson's Rule $(n = 4)$: $\int_0^{\pi} \sqrt{1+\sin^2 x}\,dx \approx \frac{\pi}{12}\left[1 + 2\sqrt{6} + 2\sqrt{2} + 2\sqrt{6} + 1\right] \approx 3.8292$

Graphing utility: 3.8202

49. Using Formula 4: $(a = 4, b = 5)$

$$\int \frac{x}{(4 + 5x)^2}\,dx = \frac{1}{25}\left(\frac{4}{4 + 5x} + \ln|4 + 5x|\right) + C$$

50. Using Formula 21: $(a = 4, b = 5)$

$$\int \frac{x}{\sqrt{4 + 5x}}\,dx = \frac{-2(8 - 5x)}{75}\sqrt{4 + 5x} + C$$
$$= \frac{10x - 16}{75}\sqrt{4 + 5x} + C$$

51. Let $u = x^2$, $du = 2x\,dx$.

$$\int_0^{\sqrt{\pi/2}} \frac{x}{1 + \sin x^2}\,dx = \frac{1}{2}\int_0^{\pi/4} \frac{1}{1 + \sin u}\,du$$
$$= \frac{1}{2}\big[\tan u - \sec u\big]_0^{\pi/4}$$
$$= \frac{1}{2}\Big[\big(1 - \sqrt{2}\big) - (0 - 1)\Big]$$
$$= 1 - \frac{\sqrt{2}}{2}$$

52. Let $u = x^2$, $du = 2x\,dx$.

$$\int_0^1 \frac{x}{1 + e^{x^2}}\,dx = \frac{1}{2}\int_0^1 \frac{1}{1 + e^u}\,du$$
$$= \frac{1}{2}\Big[u - \ln\big(1 + e^u\big)\Big]_0^1$$
$$= \frac{1}{2}\Big[\big(1 - \ln(1 + e)\big) + \ln 2\Big]$$
$$= \frac{1}{2}\left[1 + \ln\left(\frac{2}{1 + e}\right)\right]$$

53. $\displaystyle\int \frac{x}{x^2 + 4x + 8}\,dx = \frac{1}{2}\left[\ln|x^2 + 4x + 8| - 4\int \frac{1}{x^2 + 4x + 8}\,dx\right]$ (Formula 15)

$\displaystyle= \frac{1}{2}\Big[\ln|x^2 + 4x + 8|\Big] - 2\left[\frac{2}{\sqrt{32 - 16}}\arctan\left(\frac{2x + 4}{\sqrt{32 - 16}}\right)\right] + C$ (Formula 14)

$\displaystyle= \frac{1}{2}\ln|x^2 + 4x + 8| - \arctan\left(1 + \frac{x}{2}\right) + C$

54. $\displaystyle\int \frac{3}{2x\sqrt{9x^2 - 1}}\,dx = \frac{3}{2}\int \frac{1}{3x\sqrt{(3x)^2 - 1}}\,3\,dx$ $(u = 3x)$

$\displaystyle= \frac{3}{2}\operatorname{arcsec}|3x| + C$ (Formula 33)

55. $\displaystyle\int \frac{1}{\sin \pi x \cos \pi x}\,dx = \frac{1}{\pi}\int \frac{1}{\sin \pi x \cos \pi x}(\pi)\,dx$ $(u = \pi x)$

$\displaystyle= \frac{1}{\pi}\ln|\tan \pi x| + C$ (Formula 58)

56. $\displaystyle\int \frac{1}{1 + \tan \pi x}\,dx = \frac{1}{\pi}\int \frac{1}{1 + \tan \pi x}(\pi)\,dx$ $(u = \pi x)$

$\displaystyle= \frac{1}{\pi}\left[\frac{1}{2}\big(\pi x + \ln|\cos \pi x + \sin \pi x|\big)\right] + C$ (Formula 71)

57. $\displaystyle\int \theta \sin \theta \cos \theta\,d\theta = \frac{1}{2}\int \theta \sin 2\theta\,d\theta$

$$= -\frac{1}{4}\theta \cos 2\theta + \frac{1}{4}\int \cos 2\theta\,d\theta = -\frac{1}{4}\theta \cos 2\theta + \frac{1}{8}\sin 2\theta + C = \frac{1}{8}\big(\sin 2\theta - 2\theta \cos 2\theta\big) + C$$

$dv = \sin 2\theta\,d\theta \Rightarrow v = -\frac{1}{2}\cos 2\theta$

$u = \theta \Rightarrow du = d\theta$

58. $\int \frac{\csc\sqrt{2x}}{\sqrt{x}}\,dx = \sqrt{2}\int \csc\sqrt{2x}\left(\frac{1}{\sqrt{2x}}\right)dx = -\sqrt{2}\ln\left|\csc\sqrt{2x} + \cot\sqrt{2x}\right| + C$

$u = \sqrt{2x},\ du = \frac{1}{\sqrt{2x}}\,dx$

59. $\int \frac{x^{1/4}}{1 + x^{1/2}}\,dx = 4\int \frac{u(u^3)}{1 + u^2}\,du$

$$= 4\int\left(u^2 - 1 + \frac{1}{u^2 + 1}\right)du$$

$$= 4\left(\frac{1}{3}u^3 - u + \arctan u\right) + C$$

$$= \frac{4}{3}\left[x^{3/4} - 3x^{1/4} + 3\arctan\left(x^{1/4}\right)\right] + C$$

