CHAPTER 7
Applications of Integration

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Section 7.1 Area of a Region Between Two Curves

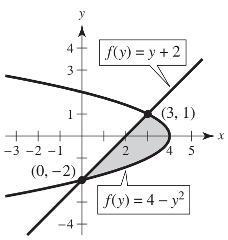
1. In variable x, the area of the region between two graphs is the area under the graph of the top function minus the area under the graph of the bottom function.

2. If $f(x) > g(x)$ on $[a, b]$, then the area is

$$A = \int_a^b \left[f(x) - g(x)\right] dx.$$

3. The points of intersection are used to determine the vertical lines that bound the region.

4. Answers will vary. *Sample answer*:



5. $A = \int_0^6 \left[0 - \left(x^2 - 6x\right)\right] dx = -\int_0^6 \left(x^2 - 6x\right) dx$

6. $A = \int_{-2}^2 \left[(2x + 5) - \left(x^2 + 2x + 1\right)\right] dx$

$= \int_{-2}^2 \left(-x^2 + 4\right) dx$

7. $A = \int_0^3 \left[\left(-x^2 + 2x + 3\right) - \left(x^2 - 4x + 3\right)\right] dx$

$= \int_0^3 \left(-2x^2 + 6x\right) dx$

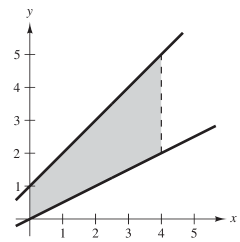
8. $A = \int_0^1 \left(x^2 - x^3\right) dx$

9. $A = 2\int_{-1}^0 3\left(x^3 - x\right) dx = 6\int_{-1}^0 \left(x^3 - x\right) dx$

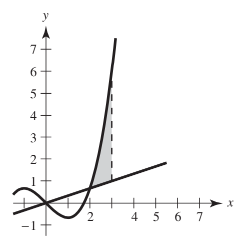
or $-6\int_0^1 \left(x^3 - x\right) dx$

10. $A = 2\int_0^1 \left[(x - 1)^3 - (x - 1)\right] dx$

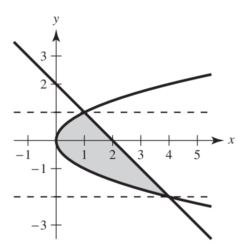
11. $\int_0^4 \left[(x + 1) - \dfrac{x}{2}\right] dx$



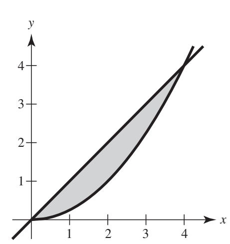
12. $\int_2^3 \left[\left(\dfrac{x^3}{3} - x\right) - \dfrac{x}{3}\right] dx$



13. $\int_{-2}^1 \left[(2 - y) - y^2\right] dy$



14. $\int_0^4 \left(2\sqrt{y} - y\right) dy$

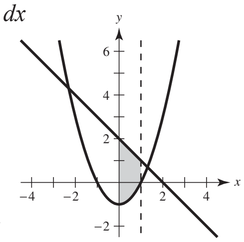


15. $A = \int_0^1 \left[(-x + 2) - (x^2 - 1)\right] dx$

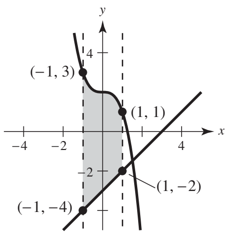
$= \int_0^1 (-x^2 - x + 3)\, dx$

$= \left[\frac{-x^3}{3} - \frac{x^2}{2} + 3x\right]_0^1$

$= \left(-\frac{1}{3} - \frac{1}{2} + 3\right) - 0 = \frac{13}{6}$



16.



$A = \int_{-1}^1 \left[(-x^3 + 2) - (x - 3)\right] dx$

$= \int_{-1}^1 (-x^3 - x + 5)\, dx$

$= \left[\frac{-x^4}{4} - \frac{x^2}{2} + 5x\right]_{-1}^1$

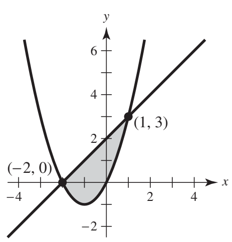
$= \left(-\frac{1}{4} - \frac{1}{2} + 5\right) - \left(-\frac{1}{4} - \frac{1}{2} - 5\right) = 10$

17. The points of intersection are given by:

$x^2 + 2x = x + 2$

$x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$ when $x = -2, 1$



$A = \int_{-2}^1 \left[g(x) - f(x)\right] dx$

$= \int_{-2}^1 \left[(x + 2) - (x^2 + 2x)\right] dx$

$= \left[\frac{-x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^1$

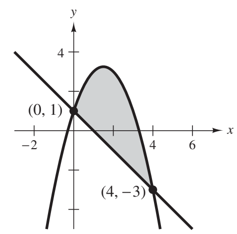
$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \frac{9}{2}$

18. The points of intersection are given by:

$-x^2 + 3x + 1 = -x + 1$

$-x^2 + 4x = 0$

$x(4 - x) = 0$ when $x = 0, 4$



$A = \int_0^4 \left[(-x^2 + 3x + 1) - (1 - x)\right] dx$

$= \int_0^4 (-x^2 + 4x)\, dx$

$= \left[\frac{-x^3}{3} + 2x^2\right]_0^4$

$= -\frac{64}{3} + 32 = \frac{32}{3}$

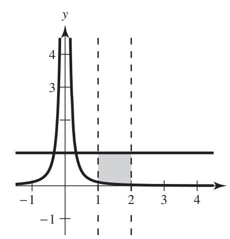
19. $A = \int_1^2 \left[1 - \frac{1}{9x^2}\right] dx$

$= \int_1^2 \left[1 - \frac{1}{9}x^{-2}\right] dx$

$= \left[x + \frac{1}{9x}\right]_1^2$

$= \left(2 + \frac{1}{18}\right) - \left(1 + \frac{1}{9}\right)$

$= \frac{17}{18}$



20. $A = \int_{-3}^{-1} \frac{-4}{x^3}\, dx$

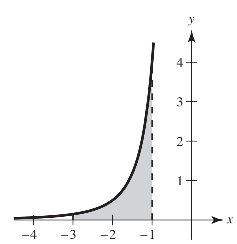
$= -4\int_{-3}^{-1} x^{-3}\, dx$

$= -4\left[\left(\frac{x^{-2}}{-2}\right)\right]_{-3}^{-1}$

$= \left[\frac{2}{x^2}\right]_{-3}^{-1}$

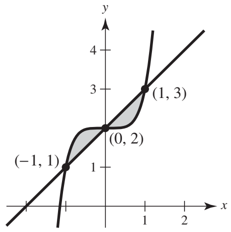
$= \frac{2}{1} - \frac{2}{9}$

$= \frac{16}{9}$



21. The points of intersection are given by:

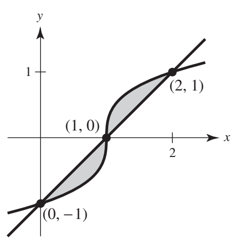
$$x^5 + 2 = x + 2$$
$$x^5 = x \text{ when } x = -1, 0, 1$$



$$A = \int_{-1}^{0} \left[\left(x^5 + 2\right) - (x + 2)\right] dx + \int_{0}^{1} \left[(x + 2) - \left(x^5 + 2\right)\right] dx$$
$$= \int_{-1}^{0} \left(x^5 - x\right) dx + \int_{0}^{1} \left(x - x^5\right) dx$$
$$= \left[\frac{x^6}{6} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^6}{6}\right]_{0}^{1}$$
$$= \left(-\frac{1}{6} + \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{6}\right)$$
$$= \frac{2}{3}$$

22. The points of intersection are given by:

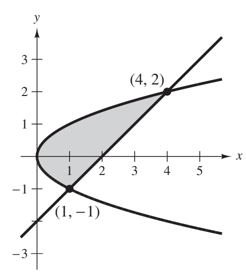
$$\sqrt[3]{x - 1} = x - 1$$
$$x - 1 = (x - 1)^3 = x^3 - 3x^2 + 3x - 1$$
$$x^3 - 3x^2 + 2x = 0$$
$$x\left(x^2 - 3x + 2\right) = 0$$
$$x(x - 2)(x - 1) = 0 \quad \text{when } x = 0, 1, 2$$



$$A = 2\int_{0}^{1} \left[(x - 1) - \sqrt[3]{x - 1}\right] dx$$
$$= 2\left[\frac{x^2}{2} - x - \frac{3}{4}(x - 1)^{4/3}\right]_{0}^{1}$$
$$= 2\left[\left(\frac{1}{2} - 1 - 0\right) - \left(-\frac{3}{4}\right)\right] = \frac{1}{2}$$

23. The points of intersection are given by:

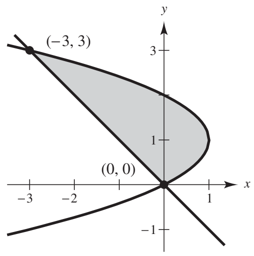
$$y^2 = y + 2$$
$$(y - 2)(y + 1) = 0 \quad \text{when } y = -1, 2$$



$$A = \int_{-1}^{2} \left[g(y) - f(y)\right] dy$$
$$= \int_{-1}^{2} \left[(y + 2) - y^2\right] dy$$
$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$

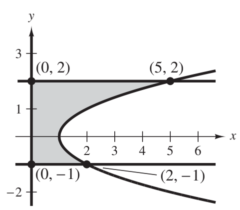
24. The points of intersection are given by:

$$2y - y^2 = -y$$
$$y(y - 3) = 0 \quad \text{when } y = 0, 3$$



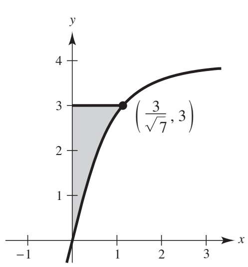
$$A = \int_0^3 [f(y) - g(y)]\,dy$$
$$= \int_0^3 [(2y - y^2) - (-y)]\,dy$$
$$= \int_0^3 (3y - y^2)\,dy$$
$$= \left[\tfrac{3}{2}y^2 - \tfrac{1}{3}y^3\right]_0^3 = \tfrac{9}{2}$$

25.



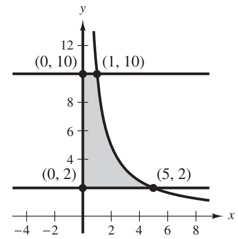
$$A = \int_{-1}^{2} [f(y) - g(y)]\,dy$$
$$= \int_{-1}^{2} [(y^2 + 1) - 0]\,dy$$
$$= \left[\frac{y^3}{3} + y\right]_{-1}^{2} = 6$$

26.



$$A = \int_0^3 [f(y) - g(y)]\,dy$$
$$= \int_0^3 \left[\frac{y}{\sqrt{16 - y^2}} - 0\right] dy$$
$$= -\frac{1}{2}\int_0^3 (16 - y^2)^{-1/2}(-2y)\,dy$$
$$= \left[-\sqrt{16 - y^2}\right]_0^3 = 4 - \sqrt{7} \approx 1.354$$

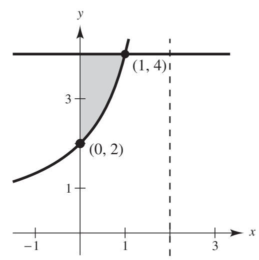
27. $y = \dfrac{10}{x} \Rightarrow x = \dfrac{10}{y}$



$$A = \int_2^{10} \frac{10}{y}\,dy$$
$$= [10 \ln y]_2^{10}$$
$$= 10(\ln 10 - \ln 2)$$
$$= 10 \ln 5 \approx 16.0944$$

28. The point of intersection is given by:

$$\frac{4}{2 - x} = 4$$
$$\frac{4}{2 - x} - 4 = 0 \qquad \text{when } x = 1$$



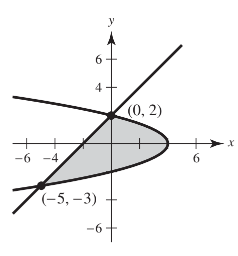
$$A = \int_0^1 \left(4 - \frac{4}{2 - x}\right) dx$$
$$= \left[4x + 4 \ln|2 - x|\right]_0^1$$
$$= 4 - 4 \ln 2$$
$$\approx 1.227$$

29. (a)

$$x = 4 - y^2$$
$$x = y - 2$$
$$4 - y^2 = y - 2$$
$$y^2 + y - 6 = 0$$
$$(y + 3)(y - 2) = 0$$

Intersection points: $(0, 2)$ and $(-5, -3)$

$$A = \int_{-5}^{0} \left[(x + 2) + \sqrt{4 - x}\right] dx + \int_0^4 2\sqrt{4 - x}\,dx$$
$$= \tfrac{61}{6} + \tfrac{32}{3} = \tfrac{125}{6}$$



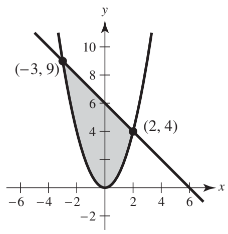
(b) $A = \int_{-3}^{2} [(4 - y^2) - (y - 2)]\,dy = \frac{125}{6}$

(c) The second method is simpler. Explanations will vary.

30. (a) $y = x^2$ and $y = 6 - x$

$x^2 = 6 - x \Rightarrow x^2 + x - 6 = 0 \Rightarrow (x + 3)(x - 2) = 0$

Intersection points: $(2, 4)$ and $(-3, 9)$

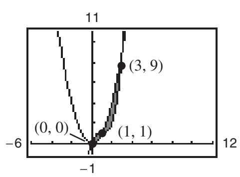


$$A = \int_{-3}^{2}\left[(6 - x) - x^2\right] dx = \frac{125}{6}$$

(b) $A = \int_0^4 2\sqrt{y}\, dy + \int_4^9\left[(6 - y) + \sqrt{y}\right] dy = \frac{32}{3} + \frac{61}{6} = \frac{125}{6}$

(c) The first method is simpler. Explanations will vary.

31. (a)



(b) The points of intersection are given by:

$x^3 - 3x^2 + 3x = x^2$

$x(x - 1)(x - 3) = 0$ when $x = 0, 1, 3$

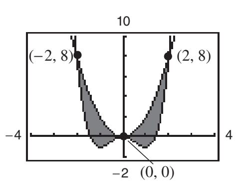
$$A = \int_0^1\left[f(x) - g(x)\right] dx + \int_1^3\left[g(x) - f(x)\right] dx$$

$$= \int_0^1\left[\left(x^3 - 3x^2 + 3x\right) - x^2\right] dx + \int_1^3\left[x^2 - \left(x^3 - 3x^2 + 3x\right)\right] dx$$

$$= \int_0^1\left(x^3 - 4x^2 + 3x\right) dx + \int_1^3\left(-x^3 + 4x^2 - 3x\right) dx = \left[\frac{x^4}{4} - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_0^1 + \left[\frac{-x^4}{4} + \frac{4}{3}x^3 - \frac{3}{2}x^2\right]_1^3 = \frac{5}{12} + \frac{8}{3} = \frac{37}{12}$$

(c) Numerical approximation: $0.417 + 2.667 \approx 3.083$

32. (a)



(b) The points of intersection are given by:

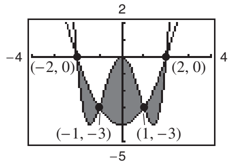
$x^4 - 2x^2 = 2x^2$

$x^2\left(x^2 - 4\right) = 0$ when $x = 0, \pm 2$

$$A = 2\int_0^2\left[2x^2 - \left(x^4 - 2x^2\right)\right] dx = 2\int_0^2\left(4x^2 - x^4\right) dx = 2\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \frac{128}{15}$$

(c) Numerical approximation: 8.533

33. (a) $f(x) = x^4 - 4x^2, \quad g(x) = x^2 - 4$



(b) The points of intersection are given by:

$$x^4 - 4x^2 = x^2 - 4$$
$$x^4 - 5x^2 + 4 = 0$$
$$(x^2 - 4)(x^2 - 1) = 0 \quad \text{when } x = \pm 2, \pm 1$$

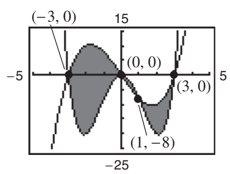
By symmetry:

$$A = 2\int_0^1 \left[(x^4 - 4x^2) - (x^2 - 4)\right] dx + 2\int_1^2 \left[(x^2 - 4) - (x^4 - 4x^2)\right] dx$$
$$= 2\int_0^1 (x^4 - 5x^2 + 4)\, dx + 2\int_1^2 (-x^4 + 5x^2 - 4)\, dx$$
$$= 2\left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right]_0^1 + 2\left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x\right]_1^2$$
$$= 2\left[\frac{1}{5} - \frac{5}{3} + 4\right] + 2\left[\left(-\frac{32}{5} + \frac{40}{3} - 8\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right)\right] = 8$$

(c) Numerical approximation:

$5.067 + 2.933 = 8.0$

34. (a)



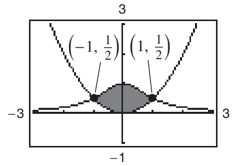
(b) The points of intersection are given by:

$$x^4 - 9x^2 = x^3 - 9x$$
$$x^4 - x^3 - 9x^2 + 9x = 0$$
$$x(x - 3)(x - 1)(x + 3) = 0 \quad \text{when } x = -3, 0, 1, 3$$

$$A = \int_{-3}^0 \left[(x^3 - 9x) - (x^4 - 9x^2)\right] dx + \int_0^1 \left[(x^4 - 9x^2) - (x^3 - 9x)\right] dx + \int_1^3 \left[(x^3 - 9x) - (x^4 - 9x^2)\right] dx$$
$$= \left[\frac{x^4}{4} - \frac{9x^2}{2} - \frac{x^5}{5} + 3x^3\right]_{-3}^0 + \left[\frac{x^5}{5} - 3x^3 - \frac{x^4}{4} + \frac{9x^2}{2}\right]_0^1 + \left[\frac{x^4}{4} - \frac{9x^2}{2} - \frac{x^5}{5} + 3x^3\right]_1^3$$
$$= \frac{1053}{20} + \frac{29}{20} + \frac{68}{5} = \frac{677}{10}$$

(c) Numerical approximation: 67.7

35. (a)



(b) The points of intersection are given by:

$$\frac{1}{1+x^2} = \frac{x^2}{2}$$

$$x^4 + x^2 - 2 = 0$$

$$(x^2+2)(x^2-1) = 0 \quad \text{when } x = \pm 1$$

$$A = 2\int_0^1 \left[f(x) - g(x)\right] dx$$

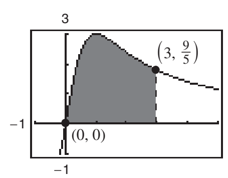
$$= 2\int_0^1 \left[\frac{1}{1+x^2} - \frac{x^2}{2}\right] dx$$

$$= 2\left[\arctan x - \frac{x^3}{6}\right]_0^1$$

$$= 2\left(\frac{\pi}{4} - \frac{1}{6}\right) = \frac{\pi}{2} - \frac{1}{3} \approx 1.237$$

(c) Numerical approximation: 1.237

36. (a)



(b) $A = \int_0^3 \left[\frac{6x}{x^2+1} - 0\right] dx$

$$= \left[3\ln(x^2+1)\right]_0^3$$

$$= 3\ln 10$$

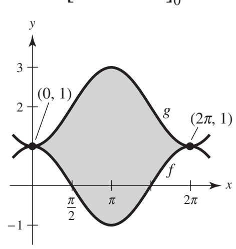
$$\approx 6.908$$

(c) Numerical approximation: 6.908

37. $A = \int_0^{2\pi} \left[(2 - \cos x) - \cos x\right] dx$

$$= 2\int_0^{2\pi} (1 - \cos x)\, dx$$

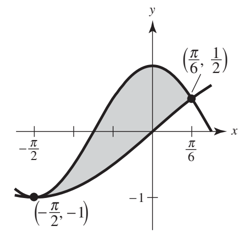
$$= 2\left[x - \sin x\right]_0^{2\pi} = 4\pi \approx 12.566$$



38. $A = \int_{-\pi/2}^{\pi/6} (\cos 2x - \sin x)\, dx$

$$= \left[\frac{1}{2}\sin 2x + \cos x\right]_{-\pi/2}^{\pi/6}$$

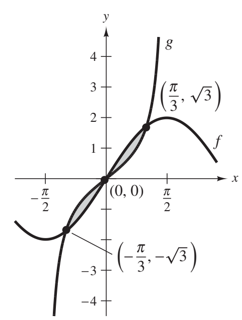
$$= \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}\right) - (0) = \frac{3\sqrt{3}}{4} \approx 1.299$$



39. $A = 2\int_0^{\pi/3} \left[f(x) - g(x)\right] dx$

$$= 2\int_0^{\pi/3} (2\sin x - \tan x)\, dx$$

$$= 2\left[-2\cos x + \ln\left|\cos x\right|\right]_0^{\pi/3} = 2(1 - \ln 2) \approx 0.614$$

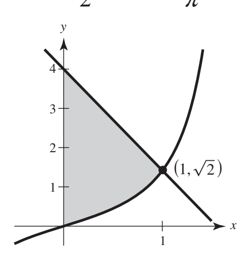


40. $A = \int_0^1 \left[\left(\sqrt{2} - 4\right)x + 4 - \sec\frac{\pi x}{4}\tan\frac{\pi x}{4}\right] dx$

$$= \left[\frac{\sqrt{2}-4}{2}x^2 + 4x - \frac{4}{\pi}\sec\frac{\pi x}{4}\right]_0^1$$

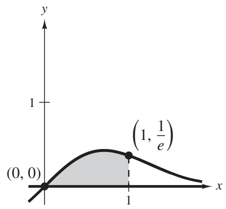
$$= \left(\frac{\sqrt{2}-4}{2} + 4 - \frac{4}{\pi}\sqrt{2}\right) - \left(-\frac{4}{\pi}\right)$$

$$= \frac{\sqrt{2}}{2} + 2 + \frac{4}{\pi}\left(1 - \sqrt{2}\right) \approx 2.1797$$



41. $A = \int_0^1 \left[xe^{-x^2} - 0\right] dx$

$= \left[-\frac{1}{2}e^{-x^2}\right]_0^1 = \frac{1}{2}\left(1 - \frac{1}{e}\right) \approx 0.316$



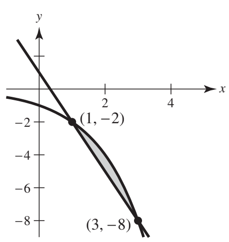
42. $A = \int_1^3 \left[-2^x - (1 - 3x)\right] dx$

$= \int_1^3 \left(3x - 1 - 2^x\right) dx$

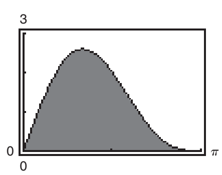
$= \left[\frac{3x^2}{2} - x - \frac{2^x}{\ln 2}\right]_1^3$

$= \left(\frac{27}{2} - 3 - \frac{8}{\ln 2}\right) - \left(\frac{3}{2} - 1 - \frac{2}{\ln 2}\right)$

$= 10 - \frac{6}{\ln 2} \approx 1.344$



43. (a)



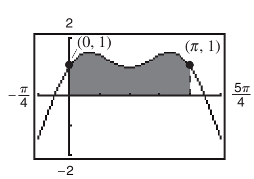
(b) $A = \int_0^\pi (2 \sin x + \sin 2x)\, dx$

$= \left[-2 \cos x - \frac{1}{2} \cos 2x\right]_0^\pi$

$= \left(2 - \frac{1}{2}\right) - \left(-2 - \frac{1}{2}\right) = 4$

(c) Numerical approximation: 4.0

44. (a)

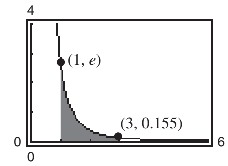


(b) $A = \int_0^\pi (2 \sin x + \cos 2x)\, dx$

$= \left[-2 \cos x + \frac{1}{2} \sin 2x\right]_0^\pi = 4$

(c) Numerical approximation: 4

45. (a)



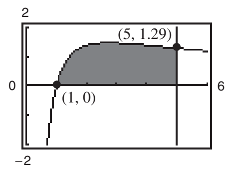
(b) $A = \int_1^3 \frac{1}{x^2} e^{1/x}\, dx$

$= \left[-e^{-1/x}\right]_1^3$

$= e - e^{1/3}$

(c) Numerical approximation: 1.323

46. (a)



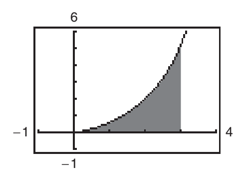
(b) $A = \int_1^5 \frac{4 \ln x}{x}\, dx$

$= \left[2(\ln x)^2\right]_1^5$

$= 2(\ln 5)^2$

(c) Numerical approximation: 5.181

47. (a)



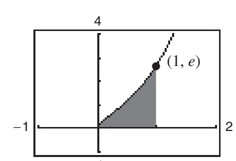
(b) The integral

$A = \int_0^3 \sqrt{\frac{x^3}{4 - x}}\, dx$

does not have an elementary antiderivative.

(c) $A \approx 4.7721$

48. (a)



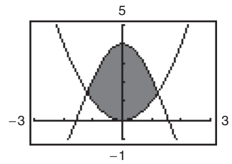
(b) The integral

$A = \int_0^1 \sqrt{x} e^x\, dx$

does not have an elementary antiderivative.

(c) 1.2556

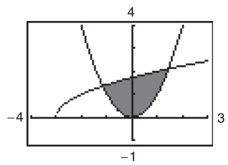
49. (a)



(b) The intersection points are difficult to determine by hand.

(c) Area $= \int_{-c}^{c}\left[4\cos x - x^2\right]dx \approx 6.3043$ where $c \approx 1.201538$.

50. (a)



(b) The intersection points are difficult to determine.

(c) Intersection points: $(-1.164035, 1.3549778)$ and $(1.4526269, 2.1101248)$

$$A = \int_{-1.164035}^{1.4526269}\left[\sqrt{3+x} - x^2\right]dx \approx 3.0578$$

51.
$$\begin{aligned} A &= \int_0^1 (y_1 - y_3)\,dx + \int_1^2 (y_2 - y_3)\,dx \\ &= \int_0^1 \left[(x^2+2) - (2-x)\right]dx + \int_1^2 \left[(4-x^2) - (2-x)\right]dx \\ &= \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 + \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_1^2 \\ &= \left(\frac{1}{3} + \frac{1}{2}\right) + \left(-\frac{8}{3} + 2 + 4\right) - \left(-\frac{1}{3} + \frac{1}{2} + 2\right) \\ &= 2 \end{aligned}$$

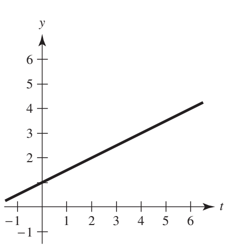
52.
$$\begin{aligned} A &= \int_0^{\pi/4} (y_3 - y_2)\,dx + \int_{\pi/4}^{\pi/2} (y_3 - y_1)\,dx \\ &= \int_0^{\pi/4} \left[(\sin x + \cos x) - \cos x\right]dx + \int_{\pi/4}^{\pi/2} \left[(\sin x + \cos x) - \sin x\right]dx \\ &= \int_0^{\pi/4} \sin x\,dx + \int_{\pi/4}^{\pi/2} \cos x\,dx \\ &= \left[-\cos x\right]_0^{\pi/4} + \left[\sin x\right]_{\pi/4}^{\pi/2} \\ &= \left(-\frac{\sqrt{2}}{2} + 1\right) + \left(1 - \frac{\sqrt{2}}{2}\right) \\ &= 2 - \sqrt{2} \end{aligned}$$

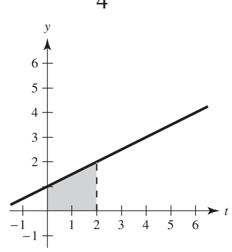
53. $F(x) = \int_0^x \left(\frac{1}{2}t + 1\right)dt = \left[\frac{t^2}{4} + t\right]_0^x = \frac{x^2}{4} + x$

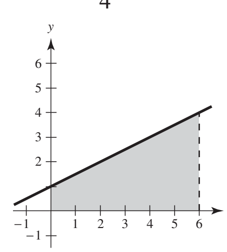
(a) $F(0) = 0$

(b) $F(2) = \frac{2^2}{4} + 2 = 3$

(c) $F(6) = \frac{6^2}{4} + 6 = 15$

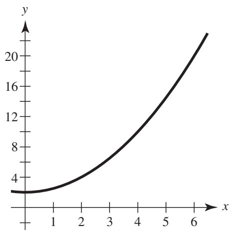




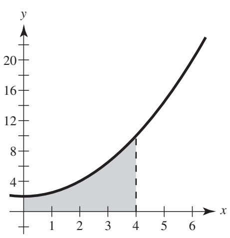


54. $F(x) = \int_0^x \left(\frac{1}{2}t^2 + 2\right) dt = \left[\frac{1}{6}t^3 + 2t\right]_0^x = \frac{x^3}{6} + 2x$

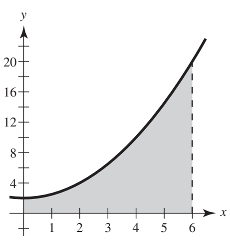
(a) $F(0) = 0$



(b) $F(4) = \frac{4^3}{6} + 2(4) = \frac{56}{3}$

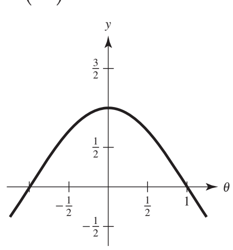


(c) $F(6) = 36 + 12 = 48$

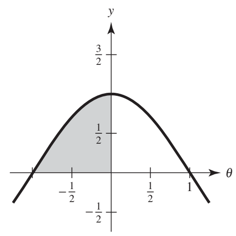


55. $F(\alpha) = \int_{-1}^{\alpha} \cos\frac{\pi\theta}{2}\,d\theta = \left[\frac{2}{\pi}\sin\frac{\pi\theta}{2}\right]_{-1}^{\alpha} = \frac{2}{\pi}\sin\frac{\pi\alpha}{2} + \frac{2}{\pi}$

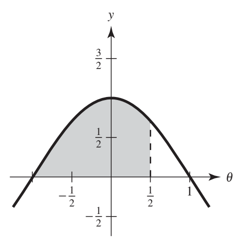
(a) $F(-1) = 0$



(b) $F(0) = \frac{2}{\pi} \approx 0.6366$

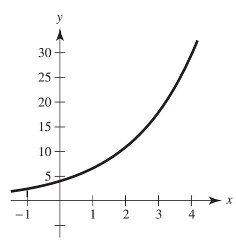


(c) $F\left(\frac{1}{2}\right) = \frac{2 + \sqrt{2}}{\pi} \approx 1.0868$

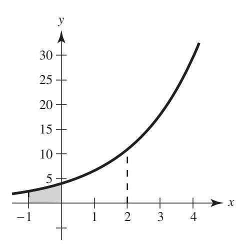


56. $F(y) = \int_{-1}^{y} 4e^{x/2}\,dx = \left[8e^{x/2}\right]_{-1}^{y} = 8e^{y/2} - 8e^{-1/2}$

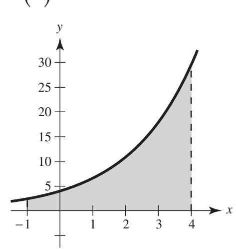
(a) $F(-1) = 0$



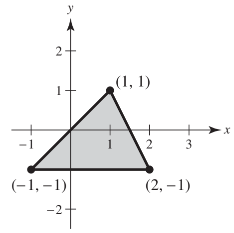
(b) $F(0) = 8 - 8e^{-1/2} \approx 3.1478$



(c) $F(4) = 8e^2 - 8e^{-1/2} \approx 54.2602$

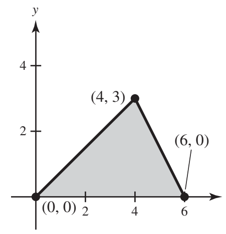


57.