$u = \sqrt[4]{x},\ x = u^4,\ dx = 4u^3\,du$

60-. $\int\sqrt{1 + \sqrt{x}}\,dx = \int u\left(4u^3 - 4u\right)du = \int\left(4u^4 - 4u^2\right)du = \frac{4u^5}{5} - \frac{4u^3}{3} + C = \frac{4}{15}\left(1 + \sqrt{x}\right)^{3/2}\left(3\sqrt{x} - 2\right) + C$

$u = \sqrt{1 + \sqrt{x}},\ x = u^4 - 2u^2 + 1,\ dx = \left(4u^3 - 4u\right)du$

61. $\int\sqrt{1 + \cos x}\,dx = \int \frac{\sqrt{1 + \cos x}}{1}\cdot\frac{\sqrt{1 - \cos x}}{\sqrt{1 - \cos x}}\,dx$

$$= \int \frac{\sin x}{\sqrt{1 - \cos x}}\,dx$$

$$= \int (1 - \cos x)^{-1/2}(\sin x)\,dx$$

$$= 2\sqrt{1 - \cos x} + C$$

$u = 1 - \cos x,\ du = \sin x\,dx$

62. $\frac{3x^3 + 4x}{\left(x^2 + 1\right)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{\left(x^2 + 1\right)^2}$

$3x^3 + 4x = (Ax + B)\left(x^2 + 1\right) + Cx + D = Ax^3 + Bx^2 + (A + C)x + (B + D)$

$A = 3,\ B = 0,\ A + C = 4 \Rightarrow C = 1,$

$B + D = 0 \Rightarrow D = 0$

$\int \frac{3x^3 + 4x}{\left(x^2 + 1\right)^2}\,dx = 3\int \frac{x}{x^2 + 1}\,dx + \int \frac{x}{\left(x^2 + 1\right)^2}\,dx = \frac{3}{2}\ln\left(x^2 + 1\right) - \frac{1}{2\left(x^2 + 1\right)} + C$

63. $\int \cos x \ln(\sin x)\,dx = \sin x \ln(\sin x) - \int \cos x\,dx = \sin x \ln(\sin x) - \sin x + C$

$dv = \cos x\,dx \Rightarrow v = \sin x$

$u = \ln(\sin x) \Rightarrow du = \frac{\cos x}{\sin x}\,dx$

64. $\int (\sin\theta + \cos\theta)^2\,d\theta = \int\left(\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta\right)d\theta$

$$= \int (1 + \sin 2\theta)\,d\theta = \theta - \frac{1}{2}\cos 2\theta + C = \frac{1}{2}(2\theta - \cos 2\theta) + C$$

65. $y = \int \frac{25}{x^2 - 25}\,dx = 25\left(\frac{1}{10}\right)\ln\left|\frac{x-5}{x+5}\right| + C$

$= \frac{5}{2}\ln\left|\frac{x-5}{x+5}\right| + C$

(Formula 24)

66. $y = \int \frac{\sqrt{4 - x^2}}{2x}\,dx = \int \frac{2\cos\theta(2\cos\theta)\,d\theta}{4\sin\theta}$

$= \int(\csc\theta - \sin\theta)\,d\theta$

$= \left[-\ln\left|\csc\theta + \cos\theta\right| + \cos\theta\right] + C$

$= -\ln\left|\frac{2 + \sqrt{4 - x^2}}{x}\right| + \frac{\sqrt{4 - x^2}}{2} + C$

$x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta,\ \sqrt{4 - x^2} = 2\cos\theta$

67. $y = \int \ln(x^2 + x)\,dx = x\ln\left|x^2 + x\right| - \int \frac{2x^2 + x}{x^2 + x}\,dx$

$= x\ln\left|x^2 + x\right| - \int \frac{2x + 1}{x + 1}\,dx$

$= x\ln\left|x^2 + x\right| - \int 2\,dx + \int \frac{1}{x + 1}\,dx$

$= x\ln\left|x^2 + x\right| - 2x + \ln\left|x + 1\right| + C$

$dv = dx \Rightarrow v = x$

$u = \ln(x^2 + x) \Rightarrow du = \frac{2x + 1}{x^2 + x}\,dx$

68. $y = \int \sqrt{1 - \cos\theta}\,d\theta$

$= \int \frac{\sin\theta}{\sqrt{1 + \cos\theta}}\,d\theta$

$= -\int (1 + \cos\theta)^{-1/2}(-\sin\theta)\,d\theta$

$= -2\sqrt{1 + \cos\theta} + C$

$u = 1 + \cos\theta,\ du = -\sin\theta\,d\theta$

69. $\int_2^{\sqrt{5}} x(x^2 - 4)^{3/2}\,dx = \left[\frac{1}{5}(x^2 - 4)^{5/2}\right]_2^{\sqrt{5}} = \frac{1}{5}$

70. $\int_0^1 \frac{x}{(x - 2)(x - 4)}\,dx = \left[2\ln\left|x - 4\right| - \ln\left|x - 2\right|\right]_0^1$

$= 2\ln 3 - 2\ln 4 + \ln 2$

$= \ln\frac{9}{8} \approx 0.118$

71. $\int_1^4 \frac{\ln x}{x}\,dx = \left[\frac{1}{2}(\ln x)^2\right]_1^4 = \frac{1}{2}(\ln 4)^2 \approx 0.961$

72. $\int_0^2 xe^{3x}\,dx = \left[\frac{e^{3x}}{9}(3x - 1)\right]_0^2 = \frac{1}{9}(5e^6 + 1) \approx 224.238$

73. $\int (x^2 - 4)\sin x\,dx = (x^2 - 4)(-\cos x) - 2x(-\sin x) + 2\cos x + C$

$\int_2^{\pi} (x^2 - 4)\sin x\,dx = \left[(4 - x^2)\cos x + 2x\sin x + 2\cos x\right]_2^{\pi}$

$= \left[(4 - \pi^2)(-1) - 2\right] - \left[4\sin 2 + 2\cos 2\right]$

$= \pi^2 - 4\sin 2 - 2\cos 2 - 6 \approx 1.0647$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	$x^2 - 4$	$\sin x$
−	$2x$	$-\cos x$
+	2	$-\sin x$
−	0	$\cos x$

74. $\int_0^5 \frac{x}{\sqrt{4+x}}\,dx = \left[\frac{2x-16}{3}\sqrt{4+x}\right]_0^5$

$= -2(3) + \frac{16}{3}(2) = \frac{14}{3}$

75. $A = \int_0^{3/2} x\sqrt{3-2x}\,dx$

Let $3 - 2x = u, -2\,dx = du, x = \frac{3-u}{2}$

$A = \int_3^0 \left(\frac{3-u}{2}\right)u^{1/2}\left(-\frac{1}{2}\,du\right)$

$= \frac{1}{4}\int_0^3 \left(3u^{1/2} - u^{3/2}\right)du$

$= \frac{1}{4}\left[2u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^3$

$= \frac{1}{4}\left[2\left(3^{3/2}\right) - \frac{2}{5}\left(3^{5/2}\right)\right]$

$= \frac{1}{4}\left[6\sqrt{3} - \frac{2}{5}9\sqrt{3}\right]$

$= 3\frac{\sqrt{3}}{5} \approx 1.0392$

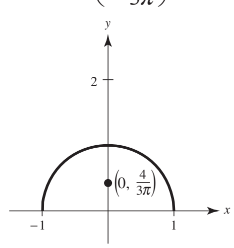
76. $A = \int_0^4 \frac{1}{25-x^2}\,dx$

$= \left[-\frac{1}{10}\ln\left|\frac{x-5}{x+5}\right|\right]_0^4 = -\frac{1}{10}\ln\frac{1}{9} = \frac{1}{10}\ln 9 \approx 0.220$

77. By symmetry, $\bar{x} = 0$, $A = \frac{1}{2}\pi$.

$\bar{y} = \frac{2}{\pi}\left(\frac{1}{2}\right)\int_{-1}^{1}\left(\sqrt{1-x^2}\right)^2 dx = \frac{1}{\pi}\left[x - \frac{1}{3}x^3\right]_{-1}^{1} = \frac{4}{3\pi}$

$(\bar{x}, \bar{y}) = \left(0, \frac{4}{3\pi}\right)$



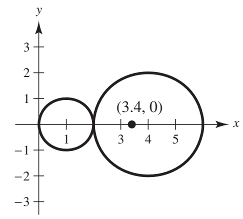
78. By symmetry, $\bar{y} = 0$.

$A = \pi + 4\pi = 5\pi$

$\bar{x} = \frac{1(\pi) + 4(4\pi)}{\pi + 4\pi}$

$= \frac{17\pi}{5\pi} = 3.4$

$(\bar{x}, \bar{y}) = (3.4, 0)$



79. $\int_0^{16} \frac{1}{\sqrt[4]{x}}\,dx = \lim_{b\to 0^+}\left[\frac{4}{3}x^{3/4}\right]_b^{16} = \frac{32}{3}$

80. $\int_0^2 \frac{7}{x-2}\,dx = \lim_{b\to 2^-}\left[7\ln|x-2|\right]_0^b$

$= -\infty$ Diverges

81. $\int_1^\infty x^2 \ln x\,dx = \lim_{b\to\infty}\left[\frac{x^3}{9}(-1 + 3\ln x)\right]_1^b = \infty$

Diverges

82. $\int_0^\infty \frac{e^{-1/x}}{x^2}\,dx = \lim_{\substack{a\to 0^+\\ b\to\infty}}\left[e^{-1/x}\right]_a^b = 1 - 0 = 1$

83. Let $u = \ln x, du = \frac{1}{x}\,dx, dv = x^{-2}\,dx, v = -x^{-1}$.

$\int \frac{\ln x}{x^2}\,dx = \frac{-\ln x}{x} + \int \frac{1}{x^2}\,dx = \frac{-\ln x}{x} - \frac{1}{x} + C$