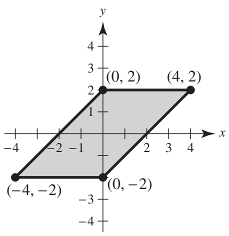


$$A = \int_{-1}^{1} (x + 1)\,dx + \int_{1}^{2} (-2x + 3 + 1)\,dx$$
$$= \left[\frac{x^2}{2} + x\right]_{-1}^{1} + \left[-x^2 + 4x\right]_{1}^{2}$$
$$= \left(\frac{1}{2} + 1\right) - \left(\frac{1}{2} - 1\right) + (-4 + 8) - (-1 + 4)$$
$$= 2 + 1 = 3$$

58. $A = \int_{0}^{4} \frac{3}{4}x\,dx + \int_{4}^{6} \left(9 - \frac{3}{2}x\right) dx$
$$= \left[\frac{3x^2}{8}\right]_{0}^{4} + \left[9x - \frac{3x^2}{4}\right]_{4}^{6}$$
$$= 6 + (54 - 27) - (36 - 12)$$
$$= 6 + 3 = 9$$



59.

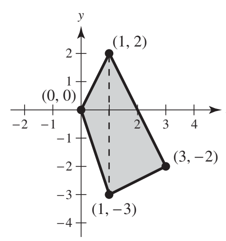


Left boundary line: $y = x + 2 \Leftrightarrow x = y - 2$

Right boundary line: $y = x - 2 \Leftrightarrow x = y + 2$

$$A = \int_{-2}^{2} \left[(y + 2) - (y - 2)\right] dy$$
$$= \int_{-2}^{2} 4\,dy = \left[4y\right]_{-2}^{2} = 8 - (-8) = 16$$

60. $A = \int_{0}^{1} \left[2x - (-3x)\right] dx + \int_{1}^{3} \left[(-2x + 4) - \left(\frac{1}{2}x - \frac{7}{2}\right)\right] dx$
$$= \int_{0}^{1} 5x\,dx + \int_{1}^{3} \left(-\frac{5}{2}x + \frac{15}{2}\right) dx$$
$$= \left[\frac{5x^2}{2}\right]_{0}^{1} + \left[-\frac{5x^2}{4} + \frac{15}{2}x\right]_{1}^{3}$$
$$= \frac{5}{2} + \left(-\frac{45}{4} + \frac{45}{2} + \frac{5}{4} - \frac{15}{2}\right)$$
$$= \frac{15}{2}$$



61. $f(x) = 2x^3 - 1$

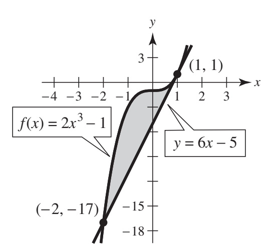
$f'(x) = 6x^2$

At $(1, 1)$, $f'(1) = 6$.

Tangent line: $y - 1 = 6(x - 1)$ or $y = 6x - 5$

The tangent line intersects $f(x) = 2x^3 - 1$ at $(-2, -17)$.

$$A = \int_{-2}^{1} \left[(2x^3 - 1) - (6x - 5)\right] dx$$
$$= \left[\frac{x^4}{2} - 3x^2 + 4x\right]_{-2}^{1}$$
$$= \left(\frac{1}{2} - 3 + 4\right) - (8 - 12 - 8)$$
$$= \frac{27}{2}$$



62. $f(x) = x - x^3$

$f'(x) = 1 - 3x^2$

At $(-1, 0)$, $f'(-1) = -2$.

Tangent line: $y = -2(x + 1)$ or $y = -2x - 2$

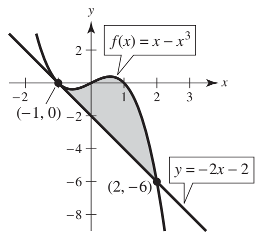
The tangent line intersects $f(x) = x - x^3$ at $(2, -6)$.

$$A = \int_{-1}^{2} \left[\left(x - x^3\right) - \left(-2x - 2\right)\right] dx$$

$$= \left[\frac{-x^4}{4} + \frac{3x^2}{2} + 2x\right]_{-1}^{2}$$

$$= (-4 + 6 + 4) - \left(-\frac{1}{4} + \frac{3}{2} - 2\right)$$

$$= \frac{27}{4}$$



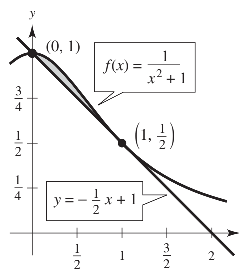
63. $f(x) = \dfrac{1}{x^2 + 1}$

$f'(x) = -\dfrac{2x}{\left(x^2 + 1\right)^2}$

At $\left(1, \dfrac{1}{2}\right)$, $f'(1) = -\dfrac{1}{2}$.

Tangent line: $y - \dfrac{1}{2} = -\dfrac{1}{2}(x - 1)$ or $y = -\dfrac{1}{2}x + 1$

The tangent line intersects $f(x) = \dfrac{1}{x^2 + 1}$ at $(0, 1)$.



$$A = \int_0^1 \left[\frac{1}{x^2 + 1} - \left(-\frac{1}{2}x + 1\right)\right] dx = \left[\arctan x + \frac{x^2}{4} - x\right]_0^1 = \frac{\pi - 3}{4} \approx 0.0354$$

64. $y = \dfrac{2}{1 + 4x^2}$, $\left(\dfrac{1}{2}, 1\right)$

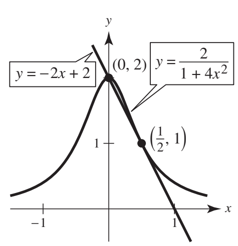
$y' = \dfrac{-16x}{\left(1 + 4x^2\right)^2}$

$y'\left(\dfrac{1}{2}\right) = \dfrac{-8}{2^2} = -2$

Tangent line: $y - 1 = -2\left(x - \dfrac{1}{2}\right)$

$y = -2x + 2$

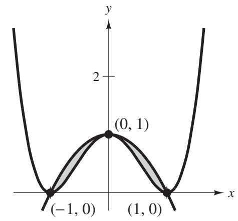
The tangent line intersects $y = \dfrac{2}{1 + 4x^2}$ at $(0, 2)$.



$$A = \int_0^{1/2} \left[\frac{2}{1 + 4x^2} - (-2x + 2)\right] dx = \left[\arctan(2x) + x^2 - 2x\right]_0^{1/2} = \arctan(1) + \frac{1}{4} - 1 = \frac{\pi}{4} - \frac{3}{4} \approx 0.0354$$

65. $x^4 - 2x^2 + 1 \le 1 - x^2$ on $[-1, 1]$

$$A = \int_{-1}^{1} \left[\left(1 - x^2\right) - \left(x^4 - 2x^2 + 1\right)\right] dx$$
$$= \int_{-1}^{1} \left(x^2 - x^4\right) dx$$
$$= \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1} = \frac{4}{15}$$



You can use a single integral because $x^4 - 2x^2 + 1 \le 1 - x^2$ on $[-1, 1]$.

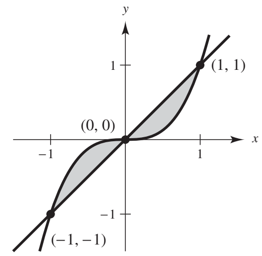
66. $x^3 \ge x$ on $[-1, 0]$, $x^3 \le x$ on $[0, 1]$

Both functions symmetric to origin.

$$\int_{-1}^{0} \left(x^3 - x\right) dx = -\int_{0}^{1} \left(x^3 - x\right) dx$$

Thus, $\int_{-1}^{1} \left(x^3 - x\right) dx = 0.$

$$A = 2\int_{0}^{1} \left(x - x^3\right) dx = 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} = \frac{1}{2}$$



67. (a) $\int_{0}^{5} \left[v_1(t) - v_2(t)\right] dt = 10$ means that Car 1 traveled 10 more meters than Car 2 on the interval $0 \le t \le 5$.

$\int_{0}^{10} \left[v_1(t) - v_2(t)\right] dt = 30$ means that Car 1 traveled 30 more meters than Car 2 on the interval $0 \le t \le 10$.

$\int_{20}^{30} \left[v_1(t) - v_2(t)\right] dt = -5$ means that Car 2 traveled 5 more meters than Car 1 on the interval $20 \le t \le 30$.

(b) No, it is not possible because you do not know the initial distance between the cars.

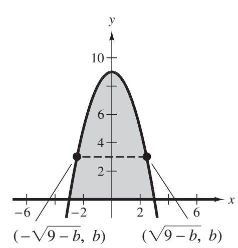
(c) At $t = 10$, Car 1 is ahead by 30 meters.

(d) At $t = 20$, Car 1 is ahead of Car 2 by 13 meters. From part (a), at $t = 30$, Car 1 is ahead by $13 - 5 = 8$ meters.

68. (a) The area between the two curves represents the difference between the accumulated deficit under the two plans.

(b) Proposal 2 is better because the cumulative deficit (the area under the curve) is less.

69.
$$A = \int_{-3}^{3} \left(9 - x^2\right) dx = 36$$
$$\int_{-\sqrt{9-b}}^{\sqrt{9-b}} \left[\left(9 - x^2\right) - b\right] dx = 18$$
$$\int_{0}^{\sqrt{9-b}} \left[(9 - b) - x^2\right] dx = 9$$
$$\left[(9 - b)x - \frac{x^3}{3}\right]_{0}^{\sqrt{9-b}} = 9$$
$$\frac{2}{3}(9 - b)^{3/2} = 9$$
$$(9 - b)^{3/2} = \frac{27}{2}$$
$$9 - b = \frac{9}{\sqrt[3]{4}}$$
$$b = 9 - \frac{9}{\sqrt[3]{4}} \approx 3.330$$



70. $A = 2\int_0^9 (9 - x)\,dx = 2\left[9x - \dfrac{x^2}{2}\right]_0^9 = 81$

$$2\int_0^{9-b} \left[(9 - x) - b\right] dx = \frac{81}{2}$$

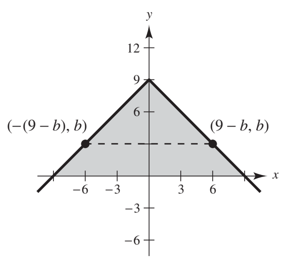
$$2\int_0^{9-b} \left[(9 - b) - x\right] dx = \frac{81}{2}$$

$$2\left[(9 - b)x - \frac{x^2}{2}\right]_0^{9-b} = \frac{81}{2}$$

$$(9 - b)(9 - b) = \frac{81}{2}$$

$$9 - b = \frac{9}{\sqrt{2}}$$

$$b = 9 - \frac{9}{\sqrt{2}} \approx 2.636$$



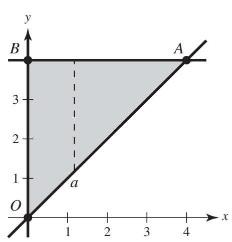
71. Area of triangle OAB is $\frac{1}{2}(4)(4) = 8$.

$$4 = \int_0^a (4 - x)\,dx = \left[4x - \frac{x^2}{2}\right]_0^a = 4a - \frac{a^2}{2}$$

$a^2 - 8a + 8 = 0$

$a = 4 \pm 2\sqrt{2}$

Because $0 < a < 4$, select $a = 4 - 2\sqrt{2} \approx 1.172$.