$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left[\frac{-\ln x}{x} - \frac{1}{x}\right]_1^b$

$= \lim_{b\to\infty}\left(\frac{-\ln b}{b} - \frac{1}{b}\right) - (-1)$

$= 0 + 1 = 1$

84. $\int_1^\infty \frac{1}{\sqrt[4]{x}}\,dx = \lim_{b\to\infty}\int_1^b x^{-1/4}\,dx$

$= \lim_{b\to\infty}\left[\frac{4}{3}x^{3/4}\right]_1^b$

$= \lim_{b\to\infty}\left[\frac{4}{3}b^{3/4} - \frac{4}{3}\right] = \infty$

Diverges

85. $$\int_2^{\infty} \frac{1}{x\sqrt{x^2-4}}\,dx = \int_2^3 \frac{1}{x\sqrt{x^2-4}}\,dx + \int_3^{\infty} \frac{1}{x\sqrt{x^2-4}}\,dx$$
$$= \lim_{b\to 2^+}\left[\frac{1}{2}\operatorname{arcsec}\left(\frac{x}{2}\right)\right]_b^3 + \lim_{c\to\infty}\left[\frac{1}{2}\operatorname{arcsec}\left(\frac{x}{2}\right)\right]_3^c$$
$$= \frac{1}{2}\operatorname{arcsec}\left(\frac{3}{2}\right) - \frac{1}{2}(0) + \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\operatorname{arcsec}\left(\frac{3}{2}\right)$$
$$= \frac{\pi}{4}$$

86. Let $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx \Rightarrow dx = 2u\,du$.

$$\int \frac{2}{\sqrt{x}(x+4)}\,dx = \int \frac{2}{u(u^2+4)}\,2u\,du = \int \frac{4}{u^2+4}\,du = 2\arctan\left(\frac{u}{2}\right) + C = 2\arctan\left(\frac{\sqrt{x}}{2}\right) + C$$

$$\int_0^{\infty} \frac{2}{\sqrt{x}(x+4)}\,dx = \lim_{b\to 0^+}\left[2\arctan\left(\frac{\sqrt{x}}{2}\right)\right]_b^1 + \lim_{c\to\infty}\left[2\arctan\left(\frac{\sqrt{x}}{2}\right)\right]_1^c = \left(2\arctan\frac{1}{2} - 0\right) + 2\left(\frac{\pi}{2}\right) - 2\arctan\frac{1}{2} = \pi$$

87. $$\int_0^{t_0} 500{,}000e^{-0.05t}\,dt = \left[\frac{500{,}000}{-0.05}e^{-0.05t}\right]_0^{t_0}$$
$$= \frac{-500{,}000}{0.05}\left(e^{-0.05t_0} - 1\right)$$
$$= 10{,}000{,}000\left(1 - e^{-0.05t_0}\right)$$

(a) $t_0 = 20$: \$6,321,205.59

(b) $t_0 \to \infty$: \$10,000,000

88. $$V = \pi\int_0^{\infty} \left(xe^{-x}\right)^2\,dx$$
$$= \pi\int_0^{\infty} x^2e^{-2x}\,dx$$
$$= \lim_{b\to\infty}\left[-\frac{\pi e^{-2x}}{4}\left(2x^2 + 2x + 1\right)\right]_0^b = \frac{\pi}{4}$$

89. (a) $P(13 \le x < \infty) = \dfrac{1}{0.95\sqrt{2\pi}}\displaystyle\int_{13}^{\infty} e^{-(x-12.9)^2/2(0.95)^2}\,dx \approx 0.4581$

(b) $P(15 \le x < \infty) = \dfrac{1}{0.95\sqrt{2\pi}}\displaystyle\int_{15}^{\infty} e^{-(x-12.9)^2/2(0.95)^2}\,dx \approx 0.0135$

Problem Solving for Chapter 8

1. (a) $$\int_{-1}^{1}\left(1 - x^2\right)dx = \left[x - \frac{x^3}{3}\right]_{-1}^{1} = 2\left(1 - \frac{1}{3}\right) = \frac{4}{3}$$

$$\int_{-1}^{1}\left(1 - x^2\right)^2\,dx = \int_{-1}^{1}\left(1 - 2x^2 + x^4\right)dx = \left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^{1} = 2\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{16}{15}$$

(b) Let $x = \sin u$, $dx = \cos u\,du$, $1 - x^2 = 1 - \sin^2 u = \cos^2 u$.

$$\int_{-1}^{1}\left(1 - x^2\right)^n\,dx = \int_{-\pi/2}^{\pi/2}\left(\cos^2 u\right)^n \cos u\,du$$
$$= \int_{-\pi/2}^{\pi/2} \cos^{2n+1} u\,du$$
$$= 2\left[\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{(2n)}{(2n+1)}\right] \quad \text{(Wallis's Formula)}$$
$$= 2\left[\frac{2^2\cdot 4^2\cdot 6^2\cdots(2n)^2}{2\cdot 3\cdot 4\cdot 5\cdots(2n)(2n+1)}\right]$$
$$= \frac{2\left(2^{2n}\right)(n!)^2}{(2n+1)!} = \frac{2^{2n+1}(n!)^2}{(2n+1)!}$$

2. (a) $\int_0^1 \ln x\,dx = \lim_{b\to 0^+}\left[x\ln - x\right]_b^1$

$= (-1) - \lim_{b\to 0^+}(b\ln b - b) = -1$

Note: $\lim_{b\to 0^+} b\ln b = \lim_{b\to 0^+}\frac{\ln b}{b^{-1}} = \lim_{b\to 0^+}\frac{1/b}{-1/b^2} = 0$

$\int_0^1 (\ln x)^2\,dx = \lim_{b\to 0^+}\left[x(\ln x)^2 - 2x\ln x + 2x\right]_b^1$

$= 2 - \lim_{b\to 0^+}\left(b(\ln b)^2 - 2b\ln b + 2b\right) = 2$

(b) Note first that $\lim_{b\to 0^+} b(\ln b)^n = 0$ (Mathematical induction).

Also, $\int (\ln x)^{n+1}\,dx = x(\ln x)^{n+1} - (n+1)\int (\ln x)^n\,dx.$

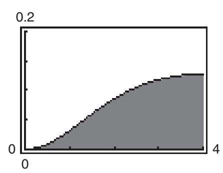
Assume $\int_0^1 (\ln x)^n\,dx = (-1)^n n!.$

Then, $\int_0^1 (\ln x)^{n+1}\,dx = \lim_{b\to 0^+}\left[x(\ln x)^{n+1}\right]_b^1 - (n+1)\int_0^1 (\ln x)^n\,dx = 0 - (n+1)(-1)^n n! = (-1)^{n+1}(n+1)!.$

3. (a) $R < I < T < L$

(b) $S(4) = \frac{4-0}{3(4)}\left[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)\right] \approx \frac{1}{3}\left[4 + 4(2) + 2(1) + 4\left(\frac{1}{2}\right) + \frac{1}{4}\right] \approx 5.417$

4. (a)



Area ≈ 0.2986

(b) Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$, $x^2 + 9 = 9\sec^2\theta$.

$$\int \frac{x^2}{(x^2+9)^{3/2}}\,dx = \int \frac{9\tan^2\theta}{(9\sec^2\theta)^{3/2}}(3\sec^2\theta\,d\theta)$$

$$= \int \frac{\tan^2\theta}{\sec\theta}\,d\theta$$

$$= \int \frac{\sin^2\theta}{\cos\theta}\,d\theta$$

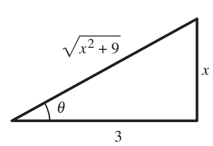
$$= \int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| - \sin\theta + C$$

$$\text{Area} = \int_0^4 \frac{x^2}{(x^2+9)^{3/2}}\,dx = \left[\ln\left|\sec\theta + \tan\theta\right| - \sin\theta\right]_0^{\tan^{-1}(4/3)}$$

$$= \left[\ln\left(\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right) - \frac{x}{\sqrt{x^2+9}}\right]_0^4$$

$$= \ln\left(\frac{5}{3} + \frac{4}{3}\right) - \frac{4}{5} = \ln 3 - \frac{4}{5}$$



(c) $x = 3\sinh u, dx = 3\cosh u\, du, x^2 + 9 = 9\sinh^2 u + 9 = 9\cosh^2 u$

$$A = \int_0^4 \frac{x^2}{(x^2+9)^{3/2}}\,dx = \int_0^{\sinh^{-1}(4/3)} \frac{9\sinh^2 u}{(9\cosh^2 u)^{3/2}}(3\cosh u\,du) = \int_0^{\sinh^{-1}(4/3)} \tanh^2 u\,du$$

$$= \int_0^{\sinh^{-1}(4/3)} (1 - \operatorname{sech}^2 u)\,du = [u - \tanh u]_0^{\text{t}\sinh^{-1}(4/3)}$$

$$= \sinh^{-1}\left(\frac{4}{3}\right) - \tanh\left(\sinh^{-1}\left(\frac{4}{3}\right)\right) = \ln\left(\frac{4}{3} + \sqrt{\frac{16}{9}+1}\right) - \tanh\left[\ln\left(\frac{4}{3} + \sqrt{\frac{16}{9}+1}\right)\right]$$

$$= \ln\left(\frac{4}{3}+\frac{5}{3}\right) - \tanh\left(\ln\left(\frac{4}{3}+\frac{5}{3}\right)\right) = \ln 3 - \tanh(\ln 3)$$

$$= \ln 3 - \frac{3-(1/3)}{3+(1/3)} = \ln 3 - \frac{4}{5}$$

5. $u = \tan\frac{x}{2}, \cos x = \frac{1-u^2}{1+u^2},$

$$2 + \cos x = 2 + \frac{1-u^2}{1+u^2} = \frac{3+u^2}{1+u^2}$$

$$dx = \frac{2\,du}{1+u^2}$$

$$\int_0^{\pi/2} \frac{1}{2+\cos x}\,dx = \int_0^1 \left(\frac{1+u^2}{3+u^2}\right)\left(\frac{2}{1+u^2}\right)du$$

$$= \int_0^1 \frac{2}{3+u^2}\,du$$

$$= \left[2\frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{6}\right) = \frac{\pi\sqrt{3}}{9} \approx 0.6046$$