72. Total area $= \int_{-2}^{2} \left(4 - y^2\right) dy = 2\int_0^2 \left(4 - y^2\right) dy$

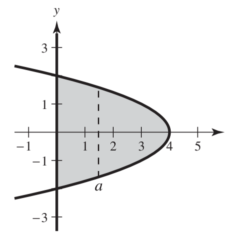
$$= 2\left[4y - \frac{y^3}{3}\right]_0^2 = 2\left[8 - \frac{8}{3}\right] = \frac{32}{3}$$

$$\frac{16}{3} = 2\int_a^4 \sqrt{4 - x}\,dx = -\frac{4}{3}(4 - x)^{3/2}\Big]_a^4 = \frac{4}{3}(4 - a)^{3/2}$$

$4 = (4 - a)^{3/2}$

$4^{2/3} = 4 - a$

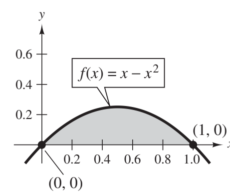
$a = 4 - 4^{2/3} \approx 1.48$



73. $\displaystyle\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \left(x_i - x_i^2\right) \Delta x$

where $x_i = \dfrac{i}{n}$ and $\Delta x = \dfrac{1}{n}$ is the same as

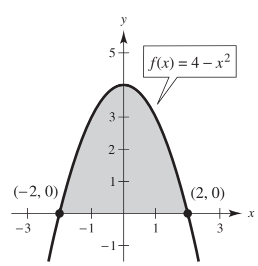
$$\int_0^1 \left(x - x^2\right) dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$



74. $\displaystyle\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \left(4 - x_i^2\right) \Delta x$

where $x_i = -2 + \dfrac{4i}{n}$ and $\Delta x = \dfrac{4}{n}$ is the same as

$$\int_{-2}^{2} \left(4 - x^2\right) dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}.$$



75. R_1 projects the greater revenue because the area under the curve is greater.

$$\int_0^5 \left[(7.21 + 0.58t) - (7.21 + 0.45t)\right] dt$$

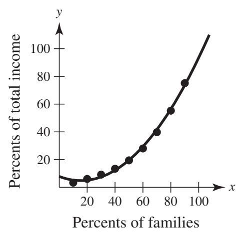
$$= \int_0^5 0.13t\,dt = \left[\frac{0.13t^2}{2}\right]_0^5 = \$1.625 \text{ million}$$

76. R_1 projects the greater revenue because the area under the curve is greater.

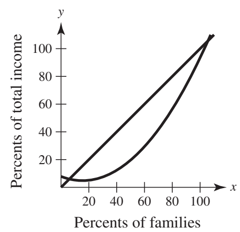
$$\int_0^5 \left[\left(7.21 + 0.26t + 0.02t^2\right) - \left(7.21 + 0.1t + 0.01t^2\right)\right] dt$$
$$= \int_0^5 \left(0.01t^2 + 0.16t\right) dt$$
$$= \left[\frac{0.01t^3}{3} + \frac{0.16t^2}{2}\right]_0^5 \approx \$2.417 \text{ million}$$

77. (a) $y_1 = 0.0124x^2 - 0.385x + 7.85$

(b)



(c)



(d) Income inequality $= \int_0^{100} \left[x - y_1\right] dx \approx 2006.7$

78. 5%: $P_1 = 15.9e^{0.05t}$ (in millions)

3.5%: $P_2 = 15.9e^{0.035t}$ (in millions)

Difference in profits over 5 years:

$$\int_0^5 (P_1 - P_2)\, dt = \int_0^5 15.9\left(e^{0.05t} - e^{0.035t}\right) dt = 15.9\left[\frac{e^{0.05t}}{0.05} - \frac{e^{0.035t}}{0.035}\right]_0^5 \approx \$3.44 \text{ million}$$

79. (a) $A = 2\left[\int_0^5 \left(1 - \frac{1}{3}\sqrt{5 - x}\right) dx + \int_5^{5.5} (1 - 0)\, dx\right]$

$$= 2\left(\left[x + \frac{2}{9}(5 - x)^{3/2}\right]_0^5 + \left[x\right]_5^{5.5}\right)$$
$$= 2\left(5 - \frac{10\sqrt{5}}{9} + 5.5 - 5\right) \approx 6.031 \text{ m}^2$$

(b) $V = 2A \approx 2(6.031) \approx 12.062 \text{ m}^3$

(c) $5000\,V \approx 5000(12.062) = 60{,}310$ pounds

80. The curves intersect at the point where the slope of y_2 equals that of y_1, 1.

$$y_2 = 0.08x^2 + k \Rightarrow y_2' = 0.16x = 1 \Rightarrow x = \frac{1}{0.16} = 6.25$$

(a) The value of k is given by

$$y_1 = y_2$$
$$6.25 = (0.08)(6.25)^2 + k$$
$$k = 3.125.$$

(b) $\text{Area} = 2\int_0^{6.25} (y_2 - y_1)\, dx$

$$= 2\int_0^{6.25} \left(0.08x^2 + 3.125 - x\right) dx$$
$$= 2\left[\frac{0.08x^3}{3} + 3.125x - \frac{x^2}{2}\right]_0^{6.25}$$
$$= 2(6.510417) \approx 13.02083$$

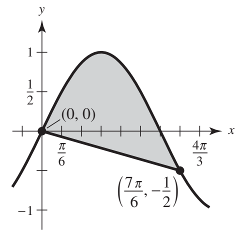
81. Line: $y = \dfrac{-3}{7\pi}x$

$$A = \int_0^{7\pi/6}\left[\sin x + \frac{3x}{7\pi}\right]dx$$

$$= \left[-\cos x + \frac{3x^2}{14\pi}\right]_0^{7\pi/6}$$

$$= \frac{\sqrt{3}}{2} + \frac{7\pi}{24} + 1$$

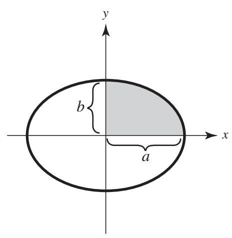
$$\approx 2.7823$$



82. $A = 4\int_0^a b\sqrt{1 - \dfrac{x^2}{a^2}}\,dx = \dfrac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$

$\int_0^a \sqrt{a^2 - x^2}\,dx$ is the area of $\dfrac{1}{4}$ of a circle $= \dfrac{\pi a^2}{4}$.

So, $A = \dfrac{4b}{a}\left(\dfrac{\pi a^2}{4}\right) = \pi ab.$



83. True. The region has been shifted C units upward (if $C > 0$), or C units downward (if $C < 0$).

84. True. This is a property of integrals.

85. False. Let $f(x) = x$ and $g(x) = 2x - x^2$, f and g intersect at $(1, 1)$, the midpoint of $[0, 2]$, but

$$\int_a^b \left[f(x) - g(x)\right]dx = \int_0^2 \left[x - \left(2x - x^2\right)\right]dx = \tfrac{2}{3} \neq 0.$$

86. True. The area under $f(x)$ between 0 and 1 is $\frac{1}{6}$. The curves intersect at $x = \frac{1}{2}^{1/3}$, and the area between $y = \left(1 - \frac{1}{2}^{1/3}\right)x$ and f on the interval $\left[0, \frac{1}{2}^{1/3}\right]$ is $\frac{1}{12}$.

87. You want to find c such that:

$$\int_0^b \left[\left(2x - 3x^3\right) - c\right]dx = 0$$

$$\left[x^2 - \tfrac{3}{4}x^4 - cx\right]_0^b = 0$$

$$b^2 - \tfrac{3}{4}b^4 - cb = 0$$

But, $c = 2b - 3b^3$ because (b, c) is on the graph.

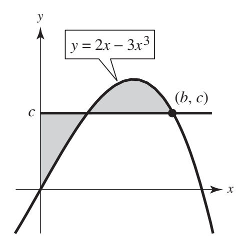
$$b^2 - \tfrac{3}{4}b^4 - \left(2b - 3b^3\right)b = 0$$

$$4 - 3b^2 - 8 + 12b^2 = 0$$

$$9b^2 = 4$$

$$b = \tfrac{2}{3}$$

$$c = \tfrac{4}{9}$$



Section 7.2 Volume: The Disk Method

1. Integrate the square of the radius of the disk over the interval, and then multiply by π.

2. The washer method requires subtracting two disk-method integrals.

3. You need more than one integral when the solid of revolution is formed by two or more distinct solids.

4. If the cross section has area $A(x)$ taken perpendicular to the x-axis, then the volume is Volume $= \int_a^b A(x)\,dx$. If the cross section has area $A(y)$ taken perpendicular to the y-axis, then the volume is Volume $= \int_c^d A(y)\,dy$.

5. $V = \pi\int_1^4 \left(\sqrt{x}\right)^2 dx = \pi\int_1^4 x\,dx = \pi\left[\dfrac{x^2}{2}\right]_1^4 = \dfrac{15\pi}{2}$

6. $V = \pi\int_0^1 (-x + 1)^2\,dx = \pi\int_0^1 \left(x^2 - 2x + 1\right)dx$

$$= \pi\left[\frac{x^3}{3} - x^2 + x\right]_0^1$$

$$= \frac{\pi}{3}$$

7. $V = \pi\int_0^1\left[\left(x^2\right)^2 - \left(x^5\right)^2\right] dx$

$= \pi\int_0^1\left(x^4 - x^{10}\right) dx$

$= \pi\left[\frac{x^5}{5} - \frac{x^{11}}{11}\right]_0^1$

$= \pi\left(\frac{1}{5} - \frac{1}{11}\right) = \frac{6\pi}{55}$

8. $2 = 4 - \frac{x^2}{4}$

$8 = 16 - x^2$

$x^2 = 8$

$x = \pm 2\sqrt{2}$

$V = \pi\int_{-2\sqrt{2}}^{2\sqrt{2}}\left[\left(4 - \frac{x^2}{4}\right)^2 - (2)^2\right] dx$

$= 2\pi\int_0^{2\sqrt{2}}\left[\frac{x^4}{16} - 2x^2 + 12\right] dx$

$= 2\pi\left[\frac{x^5}{80} - \frac{2x^3}{3} + 12x\right]_0^{2\sqrt{2}}$

$= 2\pi\left[\frac{128\sqrt{2}}{80} - \frac{32\sqrt{2}}{3} + 24\sqrt{2}\right]$

$= \frac{448\sqrt{2}}{15}\pi \approx 132.69$

9. $y = x^2 \Rightarrow x = \sqrt{y}$

$V = \pi\int_0^4\left(\sqrt{y}\right)^2 dy = \pi\int_0^4 y\, dy$

$= \pi\left[\frac{y^2}{2}\right]_0^4 = 8\pi$

10. $y = \sqrt{16 - x^2} \Rightarrow x = \sqrt{16 - y^2}$

$V = \pi\int_0^4\left(\sqrt{16 - y^2}\right)^2 dy = \pi\int_0^4\left(16 - y^2\right) dy$

$= \pi\left[16y - \frac{y^3}{3}\right]_0^4 = \frac{128\pi}{3}$

11. $y = x^{2/3} \Rightarrow x = y^{3/2}$

$V = \pi\int_0^1\left(y^{3/2}\right)^2 dy = \pi\int_0^1 y^3\, dy = \pi\left[\frac{y^4}{4}\right]_0^1 = \frac{\pi}{4}$

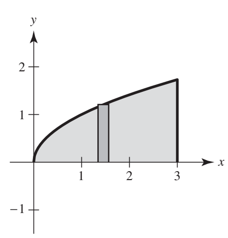
12. $V = \pi\int_1^4\left(-y^2 + 4y\right)^2 dy = \pi\int_1^4\left(y^4 - 8y^3 + 16y^2\right) dy$

$= \pi\left[\frac{y^5}{5} - 2y^4 + \frac{16y^3}{3}\right]_1^4 = \frac{459\pi}{15} = \frac{153\pi}{5}$

13. $y = \sqrt{x},\ y = 0,\ x = 3$

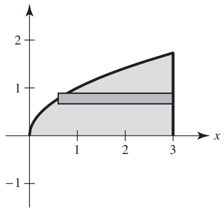
(a) $R(x) = \sqrt{x},\ r(x) = 0$

$V = \pi\int_0^3\left(\sqrt{x}\right)^2 dx = \pi\int_0^3 x\, dx = \pi\left[\frac{x^2}{2}\right]_0^3 = \frac{9\pi}{2}$



(b) $R(y) = 3,\ r(y) = y^2$

$V = \pi\int_0^{\sqrt{3}}\left[3^2 - \left(y^2\right)^2\right] dy = \pi\int_0^{\sqrt{3}}\left(9 - y^4\right) dy = \pi\left[9y - \frac{y^5}{5}\right]_0^{\sqrt{3}} = \pi\left[9\sqrt{3} - \frac{9}{5}\sqrt{3}\right] = \frac{36\sqrt{3}\pi}{5}$

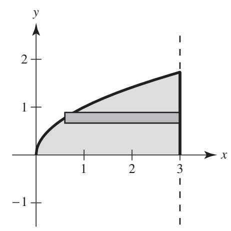


(c) $R(y) = 3 - y^2,\ r(y) = 0$

$$V = \pi\int_0^{\sqrt{3}} \left(3 - y^2\right)^2\, dy = \pi\int_0^{\sqrt{3}} \left(9 - 6y^2 + y^4\right) dy$$

$$= \pi\left[9y - 2y^3 + \frac{y^5}{5}\right]_0^{\sqrt{3}} = \pi\left[9\sqrt{3} - 6\sqrt{3} + \frac{9\sqrt{3}}{5}\right]$$

$$= \frac{24\sqrt{3}\pi}{5}$$

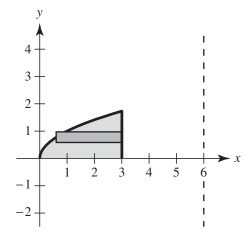


(d) $R(y) = 3 + \left(3 - y^2\right) = 6 - y^2,\ r(y) = 3$

$$V = \pi\int_0^{\sqrt{3}} \left[\left(6 - y^2\right)^2 - 3^2\right] dy = \pi\int_0^{\sqrt{3}} \left(y^4 - 12y^2 + 27\right) dy$$

$$= \pi\left[\frac{y^5}{5} - 4y^3 + 27y\right]_0^{\sqrt{3}} = \pi\left[\frac{9\sqrt{3}}{5} - 12\sqrt{3} + 27\sqrt{3}\right]$$

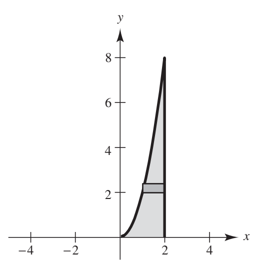
$$= \frac{84\sqrt{3}\pi}{5}$$



14. $y = 2x^2,\ y = 0,\ x = 2$

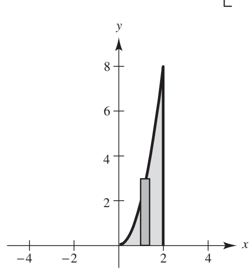
(a) $R(y) = 2,\ r(y) = \sqrt{y/2}$

$$V = \pi\int_0^8 \left(4 - \frac{y}{2}\right) dy = \pi\left[4y - \frac{y^2}{4}\right]_0^8 = 16\pi$$



(b) $R(x) = 2x^2,\ r(x) = 0$

$$V = \pi\int_0^2 4x^4\, dx = \pi\left[\frac{4x^5}{5}\right]_0^2 = \frac{128\pi}{5}$$



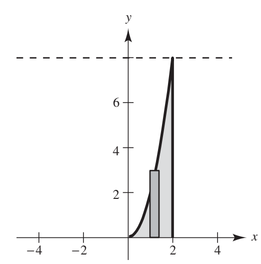
(c) $R(x) = 8,\ r(x) = 8 - 2x^2$

$$V = \pi\int_0^2 \left[64 - \left(64 - 32x^2 + 4x^4\right)\right] dx$$

$$= \pi\int_0^2 \left(32x^2 - 4x^4\right) dx = 4\pi\int_0^2 \left(8x^2 - x^4\right) dx$$

$$= 4\pi\left[\frac{8}{3}x^3 - \frac{1}{5}x^5\right]_0^2$$

$$= \frac{896\pi}{15}$$

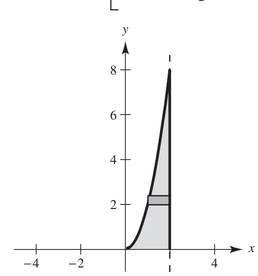


(d) $R(y) = 2 - \sqrt{y/2},\ r(y) = 0$

$$V = \pi\int_0^8 \left(2 - \sqrt{\frac{y}{2}}\right)^2 dy$$

$$= \pi\int_0^8 \left(4 - 4\sqrt{\frac{y}{2}} + \frac{y}{2}\right) dy$$

$$= \pi\left[4y - \frac{4\sqrt{2}}{3}y^{3/2} + \frac{y^2}{4}\right]_0^8 = \frac{16\pi}{3}$$



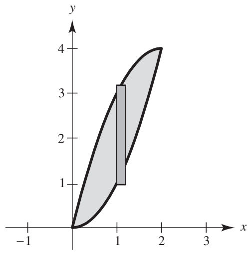
15. $y = x^2,\ y = 4x - x^2$ intersect at $(0, 0)$ and $(2, 4)$.