6. $y = \ln(1-x^2), y' = \frac{-2x}{1-x^2}$

$$1 + (y')^2 = 1 + \frac{4x^2}{(1-x^2)^2}$$

$$= \frac{1 - 2x^2 + x^4 + 4x^2}{(1-x^2)^2}$$

$$= \left(\frac{1+x^2}{1-x^2}\right)^2$$

$$\text{Arc length} = \int_0^{1/2} \sqrt{1+(y')^2}\,dx$$

$$= \int_0^{1/2} \left(\frac{1+x^2}{1-x^2}\right)dx$$

$$= \int_0^{1/2} \left(-1 + \frac{2}{1-x^2}\right)dx$$

$$= \int_0^{1/2} \left(-1 + \frac{1}{x+1} + \frac{1}{1-x}\right)dx$$

$$= [-x + \ln(1+x) - \ln(1-x)]_0^{1/2}$$

$$= \left(-\frac{1}{2} + \ln\frac{3}{2} - \ln\frac{1}{2}\right)$$

$$= -\frac{1}{2} + \ln 3 - \ln 2 + \ln 2$$

$$= \ln 3 - \frac{1}{2} \approx 0.5986$$

7. Let $u = cx, du = c\,dx$.

$$\int_0^b e^{-c^2x^2}\,dx = \int_0^{cb} e^{-u^2}\frac{du}{c} = \frac{1}{c}\int_0^{cb} e^{-u^2}\,du$$

As $b \to \infty, cb \to \infty$. So, $\int_0^\infty e^{-c^2x^2}dx = \frac{1}{c}\int_0^\infty e^{-x^2}\,dx$.

$\bar{x} = 0$ by symmetry.

$$\bar{y} = \frac{M_x}{m} = \frac{2\int_0^\infty \frac{\left(e^{-c^2x^2}\right)}{2}\,dx}{2\int_0^\infty e^{-c^2x^2}\,dx}$$

$$= \left(\frac{1}{2}\right)\frac{\int_0^\infty e^{-2c^2x^2}\,dx}{\int_0^\infty e^{-c^2x^2}\,dx}$$

$$= \left(\frac{1}{2}\right)\frac{\frac{1}{\sqrt{2}c}\int_0^\infty e^{-x^2}\,dx}{\frac{1}{c}\int_0^\infty e^{-x^2}\,dx}$$

$$= \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

So, $(\bar{x}, \bar{y}) = \left(0, \frac{\sqrt{2}}{4}\right)$.

8. $$f'(a)(b-a) - \int_a^b f''(t)(t-b)\,dt = f'(a)(b-a) - \left\{\left[f'(t)(t-b)\right]_a^b - \int_a^b f'(t)\,dt\right\}$$

$$= f'(a)(b-a) + f'(a)(a-b) + \left[f(t)\right]_a^b = f(b) - f(a)$$

$dv = f''(t)\,dt \Rightarrow \quad v = f'(t)$

$u = t - b \quad \Rightarrow du = dt$

9. (a) Let $y = f^{-1}(x), f(y) = x, dx = f'(y)\,dy$.

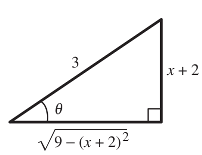
$$\int f^{-1}(x)\,dx = \int yf'(y)\,dy$$

$$= yf(y) - \int f(y)\,dy \qquad \begin{bmatrix} u = y, du = dy \\ dv = f'(y)\,dy, v = f(y) \end{bmatrix}$$

$$= xf^{-1}(x) - \int f(y)\,dy$$

(b) $f^{-1}(x) = \arcsin x = y, f(x) = \sin x$

$$\int \arcsin x\,dx = x\arcsin x - \int \sin y\,dy = x\arcsin x + \cos y + C = x\arcsin x + \sqrt{1-x^2} + C$$



(c) $f(x) = e^x, f^{-1}(x) = \ln x = y \qquad x = 1 \Leftrightarrow y = 0; x = e \Leftrightarrow y = 1$

$$\int_1^e \ln x\,dx = \left[x\ln x\right]_1^e - \int_0^1 e^y dy = e - \left[e^y\right]_0^1 = e - (e-1) = 1$$

10. $x^4 + 1 = (x^2 + ax + b)(x^2 + cx + d)$

$$= x^4 + (a + c)x^3 + (ac + b + d)x^2 + (ad + bc)x + bd$$

$a = -c, b = d = 1, a = \sqrt{2}$

$$x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$$

$$\int_0^1 \frac{1}{x^4 + 1}\,dx = \int_0^1 \frac{Ax + B}{x^2 + \sqrt{2}x + 1}\,dx + \int_0^1 \frac{Cx + D}{x^2 - \sqrt{2}x + 1}\,dx$$

$$= \int_0^1 \frac{\frac{1}{2} + \frac{\sqrt{2}}{4}x}{x^2 + \sqrt{2}x + 1}\,dx - \int_0^1 \frac{-\frac{1}{2} + \frac{\sqrt{2}}{4}x}{x^2 + \sqrt{2}x + 1}\,dx$$

$$= \frac{\sqrt{2}}{4}\Big[\arctan(\sqrt{2}x + 1) + \arctan(\sqrt{2}x - 1)\Big]_0^1 + \frac{\sqrt{2}}{8}\Big[\ln(x^2 + \sqrt{2}x + 1) - \ln(x^2 - \sqrt{2}x + 1)\Big]_0^1$$

$$= \frac{\sqrt{2}}{4}\Big[\arctan(\sqrt{2} + 1) + \arctan(\sqrt{2} - 1)\Big] + \frac{\sqrt{2}}{8}\Big[\ln(2 + \sqrt{2}) - \ln(2 - \sqrt{2})\Big] - \frac{\sqrt{2}}{4}\left[\frac{\pi}{4} - \frac{\pi}{4}\right] - \frac{\sqrt{2}}{8}[0]$$

$$\approx 0.5554 + 0.3116$$

$$\approx 0.8670$$

11. $\dfrac{N(x)}{D(x)} = \dfrac{P_1}{x - c_1} + \dfrac{P_2}{x - c_2} + \cdots + \dfrac{P_n}{x - c_n}$

$N(x) = P_1(x - c_2)(x - c_3)\ldots(x - c_n) + P_2(x - c_1)(x - c_3)\ldots(x - c_n) + \cdots + P_n(x - c_1)(x - c_2)\ldots(x - c_{n-1})$

Let $x = c_1$: $N(c_1) = P_1(c_1 - c_2)(c_1 - c_3)\ldots(c_1 - c_n)$

$$P_1 = \frac{N(c_1)}{(c_1 - c_2)(c_1 - c_3)\ldots(c_1 - c_n)}$$

Let $x = c_2$: $N(c_2) = P_2(c_2 - c_1)(c_2 - c_3)\ldots(c_2 - c_n)$

$$P_2 = \frac{N(c_2)}{(c_2 - c_1)(c_2 - c_3)\ldots(c_2 - c_n)}$$

$\vdots \qquad \vdots$

Let $x = c_n$: $N(c_n) = P_n(c_n - c_1)(c_n - c_2)\ldots(c_n - c_{n-1})$

$$P_n = \frac{N(c_n)}{(c_n - c_1)(c_n - c_2)\ldots(c_n - c_{n-1})}$$

If $D(x) = (x - c_1)(x - c_2)(x - c_3)\ldots(x - c_n)$, then by the Product Rule

$D'(x) = (x - c_2)(x - c_3)\ldots(x - c_n) + (x - c_1)(x - c_3)\ldots(x - c_n) + \cdots + (x - c_1)(x - c_2)(x - c_3)\ldots(x - c_{n-1})$

and

$D'(c_1) = (c_1 - c_2)(c_1 - c_3)\ldots(c_1 - c_n)$

$D'(c_2) = (c_2 - c_1)(c_2 - c_3)\ldots(c_2 - c_n)$

$\vdots$

$D'(c_n) = (c_n - c_1)(c_n - c_2)\ldots(c_n - c_{n-1})$.

So, $P_k = N(c_k)/D'(c_k)$ for $k = 1, 2, \ldots, n$.

12. $\dfrac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x} = \dfrac{P_1}{x} + \dfrac{P_2}{x-1} + \dfrac{P_3}{x+4} + \dfrac{P_4}{x-3} \Rightarrow c_1 = 0, c_2 = 1, c_3 = -4, c_4 = 3$

$N(x) = x^3 - 3x^2 + 1$

$D'(x) = 4x^3 - 26x + 12$

$P_1 = \dfrac{N(0)}{D'(0)} = \dfrac{1}{12}$

$P_2 = \dfrac{N(1)}{D'(1)} = \dfrac{-1}{-10} = \dfrac{1}{10}$

$P_3 = \dfrac{N(-4)}{D'(-4)} = \dfrac{-111}{-140} = \dfrac{111}{140}$

$P_4 = \dfrac{N(3)}{D'(3)} = \dfrac{1}{42}$

So, $\dfrac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x} = \dfrac{1/12}{x} + \dfrac{1/10}{x-1} + \dfrac{111/140}{x+4} + \dfrac{1/42}{x-3}$.

13. (a) Let $x = \dfrac{\pi}{2} - u, dx = du$.

$$I = \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x} dx = \int_{\pi/2}^0 \frac{\sin\left(\frac{\pi}{2} - u\right)}{\cos\left(\frac{\pi}{2} - u\right) + \sin\left(\frac{\pi}{2} - u\right)}(-du) = \int_0^{\pi/2} \frac{\cos u}{\sin u + \cos u} du$$

So,

$$2I = \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x} dx + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} dx = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}.$$

(b) $I = \displaystyle\int_{\pi/2}^0 \frac{\sin^n\left(\frac{\pi}{2} - u\right)}{\cos^n\left(\frac{\pi}{2} - u\right) + \sin^n\left(\frac{\pi}{2} - u\right)}(-du) = \int_0^{\pi/2} \frac{\cos^n u}{\sin^n u + \cos^n u} du$

So, $2I = \displaystyle\int_0^{\pi/2} 1\, dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$.