(a) $R(x) = 4x - x^2,\ r(x) = x^2$

$$V = \pi\int_0^2 \left[\left(4x - x^2\right)^2 - x^4\right] dx$$

$$= \pi\int_0^2 \left(16x^2 - 8x^3\right) dx$$

$$= \pi\left[\frac{16}{3}x^3 - 2x^4\right]_0^2 = \frac{32\pi}{3}$$

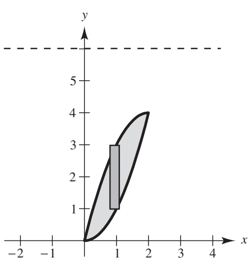


(b) $R(x) = 6 - x^2,\ r(x) = 6 - \left(4x - x^2\right)$

$$V = \pi\int_0^2 \left[\left(6 - x^2\right)^2 - \left(6 - 4x + x^2\right)^2\right] dx$$

$$= 8\pi\int_0^2 \left(x^3 - 5x^2 + 6x\right) dx$$

$$= 8\pi\left[\frac{x^4}{4} - \frac{5}{3}x^3 + 3x^2\right]_0^2 = \frac{64\pi}{3}$$



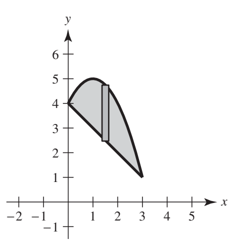
16. $y = 4 + 2x - x^2,\ y = 4 - x$ intersect at $(0, 4)$ and $(3, 1)$.

(a) $R(x) = 4 + 2x - x^2,\ r(x) = 4 - x$

$$V = \pi\int_0^3 \left[\left(4 + 2x - x^2\right)^2 - (4 - x)^2\right] dx$$

$$= \pi\int_0^3 \left(x^4 - 4x^3 - 5x^2 + 24x\right) dx$$

$$= \pi\left[\frac{x^5}{5} - x^4 - \frac{5x^3}{3} + 12x^2\right]_0^3 = \frac{153\pi}{5}$$

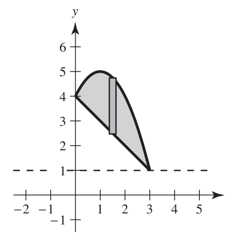


(b) $R(x) = \left(4 + 2x - x^2\right) - 1,\ r(x) = (4 - x) - 1$

$$V = \pi\int_0^3\left[\left(3 + 2x - x^2\right)^2 - (3 - x)^2\right] dx$$

$$= \pi\int_0^3\left(x^4 - 4x^3 - 3x^2 + 18x\right) dx$$

$$= \pi\left[\frac{x^5}{5} - x^4 - x^3 + 9x^2\right]_0^3 = \frac{108\pi}{5}$$



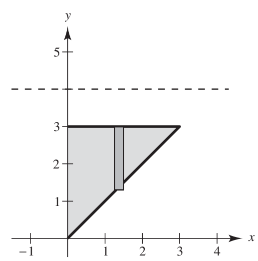
17. $R(x) = 4 - x,\ r(x) = 1$

$$V = \pi\int_0^3\left[(4 - x)^2 - (1)^2\right] dx$$

$$= \pi\int_0^3\left(x^2 - 8x + 15\right) dx$$

$$= \pi\left[\frac{x^3}{3} - 4x^2 + 15x\right]_0^3$$

$$= 18\pi$$



18. $R(x) = 4 - \dfrac{x^3}{2},\ r(x) = 0$

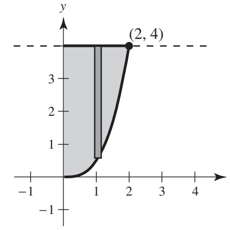
$$V = \pi\int_0^1\left(4 - \frac{x^3}{2}\right)^2 dx$$

$$= \pi\int_0^2\left[16 - 4x^3 + \frac{x^6}{4}\right] dx$$

$$= \pi\left[16x - x^4 + \frac{x^7}{28}\right]_0^2$$

$$= \pi\left(32 - 16 + \frac{128}{28}\right)$$

$$= \frac{144}{7}\pi$$



19. $R(x) = 4,\ r(x) = 4 - \dfrac{2}{1+x}$

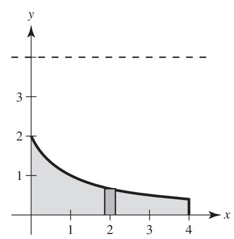
$$V = \pi \int_0^4 \left[4^2 - \left(4 - \frac{2}{1+x}\right)^2\right] dx$$

$$= \pi \int_0^4 \left[\frac{16}{1+x} - \frac{4}{(1+x)^2}\right] dx$$

$$= \pi\left[16 \ln(1+x) + \frac{4}{1+x}\right]_0^4$$

$$= \pi\left[\left(16 \ln 5 + \frac{4}{5}\right) - 4\right]$$

$$= \pi\left(16 \ln 5 - \frac{16}{5}\right)$$



20. $R(x) = 4,\ r(x) = 4 - \sqrt{1-x}$

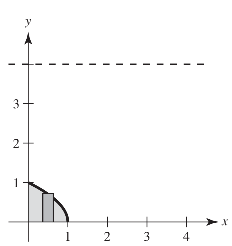
$$V = \pi \int_0^1 \left[4^2 - \left(4 - \sqrt{1-x}\right)^2\right] dx$$

$$= \pi \int_0^1 \left[8\sqrt{1-x} - (1-x)\right] dx$$

$$= \pi\left[\frac{-16}{3}(1-x)^{3/2} - x + \frac{x^2}{2}\right]_0^1$$

$$= \pi\left[\left(-1 + \frac{1}{2}\right) - \left(-\frac{16}{3}\right)\right]$$

$$= \frac{29\pi}{6}$$



21. $y = x$

$R(y) = 5 - y,\ r(y) = 0$

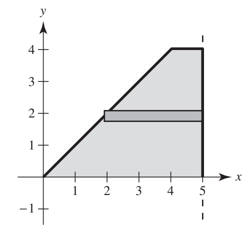
$$V = \pi\int_0^4 (5-y)^2\, dy$$

$$= \pi\int_0^4 \left(25 - 10y + y^2\right) dy$$

$$= \pi\left[25y - 5y^2 + \frac{y^3}{3}\right]_0^4$$

$$= \pi\left[100 - 80 + \frac{64}{3}\right]$$

$$= \frac{124\pi}{3}$$



22. $y = 2 - \dfrac{x}{2} \Rightarrow x = 4 - 2y$

$R(y) = 5,\ r(y) = 5 - (4 - 2y) = 1 + 2y$

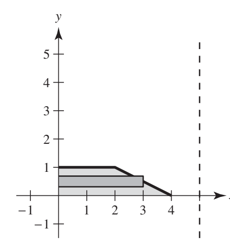
$$V = \pi \int_0^1 \left[5^2 - (1+2y)^2\right] dy$$

$$= \pi \int_0^1 \left[24 - 4y - 4y^2\right] dy$$

$$= \pi\left[24y - 2y^2 - \frac{4}{3}y^3\right]_0^1$$

$$= \pi\left(24 - 2 - \frac{4}{3}\right)$$

$$= \frac{62\pi}{3}$$



23. $x = y^2$

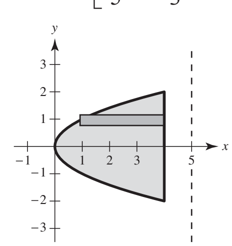
$R(y) = 5 - y^2,\ r(y) = 1$

$$V = \pi\int_{-2}^{2}\left[\left(5 - y^2\right)^2 - 1\right] dy$$

$$= 2\pi\int_0^2 \left[y^4 - 10y^2 + 24\right] dy$$

$$= 2\pi\left[\frac{y^5}{5} - \frac{10y^3}{3} + 24y\right]_0^2$$

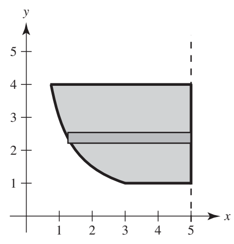
$$= 2\pi\left[\frac{32}{5} - \frac{80}{3} + 48\right] = \frac{832\pi}{15}$$



24. $xy = 3, x = \dfrac{3}{y}$

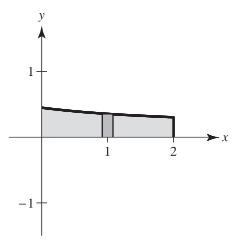
$R(y) = 5 - \dfrac{3}{y}, r(y) = 0$

$$V = \pi\int_1^4 \left(5 - \frac{3}{y}\right)^2 dy$$
$$= \pi\int_1^4 \left(25 + \frac{9}{y^2} - \frac{30}{y}\right) dy$$
$$= \pi\left[25y - \frac{9}{y} - 30 \ln y\right]_1^4$$
$$= \pi\left[\left(100 - \frac{9}{4} - 30 \ln 4\right) - (25 - 9)\right]$$
$$= \pi\left[\frac{327}{4} - 30 \ln 4\right] \approx 126.17$$



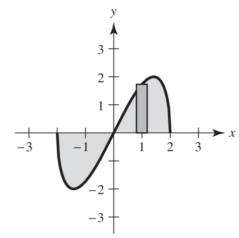
25. $R(x) = \dfrac{1}{\sqrt{3x + 5}}, r(x) = 0$

$$V = \pi \int_0^2 \left(\frac{1}{\sqrt{3x + 5}}\right)^2 dx$$
$$= \pi \int_0^2 \frac{1}{3x + 5} dx$$
$$= \frac{\pi}{3}\left[\ln(3x + 5)\right]_0^2$$
$$= \frac{\pi}{3}(\ln 11 - \ln 5) = \frac{\pi}{3} \ln \frac{11}{5}$$



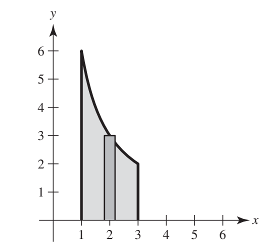
26. $R(x) = x\sqrt{4 - x^2}, r(x) = 0$

$$V = 2\pi\int_0^2 \left(x\sqrt{4 - x^2}\right)^2 dx$$
$$= 2\pi\int_0^2 \left(4x^2 - x^4\right) dx$$
$$= 2\pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2$$
$$= 2\pi\left[\frac{32}{3} - \frac{32}{5}\right] = \frac{128\pi}{15}$$



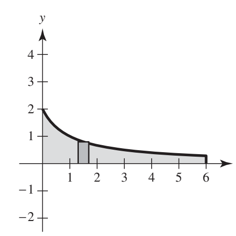
27. $R(x) = \dfrac{6}{x}, r(x) = 0$

$$V = \pi \int_1^3 \left(\frac{6}{x}\right)^2 dx = \pi\left[-\frac{36}{x}\right]_1^3 = 36\pi\left[-\frac{1}{3} + 1\right] = 24\pi$$



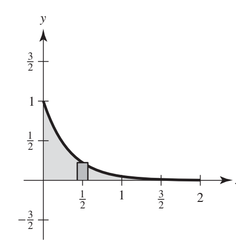
28. $R(x) = \dfrac{2}{x + 1}, r(x) = 0$

$$V = \pi\int_0^6 \left(\frac{2}{x + 1}\right)^2 dx$$
$$= 4\pi\int_0^6 (x + 1)^{-2} dx$$
$$= 4\pi\left[\frac{-1}{x + 1}\right]_0^6$$
$$= 4\pi\left[-\frac{1}{7} + 1\right] = \frac{24\pi}{7}$$



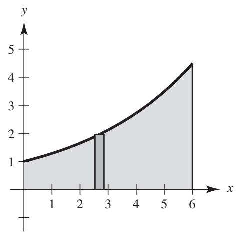
29. $R(x) = e^{-3x}, r(x) = 0$

$$V = \pi \int_0^2 \left(e^{-3x}\right)^2 dx$$
$$= \pi \int_0^2 e^{-6x} dx$$
$$= \pi\left[\left(-\frac{1}{6}\right)e^{-6x}\right]_0^2$$
$$= -\frac{1}{6}\pi\left(e^{-12} - 1\right)$$
$$= \frac{\pi\left(1 - e^{-12}\right)}{6}$$



30. $R(x) = e^{x/4}, r(x) = 0$

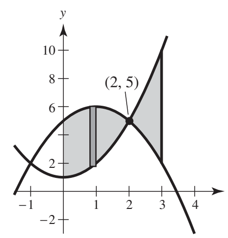
$$V = \pi\int_0^6 \left(e^{x/4}\right)^2 \, dx$$
$$= \pi\int_0^6 e^{x/2} \, dx$$
$$= \pi\left[2e^{x/2}\right]_0^6$$
$$= \pi\left(2e^3 - 2\right) \approx 119.92$$



31.
$$x^2 + 1 = -x^2 + 2x + 5$$
$$2x^2 - 2x - 4 = 0$$
$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$

The curves intersect at $(-1, 2)$ and $(2, 5)$.

$$V = \pi\int_0^2\left[\left(5 + 2x - x^2\right)^2 - \left(x^2 + 1\right)^2\right] dx + \pi\int_2^3\left[\left(x^2 + 1\right)^2 - \left(5 + 2x - x^2\right)^2\right] dx$$
$$= \pi\int_0^2 \left(-4x^3 - 8x^2 + 20x + 24\right) dx + \pi\int_2^3 \left(4x^3 + 8x^2 - 20x - 24\right) dx$$
$$= \pi\left[-x^4 - \frac{8}{3}x^3 + 10x^2 + 24x\right]_0^2 + \pi\left[x^4 + \frac{8}{3}x^3 - 10x^2 - 24x\right]_2^3$$
$$= \pi\frac{152}{3} + \pi\frac{125}{3} = \frac{277\pi}{3}$$

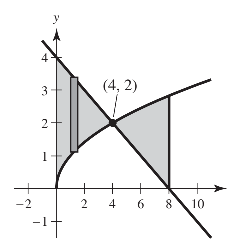


32. $\sqrt{x} = -\frac{1}{2}x + 4$

$$x = \frac{1}{4}x^2 - 4x + 16$$
$$0 = x^2 - 20x + 64$$
$$0 = (x - 4)(x - 16)$$

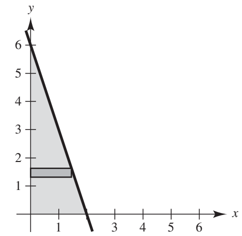
The curves intersect at $(4, 2)$. (Note $x = 16$ is an extraneous root.)

$$V = \pi\int_0^4\left[\left(4 - \frac{1}{2}x\right)^2 - \left(\sqrt{x}\right)^2\right] dx + \pi\int_4^8\left[\left(\sqrt{x}\right)^2 - \left(4 - \frac{1}{2}x\right)^2\right] dx$$
$$= \pi\int_0^4\left(\frac{x^2}{4} - 5x + 16\right) dx + \pi\int_4^8\left(-\frac{x^2}{4} + 5x - 16\right) dx$$
$$= \pi\left[\frac{x^3}{12} - \frac{5x^2}{2} + 16x\right]_0^4 + \pi\left[-\frac{x^3}{12} + \frac{5x^2}{2} - 16x\right]_4^8$$
$$= \frac{88}{3}\pi + \frac{56}{3}\pi = 48\pi$$



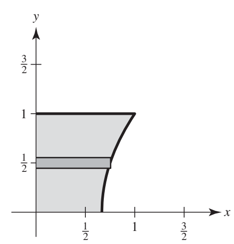
33. $y = 6 - 3x \Rightarrow x = \frac{1}{3}(6 - y)$

$$V = \pi\int_0^6 \left[\frac{1}{3}(6 - y)\right]^2 dy$$
$$= \frac{\pi}{9}\int_0^6 \left[36 - 12y + y^2\right] dy$$
$$= \frac{\pi}{9}\left[36y - 6y^2 + \frac{y^3}{3}\right]_0^6$$
$$= \frac{\pi}{9}\left[216 - 216 + \frac{216}{3}\right]$$
$$= 8\pi = \frac{1}{3}\pi r^2 h, \text{ Volume of cone}$$



34. $y = \sqrt{3x - 2} \Rightarrow y^2 = 3x - 2 \Rightarrow x = \frac{1}{3}(y^2 + 2), y \geq 0$

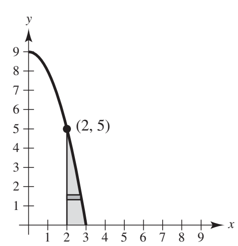
$$V = \pi\int_0^1 \left[\frac{1}{3}(y^2 + 2)\right]^2 dy$$
$$= \frac{\pi}{9}\int_0^1 \left[y^4 + 4y^2 + 4\right] dy$$
$$= \frac{\pi}{9}\left[\frac{y^5}{5} + \frac{4y^3}{3} + 4y\right]_0^1$$
$$= \frac{\pi}{9}\left[\frac{1}{5} + \frac{4}{3} + 4\right]$$
$$= \frac{83\pi}{135}$$



35. $y = 9 - x^2, y = 0, x = 2, x = 3$

$x = \sqrt{9 - y}$

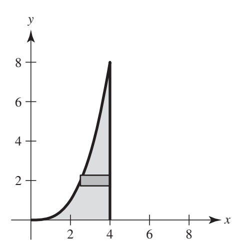
$$V = \pi\int_0^5 \left[\left(\sqrt{9 - y}\right)^2 - 2^2\right] dy$$
$$= \pi\int_0^5 (5 - y)\, dy$$
$$= \pi\left[5y - \frac{y^2}{2}\right]_0^5 = \pi\left(25 - \frac{25}{2}\right) = \frac{25\pi}{2}$$



36. $y = \frac{x^3}{8} \Rightarrow x^3 = 8y \Rightarrow x = 2y^{1/3}$

$R(x) = 4, r(x) = 4 - 2y^{1/3}$

$$V = \pi\int_0^8 \left[4^2 - \left(2y^{1/3}\right)^2\right] dy$$
$$= \pi\int_0^8 \left[16 - 4y^{2/3}\right] dy$$
$$= \pi\left[16y - \frac{12}{5}y^{5/3}\right]_0^8$$
$$= \pi\left[128 - \frac{384}{5}\right]$$
$$= \frac{256\pi}{5}$$

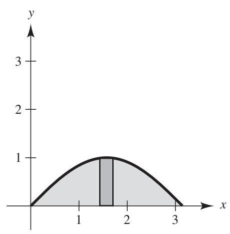


37. $V = \pi\int_0^{\pi} (\sin x)^2\, dx$

$= \pi\int_0^{\pi} \frac{1 - \cos 2x}{2}\, dx$

$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi}{2}[\pi] = \frac{\pi^2}{2}$

Numerical approximation: 4.9348



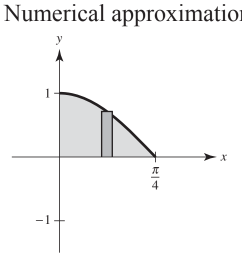
38. $V = \pi\int_0^{\pi/4} \cos^2 2x\, dx$

$= \pi\int_0^{\pi/4} \frac{1 + \cos 4x}{2}\, dx$

$= \frac{\pi}{2}\left[x + \frac{\sin 4x}{4}\right]_0^{\pi/4}$

$= \frac{\pi}{2}\left[\frac{\pi}{4}\right] = \frac{\pi^2}{8}$

Numerical approximation: 1.2337

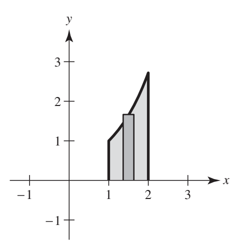


39. $V = \pi\int_1^{2} \left(e^{x-1}\right)^2\, dx$

$= \pi\int_1^{2} e^{2x-2}\, dx$

$= \frac{\pi}{2}e^{2x-2}\Big]_1^2$

$= \frac{\pi}{2}\left(e^2 - 1\right)$



Numerical approximation: 10.0359

40. $V = \pi\int_{-1}^{2} \left(e^{x/2} + e^{-x/2}\right)^2\, dx$

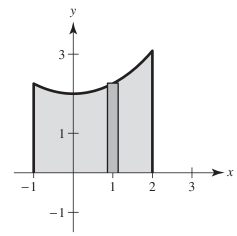
$= \pi\int_{-1}^{2} \left(e^x + e^{-x} + 2\right) dx$

$= \pi\left[e^x - e^{-x} + 2x\right]_{-1}^2$

$= \pi\left[\left(e^2 - e^{-2} + 4\right) - \left(e^{-1} - e - 2\right)\right]$

$= \pi\left(e^2 + e + 6 - e^{-2} - e^{-1}\right)$

Numerical approximation: 49.0218



41. $V = \pi\int_0^1 x^2\, dx = \pi\left[\frac{x^3}{3}\right]_0^1 = \frac{\pi}{3}$

42. $V = \pi\int_0^1 \left[1^2 - (1 - y)^2\right] dy$

$= \pi\int_0^1 \left[2y - y^2\right] dy$

$= \pi\left[y^2 - \frac{y^3}{3}\right]_0^1$

$= \pi\left(1 - \frac{1}{3}\right) = \frac{2}{3}\pi$

43. $V = \pi\int_0^1 y^2\, dy = \pi\frac{y^3}{3}\Big]_0^1 = \frac{\pi}{3}$

44. $V = \pi\int_0^1 \left[\left(1 - x^2\right)^2 - (1 - x)^2\right] dx$

$= \pi\int_0^1 \left[1 - 2x^2 + x^4 - 1 + 2x - x^2\right] dx$

$= \pi\int_0^1 \left[2x - 3x^2 + x^4\right] dx$

$= \pi\left[x^2 - x^3 + \frac{x^5}{5}\right]_0^1$

$= \pi\left(\frac{1}{5}\right) = \frac{\pi}{5}$

45. $V = \pi\int_0^1 \left(x^2 - x^4\right) dx$

$= \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$

$= \pi\left(\frac{1}{3} - \frac{1}{5}\right)$

$= \frac{2\pi}{15}$

46. $V = \pi\int_0^1 \left(1 - \sqrt{y}\right)^2 dy$

$= \pi\int_0^1 \left(1 - 2\sqrt{y} + y\right) dy$

$= \pi\left[y - \frac{4}{3}y^{3/2} + \frac{y^2}{2}\right]_0^1$

$= \pi\left(1 - \frac{4}{3} + \frac{1}{2}\right)$

$= \frac{\pi}{6}$

47. $V = \pi\int_0^1 (1 - y)\,dy$

$= \pi\left[y - \frac{y^2}{2}\right]_0^1 = \pi\left(1 - \frac{1}{2}\right) = \frac{\pi}{2}$

48. $V = \pi\int_0^1 \left[1^2 - \left(1 - x^2\right)^2\right] dx$

$= \pi\int_0^1 \left(2x^2 - x^4\right) dx$

$= \pi\left[\frac{2x^3}{3} - \frac{x^5}{5}\right]_0^1$

$= \pi\left[\frac{2}{3} - \frac{1}{5}\right] = \frac{7}{15}\pi$

49. $V = \pi\int_0^2 \left[e^{-x^2}\right]^2 dx \approx 1.9686$

50. $V = \pi\int_1^3 [\ln x]^2\,dx \approx 3.2332$

51. $V = \pi\int_0^5 \left[2\arctan(0.2x)\right]^2 dx$

≈ 15.4115

52. $x^2 = \sqrt{2x}$

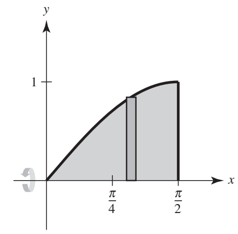
$x^4 = 2x$

$x^3 = 2$

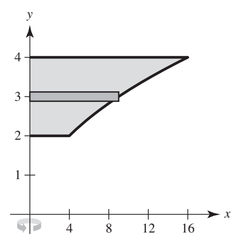
$x = 2^{1/3} \approx 1.2599$

$V = \pi\int_0^{2^{1/3}} \left[\left(\sqrt{2x}\right)^2 - \left(x^2\right)^2\right] dx \approx 2.9922$

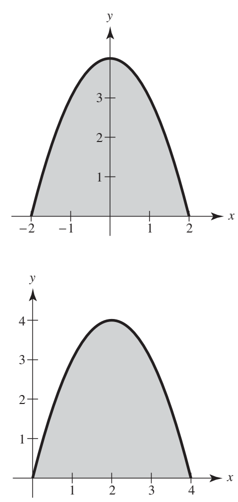
53. (a) $\pi\int_0^{\pi/2} \sin^2 x\,dx$ represents the volume of the solid generated by revolving the region bounded by $y = \sin x$, $y = 0$, $x = 0$, $x = \pi/2$ about the x-axis.



(b) $\pi\int_2^4 y^4\,dy$ represents the volume of the solid generated by revolving the region bounded by $x = y^2$, $x = 0$, $y = 2$, $y = 4$ about the y-axis.

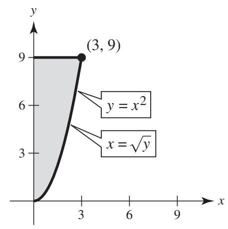


54.



The volumes are the same because the solid has been translated horizontally. $\left(4x - x^2 = 4 - (x - 2)^2\right)$

55.



(a) Around x-axis:

$$V = \pi\int_0^3 \left[9^2 - \left(x^2\right)^2\right] dx = \frac{972}{5}\pi = 194.4\pi$$

(b) Around y-axis:

$$V = \pi\int_0^9 \left(\sqrt{y}\right)^2 dy = \frac{81}{2}\pi = 40.5\pi$$

(c) Around $x = 3$:

$$V = \pi\left(3^2\right)9 - \int_0^9 \pi\left(\sqrt{y} - 3\right)^2 dy = 81\pi - \frac{27}{2}\pi = \frac{135\pi}{2} \approx 67.5\pi$$

So, $b < c < a$.

56. (a) Matches (ii) because the axis of rotation is vertical, and this is the washer method.

(b) Matches (iv) because the axis of rotation is horizontal, and this is the washer method.

(c) Matches (i) because the axis of rotation is horizontal.

(d) Matches (iii) because the axis of rotation is vertical.

57. $V = \pi\int_1^3 \left(\sqrt{x}\right)^2 dx = \pi\frac{x^2}{2}\Big]_1^3 = 4\pi$

Let $1 < c < 3$.

$$\pi\int_1^c x\,dx = \pi\frac{x^2}{2}\Big]_1^c = \frac{\pi c^2}{2} - \frac{\pi}{2} = 2\pi$$

$$\Rightarrow c^2 - 1 = 4 \Rightarrow c = \sqrt{5}$$

58. From Exercise 57, $V = 4\pi$

Let $1 < c < 3$.

$$\pi\int_1^c x\,dx = \frac{\pi c^2}{2} - \frac{\pi}{2} = \frac{4\pi}{3} \quad \text{(one-third of volume)}$$

$$3c^2 - 3 = 8$$

$$c^2 = \frac{11}{3}$$

$$c = \sqrt{\frac{11}{3}}$$

Find the other value of x.

$$\pi\int_1^d x\,dx = \frac{\pi d^2}{2} - \frac{\pi}{2} = \frac{8\pi}{3} \quad \text{(two-thirds of volume)}$$

$$3d^2 - 3 = 16 \Rightarrow d = \sqrt{\frac{19}{3}}$$

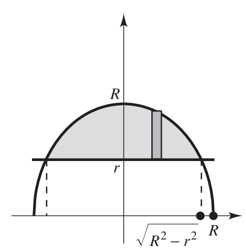
The values of x are $\sqrt{\frac{11}{3}}$ and $\sqrt{\frac{19}{3}}$.

59. $V = \pi\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \left[\left(\sqrt{R^2 - x^2}\right)^2 - r^2\right] dx$

$$= 2\pi\int_0^{\sqrt{R^2-r^2}} \left(R^2 - r^2 - x^2\right) dx$$

$$= 2\pi\left[\left(R^2 - r^2\right)x - \frac{x^3}{3}\right]_0^{\sqrt{R^2-r^2}}$$

$$= 2\pi\left[\left(R^2 - r^2\right)^{3/2} - \frac{\left(R^2 - r^2\right)^{3/2}}{3}\right] = \frac{4}{3}\pi\left(R^2 - r^2\right)^{3/2}$$



60. Let $R = 6$ in the previous Exercise.

$$\frac{4}{3}\pi\left(36 - r^2\right)^{3/2} = \frac{1}{2}\left(\frac{4}{3}\right)\pi(6)^3$$

$$\left(36 - r^2\right)^{3/2} = 108$$

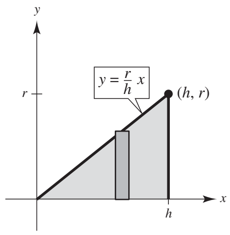
$$36 - r^2 = (108)^{2/3}$$

$$r^2 = 36 - 108^{2/3}$$

$$r = \sqrt{36 - 108^{2/3}} \approx 3.65$$

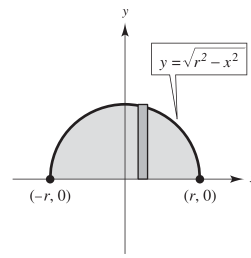
61. $R(x) = \dfrac{r}{h}x,\ r(x) = 0$

$$V = \pi\int_0^h \frac{r^2}{h^2}x^2\,dx = \left[\frac{r^2\pi}{3h^2}x^3\right]_0^h = \frac{r^2\pi}{3h^2}h^3 = \frac{1}{3}\pi r^2 h$$



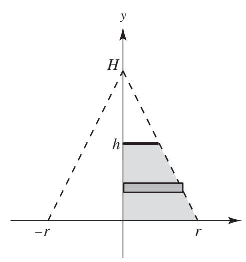
62. $R(x) = \sqrt{r^2 - x^2},\ r(x) = 0$

$$\begin{aligned} V &= \pi\int_{-r}^{r}\left(r^2 - x^2\right)dx \\ &= 2\pi\int_0^r\left(r^2 - x^2\right)dx \\ &= 2\pi\left[r^2x - \tfrac{1}{3}x^3\right]_0^r \\ &= 2\pi\left(r^3 - \tfrac{1}{3}r^3\right) = \tfrac{4}{3}\pi r^3 \end{aligned}$$



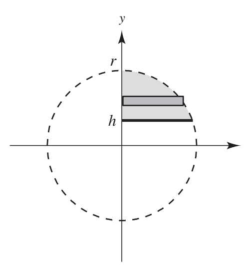
63. $x = r - \dfrac{r}{H}y = r\left(1 - \dfrac{y}{H}\right),\ R(y) = r\left(1 - \dfrac{y}{H}\right),\ r(y) = 0$

$$\begin{aligned} V = \pi\int_0^h\left[r\left(1 - \frac{y}{H}\right)\right]^2 dy &= \pi r^2\int_0^h\left(1 - \frac{2}{H}y + \frac{1}{H^2}y^2\right)dy \\ &= \pi r^2\left[y - \frac{1}{H}y^2 + \frac{1}{3H^2}y^3\right]_0^h \\ &= \pi r^2\left(h - \frac{h^2}{H} + \frac{h^3}{3H^2}\right) = \pi r^2 h\left(1 - \frac{h}{H} + \frac{h^2}{3H^2}\right) \end{aligned}$$

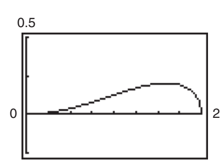


64. $x = \sqrt{r^2 - y^2},\ R(y) = \sqrt{r^2 - y^2},\ r(y) = 0$

$$\begin{aligned} V &= \pi\int_h^r\left(\sqrt{r^2 - y^2}\right)^2 dy \\ &= \pi\int_h^r\left(r^2 - y^2\right)dy \\ &= \pi\left[r^2y - \frac{y^3}{3}\right]_h^r \\ &= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(r^2h - \frac{h^3}{3}\right)\right] \\ &= \pi\left(\frac{2r^3}{3} - r^2h + \frac{h^3}{3}\right) = \frac{\pi}{3}\left(2r^3 - 3r^2h + h^3\right) \end{aligned}$$



65.