14. Consider $\displaystyle\int \frac{1}{\ln x} dx$.

Let $u = \ln x, du = \dfrac{1}{x} dx, x = e^u$. Then $\displaystyle\int \frac{1}{\ln x} dx = \int \frac{1}{u} e^u\, du = \int \frac{e^u}{u} du$.

If $\displaystyle\int \frac{1}{\ln x} dx$ were elementary, then $\displaystyle\int \frac{e^u}{u} du$ would be too, which is false.

So, $\displaystyle\int \frac{1}{\ln x} dx$ is not elementary.

15. $s(t) = \int\left[-32t + 12{,}000 \ln \dfrac{50{,}000}{50{,}000 - 400t}\right] dt = -16t^2 + 12{,}000\int\left[\ln 50{,}000 - \ln(50{,}000 - 400t)\right] dt$

$$= 16t^2 + 12{,}000t \ln 50{,}000 - 12{,}000\left[t\ln(50{,}000 - 400t) - \int \frac{-400t}{50{,}000 - 400t}\,dt\right]$$

$$= -16t^2 + 12{,}000t \ln \frac{50{,}000}{50{,}000 - 400t} + 12{,}000t\int\left[1 - \frac{50{,}000}{50{,}000 - 400t}\right]dt$$

$$= -16t^2 + 12{,}000t \ln \frac{50{,}000}{50{,}000 - 400t} + 12{,}000t + 1{,}500{,}000 \ln(50{,}000 - 400t) + C$$

$s(0) = 1{,}500{,}000 \ln 50{,}000 + C = 0$

$C = -1{,}500{,}000 \ln 50{,}000$

$$s(t) = -16t^2 + 12{,}000t\left[1 + \ln\frac{50{,}000}{50{,}000 - 400t}\right] + 1{,}500{,}000 \ln \frac{50{,}000 - 400t}{50{,}000}$$

When $t = 100$, $s(100) \approx 557{,}168.626$ feet.

16. By parts,

$$\int_a^b f(x)g''(x)\,dx = \left[f(x)g'(x)\right]_a^b - \int_a^b f'(x)g'(x)\,dx \qquad \left[u = f(x),\ dv = g''(x)\,dx\right]$$

$$= -\int_a^b f'(x)g'(x)\,dx$$

$$= \left[-f'(x)g(x)\right]_a^b + \int_a^b g(x)f''(x)\,dx \qquad \left[u = f'(x),\ dv = g'(x)\,dx\right]$$

$$= \int_a^b f''(x)g(x)\,dx.$$

17. Let $u = (x - a)(x - b)$, $du = [(x - a) + (x - b)]\,dx$, $dv = f''(x)\,dx$, $v = f'(x)$.

$$\int_a^b (x - a)(x - b)\,dx = \left[(x - a)(x - b)f'(x)\right]_a^b - \int_a^b [(x - a) + (x - b)]f'(x)\,dx$$

$$= -\int_a^b (2x - a - b)f'(x)\,dx \qquad \begin{pmatrix} u = 2x - a - b \\ dv = f'(x)\,dx \end{pmatrix}$$

$$= \left[-(2x - a - b)f(x)\right]_a^b + \int_a^b 2f(x)\,dx = 2\int_a^b f(x)\,dx$$

18.

$$\int_2^\infty \left[\frac{1}{x^5} + \frac{1}{x^{10}} + \frac{1}{x^{15}}\right]dx < \int_2^\infty \frac{1}{x^5 - 1}\,dx < \int_2^\infty \left[\frac{1}{x^5} + \frac{1}{x^{10}} + \frac{2}{x^{15}}\right]dx$$

$$\lim_{b\to\infty}\left[-\frac{1}{4x^4} - \frac{1}{9x^9} - \frac{1}{14x^{14}}\right]_2^b < \int_2^\infty \frac{1}{x^5 - 1}\,dx < \lim_{b\to\infty}\left[-\frac{1}{4x^4} - \frac{1}{9x^9} - \frac{1}{7x^{14}}\right]_2^b$$

$$0.015846 < \int_2^\infty \frac{2}{x^5 - 1}\,dx < 0.015851$$

19. $\dfrac{1}{2}V = \displaystyle\int_0^{\arcsin(c)} \pi(c - \sin x)^2\,dx + \int_{\arcsin(c)}^{\pi/2} \pi(\sin x - c)^2\,dx = \frac{2c^2\pi - 8c + \pi}{4}\pi = f(c)$

$f'(c) = \dfrac{4c\pi - 8}{4}\pi = 0 \Rightarrow c = \dfrac{2}{\pi}$

For $c = 0$, $\dfrac{1}{2}V = \dfrac{\pi^2}{4} \approx 2.4674.$

For $c = 1$, $\dfrac{1}{2}V = \dfrac{\pi}{4}(3\pi - 8) \approx 1.1190.$

For $c = \dfrac{2}{\pi}$, $\dfrac{1}{2}V = \dfrac{\pi^2 - 8}{4} \approx 0.4674.$

(a) Maximum: $c = 0$

(b) Minimum: $c = \dfrac{2}{\pi}$