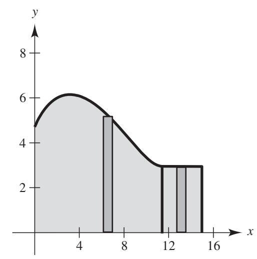
$$V = \pi\int_0^2\left(\frac{1}{8}x^2\sqrt{2 - x}\right)^2 dx = \frac{\pi}{64}\int_0^2 x^4(2 - x)\,dx = \frac{\pi}{64}\left[\frac{2x^5}{5} - \frac{x^6}{6}\right]_0^2 = \frac{\pi}{30}\text{ m}^3$$

66. $y = \begin{cases} \sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}, & 0 \le x \le 11.5 \\ 2.95, & 11.5 < x \le 15 \end{cases}$

$$V = \pi\int_0^{11.5}\left(\sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}\right)^2 dx + \pi\int_{11.5}^{15} 2.95^2\, dx$$

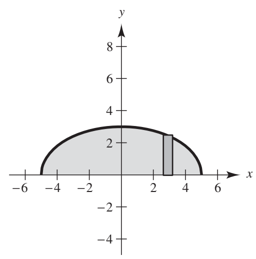
$$= \pi\left[\frac{0.1x^4}{4} - \frac{2.2x^3}{3} + \frac{10.9x^2}{2} + 22.2x\right]_0^{11.5} + \pi\left[2.95^2 x\right]_{11.5}^{15}$$

≈ 1031.9016 cubic centimeters



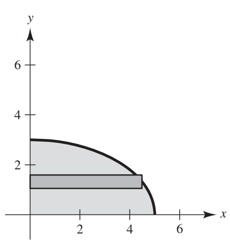
67. (a) $R(x) = \frac{3}{5}\sqrt{25 - x^2},\ r(x) = 0$

$$V = \frac{9\pi}{25}\int_{-5}^{5}\left(25 - x^2\right) dx = \frac{18\pi}{25}\int_0^5\left(25 - x^2\right) dx = \frac{18\pi}{25}\left[25x - \frac{x^3}{3}\right]_0^5 = 60\pi$$



(b) $R(y) = \frac{5}{3}\sqrt{9 - y^2},\ r(y) = 0,\ x \ge 0$

$$V = \frac{25\pi}{9}\int_0^3\left(9 - y^2\right) dy = \frac{25\pi}{9}\left[9y - \frac{y^3}{3}\right]_0^3 = 50\pi$$



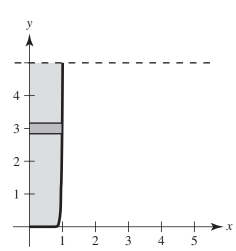
68. $V = \pi\int_0^5\left[\left(\frac{y}{4}\right)^{1/32}\right]^2 dy$

$$= \pi\int_0^5\left(\frac{y}{4}\right)^{1/16} dy$$

$$= \frac{\pi}{4^{1/16}}\left[\frac{16}{17}y^{17/16}\right]_0^5$$

$\approx 4.772\,\pi$ in.^{3}

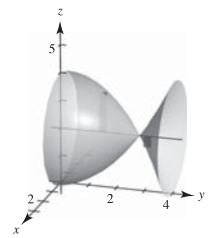
≈ 14.992 in.^{3}



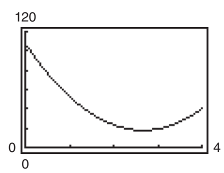
69. (a) First find where $y = b$ intersects the parabola:

$$b = 4 - \frac{x^2}{4}$$
$$x^2 = 16 - 4b = 4(4 - b)$$
$$x = 2\sqrt{4 - b}$$

$$V = \int_0^{2\sqrt{4-b}} \pi\left[4 - \frac{x^2}{4} - b\right]^2 dx + \int_{2\sqrt{4-b}}^{4} \pi\left[b - 4 + \frac{x^2}{4}\right]^2 dx$$
$$= \int_0^4 \pi\left[4 - \frac{x^2}{4} - b\right]^2 dx$$
$$= \pi\int_0^4 \left[\frac{x^4}{16} - 2x^2 + \frac{bx^2}{2} + b^2 - 8b + 16\right] dx$$
$$= \pi\left[\frac{x^5}{80} - \frac{2x^3}{3} + \frac{bx^3}{6} + b^2x - 8bx + 16x\right]_0^4$$
$$= \pi\left(\frac{64}{5} - \frac{128}{3} + \frac{32}{3}b + 4b^2 - 32b + 64\right) = \pi\left(4b^2 - \frac{64}{3}b + \frac{512}{15}\right)$$



(b) Graph of $V(b) = \pi\left(4b^2 - \frac{64}{3}b + \frac{512}{15}\right)$



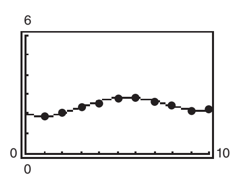
Minimum volume is 17.87 for $b = 2.67$.

(c) $V'(b) = \pi\left(8b - \frac{64}{3}\right) = 0 \Rightarrow b = \frac{64/3}{8} = \frac{8}{3} = 2\frac{2}{3}$

$V''(b) = 8\pi > 0 \Rightarrow b = \frac{8}{3}$ is a relative minimum.

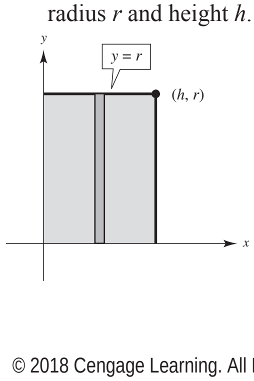
70. (a) $f(x) = 0.00249x^4 - 0.0529x^3 + 0.3314x^2 - 0.4999x + 2.112$

(b) $V \approx \int_0^{10} \pi f(x)^2\, dx \approx 186.35 \text{ cm}^3$



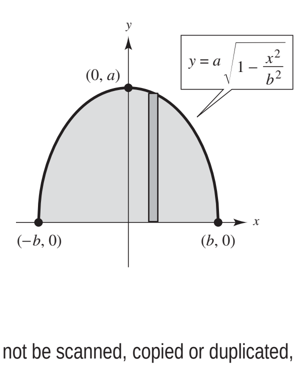
71. (a) $\pi\int_0^h r^2\, dx$ (ii)

is the volume of a right circular cylinder with radius r and height h.



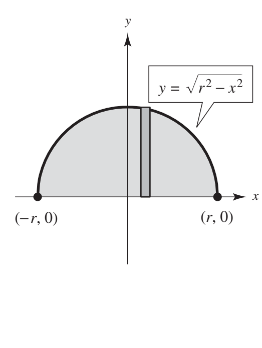
(b) $\pi\int_{-b}^{b}\left(a\sqrt{1 - \frac{x^2}{b^2}}\right)^2 dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.



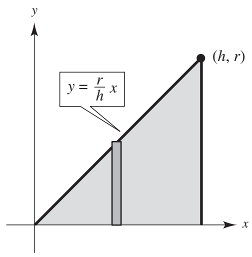
(c) $\pi\int_{-r}^{r}\left(\sqrt{r^2 - x^2}\right)^2 dx$ (iii)

is the volume of a sphere with radius r.



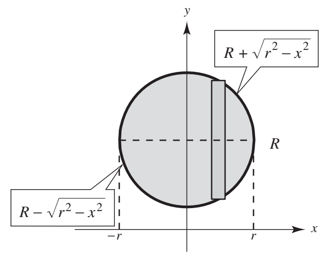
(d) $\pi\int_0^h \left(\frac{rx}{h}\right)^2 dx$ (i)

is the volume of a right circular cone with the radius of the base as r and height h.



(e) $\pi\int_{-r}^{r}\left[\left(R + \sqrt{r^2 - x^2}\right)^2 - \left(R - \sqrt{r^2 - x^2}\right)^2\right] dx$ (v)

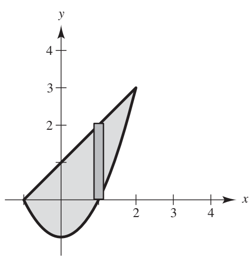
is the volume of a torus with the radius of its circular cross section as r and the distance from the axis of the torus to the center of its cross section as R.



72. Let $A_1(x)$ and $A_2(x)$ equal the areas of the cross sections of the two solids for $a \le x \le b$.

Because $A_1(x) = A_2(x)$, you have $V_1 = \int_a^b A_1(x)\,dx = \int_a^b A_2(x)\,dx = V_2$. So, the volumes are the same.

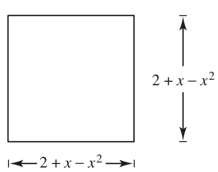
73.



Base of cross section $= (x + 1) - (x^2 - 1) = 2 + x - x^2$

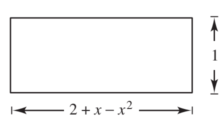
(a) $A(x) = b^2 = \left(2 + x - x^2\right)^2 = 4 + 4x - 3x^2 - 2x^3 + x^4$

$$V = \int_{-1}^{2}\left(4 + 4x - 3x^2 - 2x^3 + x^4\right) dx = \left[4x + 2x^2 - x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_{-1}^{2} = \frac{81}{10}$$

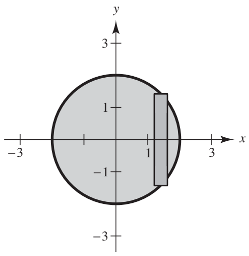


(b) $A(x) = bh = \left(2 + x - x^2\right)1$

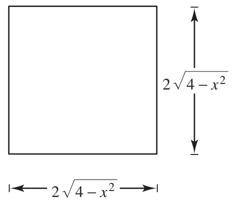
$$V = \int_{-1}^{2}\left(2 + x - x^2\right) dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$



74.



Base of cross section $= 2\sqrt{4 - x^2}$



(a) $A(x) = b^2 = \left(2\sqrt{4 - x^2}\right)^2$

$$V = \int_{-2}^{2} 4\left(4 - x^2\right)\,dx$$

$$= 4\left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

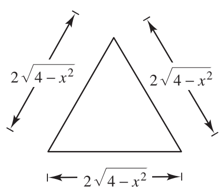
$$= \frac{128}{3}$$

(b) $A(x) = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{4 - x^2}\right)\left(\sqrt{3}\sqrt{4 - x^2}\right) = \sqrt{3}\left(4 - x^2\right)$

$$V = \sqrt{3}\int_{-2}^{2}\left(4 - x^2\right)dx$$

$$= \sqrt{3}\left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

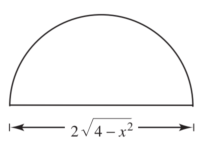
$$= \frac{32\sqrt{3}}{3}$$



(c) $A(x) = \frac{1}{2}\pi r^2 = \frac{\pi}{2}\left(\sqrt{4 - x^2}\right)^2 = \frac{\pi}{2}\left(4 - x^2\right)$

$$V = \frac{\pi}{2}\int_{-2}^{2}\left(4 - x^2\right)dx$$

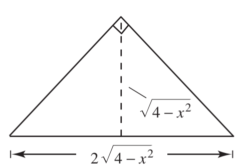
$$= \frac{\pi}{2}\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{16\pi}{3}$$



(d) $A(x) = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{4 - x^2}\right)\left(\sqrt{4 - x^2}\right) = 4 - x^2$

$$V = \int_{-2}^{2}\left(4 - x^2\right)dx$$

$$= \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}$$



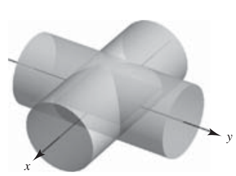
75. The cross sections are squares. By symmetry, you can set up an integral for an eighth of the volume and multiply by 8.

$A(y) = b^2 = \left(\sqrt{r^2 - y^2}\right)^2$

$$V = 8\int_0^r \left(r^2 - y^2\right)dy$$

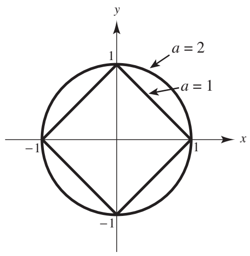
$$= 8\left[r^2y - \tfrac{1}{3}y^3\right]_0^r$$

$$= \tfrac{16}{3}r^3$$



76. (a) When $a = 1$: $|x| + |y| = 1$ represents a square.

When $a = 2$: $|x|^2 + |y|^2 = 1$ represents a circle.



(b) $|y| = \left(1 - |x|^a\right)^{1/a}$

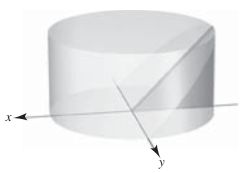
$$A = 2\int_{-1}^{1}\left(1 - |x|^a\right)^{1/a}\,dx = 4\int_0^1\left(1 - x^a\right)^{1/a}\,dx$$

To approximate the volume of the solid, from n slices, each of whose area is approximated by the integral above. Then sum the volumes of these n slices.

77. (a) Because the cross sections are isosceles right triangles:

$$A(x) = \frac{1}{2}bh = \frac{1}{2}\left(\sqrt{r^2 - y^2}\right)\left(\sqrt{r^2 - y^2}\right) = \frac{1}{2}\left(r^2 - y^2\right)$$

$$V = \frac{1}{2}\int_{-r}^{r}\left(r^2 - y^2\right)dy = \int_0^r\left(r^2 - y^2\right)dy = \left[r^2y - \frac{y^3}{3}\right]_0^r = \frac{2}{3}r^3$$



(b) $$A(x) = \frac{1}{2}bh = \frac{1}{2}\sqrt{r^2 - y^2}\left(\sqrt{r^2 - y^2}\tan\theta\right) = \frac{\tan\theta}{2}\left(r^2 - y^2\right)$$

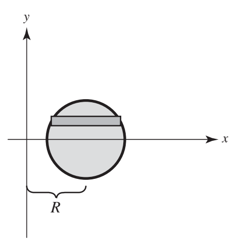
$$V = \frac{\tan\theta}{2}\int_{-r}^{r}\left(r^2 - y^2\right)dy = \tan\theta\int_0^r\left(r^2 - y^2\right)dy = \tan\theta\left[r^2y - \frac{y^3}{3}\right]_0^r = \frac{2}{3}r^3\tan\theta$$

As $\theta \to 90°$, $V \to \infty$.

78. (a) $(x - R)^2 + y^2 = r^2$

$$x = R \pm \sqrt{r^2 - y^2}$$

$$V = 2\pi\int_0^r\left(\left[R + \sqrt{r^2 - y^2}\right]^2 - \left[R - \sqrt{r^2 - y^2}\right]^2\right)dy = 2\pi\int_0^r 4R\sqrt{r^2 - y^2}\,dy = 8\pi R\int_0^r\sqrt{r^2 - y^2}\,dy$$



(b) $\int_0^r\sqrt{r^2 - y^2}\,dy$ is one-quarter of the area of a circle of radius r, $\frac{1}{4}\pi r^2$.

$$V = 8\pi R\left(\tfrac{1}{4}\pi r^2\right) = 2\pi^2r^2R$$

Section 7.3 Volume: The Shell Method

1. Determine the distance from the center of a representative rectangle to the axis of revolution, and find the height of the rectangle. Then use the formula

$$V = 2\pi\int_c^d p(y)h(y)\,dy$$

for a horizontal axis of revolution.

Use

$$V = 2\pi\int_a^b p(x)h(x)\,dx$$

for a vertical axis of revolution.

2. In the shell method, the rectangle generates a representative shell when revolved. In the disk method, the rectangle generates a representative disk when revolved. For the shell method, the representative rectangle is always parallel to the axis of revolution. For the disk method, the representative rectangle is always perpendicular to the axis of revolution.

3. $p(x) = x,\ h(x) = x$

$$V = 2\pi\int_0^2 x(x)\,dx = \left[\frac{2\pi x^3}{3}\right]_0^2 = \frac{16\pi}{3}$$

4. $p(x) = x,\ h(x) = 1 - x$

$$V = 2\pi\int_0^1 x(1-x)\,dx$$

$$= 2\pi\int_0^1 \left(x - x^2\right)dx = 2\pi\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{\pi}{3}$$

5. $p(x) = x,\ h(x) = \sqrt{x}$

$$V = 2\pi\int_0^4 x\sqrt{x}\,dx = 2\pi\int_0^4 x^{3/2}\,dx = \left[\frac{4\pi}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}$$

6. $p(x) = x,\ h(x) = 3 - \left(\frac{1}{2}x^2 + 1\right) = 2 - \frac{1}{2}x^2$

$$V = 2\pi\int_0^2 x\left(2 - \frac{1}{2}x^2\right)dx$$

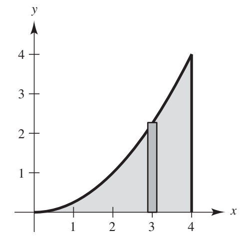
$$= 2\pi\left[x^2 - \frac{x^4}{8}\right]_0^2 = 2\pi(4-2) = 4\pi$$

7. $p(x) = x,\ h(x) = \frac{1}{4}x^2$

$$V = 2\pi\int_0^4 x\left(\frac{1}{4}x^2\right)dx$$

$$= \frac{\pi}{2}\left[\frac{x^4}{4}\right]_0^4$$

$$= 32\pi$$

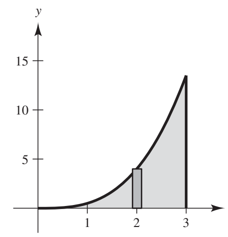


8. $p(x) = x,\ h(x) = \frac{1}{2}x^3$

$$V = 2\pi\int_0^3 x\left(\frac{x^3}{2}\right)dx$$

$$= \pi\left[\frac{x^5}{5}\right]_0^3$$

$$= \frac{243\pi}{5}$$

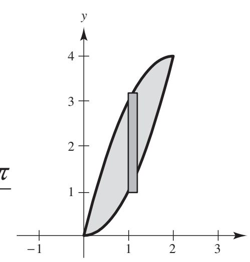


9. $p(x) = x,\ h(x) = \left(4x - x^2\right) - x^2 = 4x - 2x^2$

$$V = 2\pi\int_0^2 x\left(4x - 2x^2\right)dx$$

$$= 4\pi\int_0^2 \left(2x^2 - x^3\right)dx$$

$$= 4\pi\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 = \frac{16\pi}{3}$$



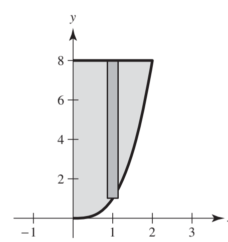
10. $p(x) = x,\ h(x) = 8 - x^3$

$$V = 2\pi\int_0^2 x\left(8 - x^3\right)dx$$

$$= 2\pi\left[4x^2 - \frac{x^5}{5}\right]_0^2$$

$$= 2\pi\left[16 - \frac{32}{5}\right]$$

$$= \frac{96\pi}{5}$$



11. $p(x) = x, h(x) = \sqrt{2x - 5}$

$$V = 2\pi \int_{5/2}^{4} x\sqrt{2x - 5}\, dx$$

Let $u = 2x - 5, x = \frac{1}{2}(u + 5), du = 2\, dx.$

When $x = \frac{5}{2}, u = 0.$

When $x = 4, u = 3.$

$$V = 2\pi \int_0^3 \frac{1}{2}(u + 5)u^{1/2} \frac{du}{2}$$

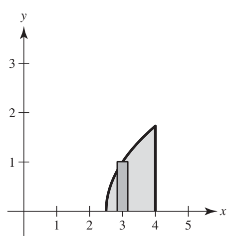
$$= \frac{\pi}{2} \int_0^3 \left(u^{3/2} + 5u^{1/2}\right) du$$

$$= \frac{\pi}{2}\left[\frac{2}{5}u^{5/2} + \frac{10}{3}u^{3/2}\right]_0^3$$

$$= \frac{\pi}{2}\left[\frac{2}{5}\left(3^{5/2}\right) + \frac{10}{3}\left(3^{3/2}\right)\right]$$

$$= \frac{\pi}{2}\left[\frac{18}{5}\sqrt{3} + 10\sqrt{3}\right]$$

$$= 34\pi\frac{\sqrt{3}}{5}$$

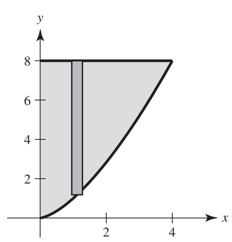


12. $p(x) = x, h(x) = 8 - x^{3/2}$

$$V = 2\pi\int_0^4 x\left(8 - x^{3/2}\right) dx$$

$$= 2\pi\left[4x^2 - \frac{2}{7}x^{7/2}\right]_0^4$$

$$= 2\pi\left[64 - \frac{2}{7}(128)\right] = \frac{384\pi}{7}$$



13. $p(y) = y, h(y) = 2 - y$

$$V = 2\pi\int_0^2 y(2 - y)\, dy$$

$$= 2\pi\int_0^2 \left(2y - y^2\right) dy$$

$$= 2\pi\left[y^2 - \frac{y^3}{3}\right]_0^2 = \frac{8\pi}{3}$$

14. $p(y) = -y \qquad \left(\text{So, } p(y) \ge 0 \text{ on } [-2, 0]\right)$

$h(y) = 3 - (1 - y) = 2 + y$

$$V = 2\pi\int_{-2}^0 (-y)(2 + y)\, dy$$

$$= 2\pi\int_{-2}^0 \left[-2y - y^2\right] dy$$

$$= 2\pi\left[-y^2 - \frac{y^3}{3}\right]_{-2}^0$$

$$= 2\pi\left[0 - \left(-4 + \frac{8}{3}\right)\right]$$

$$= 2\pi\frac{4}{3}$$

$$= \frac{8\pi}{3}$$

15. $p(y) = y$ and $h(y) = 1$ if $0 \le y < \frac{1}{2}.$

$p(y) = y$ and $h(y) = \frac{1}{y} - 1$ if $\frac{1}{2} \le y \le 1.$

$$V = 2\pi\int_0^{1/2} y\, dy + 2\pi\int_{1/2}^1 (1 - y)\, dy$$

$$= 2\pi\left[\frac{y^2}{2}\right]_0^{1/2} + 2\pi\left[y - \frac{y^2}{2}\right]_{1/2}^1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

16. $p(y) = y, h(y) = 4 - y^2$

$$V = 2\pi \int_0^2 y\left(4 - y^2\right) dy$$

$$= 2\pi\left[2y^2 - \frac{y^4}{4}\right]_0^2$$

$$= 2\pi[8 - 4] = 8\pi$$

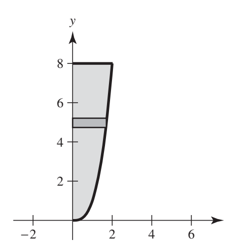
17. $p(y) = y, h(y) = \sqrt[3]{y}$

$$V = 2\pi\int_0^8 y\sqrt[3]{y}\, dy$$

$$= 2\pi\int_0^8 y^{4/3}\, dy$$

$$= \left[2\pi\left(\frac{3}{7}\right)y^{7/3}\right]_0^8$$

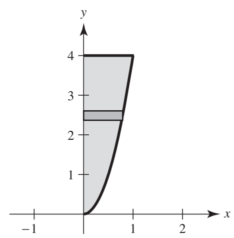
$$= \frac{6\pi}{7}\left(2^7\right) = \frac{768\pi}{7}$$



18. $y = 4x^2, x = \dfrac{\sqrt{y}}{2}$

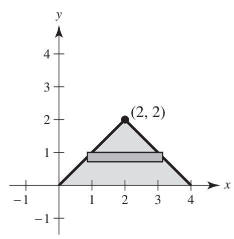
$p(y) = y, h(y) = \dfrac{\sqrt{y}}{2}$

$$V = 2\pi\int_0^4 y\left(\frac{\sqrt{y}}{2}\right) dy$$
$$= \pi\int_0^4 y^{3/2}\, dy$$
$$= \pi\left[\frac{2}{5}y^{5/2}\right]_0^4$$
$$= \frac{64\pi}{5}$$



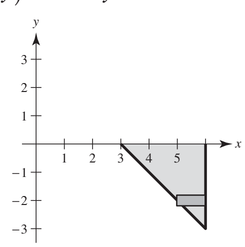
19. $p(y) = y, h(y) = (4 - y) - (y) = 4 - 2y$

$$V = 2\pi\int_0^2 y(4 - 2y)\, dy$$
$$= 2\pi\int_0^2 \left(4y - 2y^2\right) dy$$
$$= 2\pi\left[2y^2 - \frac{2}{3}y^3\right]_0^2$$
$$= 2\pi\left(8 - \frac{16}{3}\right) = \frac{16\pi}{3}$$



20. $p(y) = -y, h(y) = 6 - (3 - y) = 3 + y$

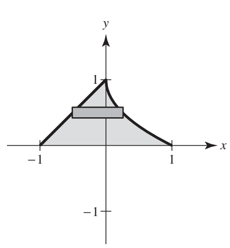
$$V = 2\pi\int_{-3}^0 -y(3 + y)\, dy$$
$$= -2\pi\left[\frac{3}{2}y^2 + \frac{y^3}{3}\right]_{-3}^0$$
$$= -2\pi\left[-\frac{27}{2} + 9\right] = 9\pi$$



21. $y = 1 - \sqrt{x} \Rightarrow \sqrt{x} = 1 - y \Rightarrow x = (1 - y)^2$

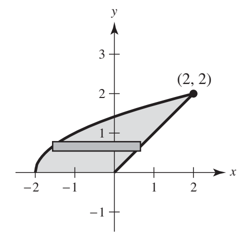
$p(y) = y, h(y) = (1 - y)^2 - (y - 1) = y^2 - 3y + 2$

$$V = 2\pi\int_0^1 y\left(y^2 - 3y + 2\right) dy$$
$$= 2\pi\left[\frac{y^4}{4} - y^3 + y^2\right]_0^1$$
$$= 2\pi\left[\frac{1}{4} - 1 + 1\right] = \frac{\pi}{2}$$



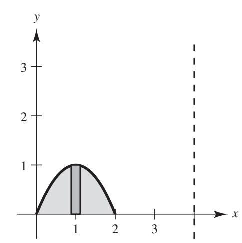
22. $p(y) = y, h(y) = y - \left(y^2 - 2\right) = 2 + y - y^2$

$$V = 2\pi\int_0^2 y\left(2 + y - y^2\right) dy$$
$$= 2\pi\int_0^2 \left(2y + y^2 - y^3\right) dy$$
$$= 2\pi\left[y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right]_0^2$$
$$= 2\pi\left(4 + \frac{8}{3} - 4\right) = \frac{16\pi}{3}$$



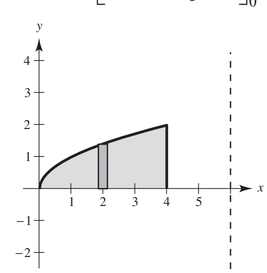
23. $p(x) = 4 - x, h(x) = 2x - x^2$

$$V = 2\pi\int_0^2 (4 - x)\left(2x - x^2\right) dx$$
$$= 2\pi\int_0^2 \left(8x - 6x^2 + x^3\right) dx$$
$$= 2\pi\left[4x^2 - 2x^3 + \frac{x^4}{4}\right]_0^2$$
$$= 2\pi[16 - 16 + 4] = 8\pi$$



24. $p(x) = 6 - x, h(x) = \sqrt{x}$

$$V = 2\pi\int_0^4 (6 - x)\sqrt{x}\, dx$$
$$= 2\pi\int_0^4 \left(6x^{1/2} - x^{3/2}\right) dx$$
$$= 2\pi\left[4x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^4 = \frac{192\pi}{5}$$



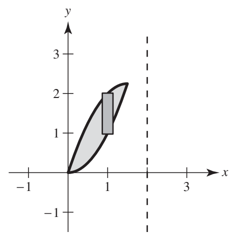
25. $3x - x^2 = x^2 \Rightarrow 3x = 2x^2 \Rightarrow x = 0, \dfrac{3}{2}$

$p(x) = 2 - x,\ h(x) = (3x - x^2) - x^2 = 3x - 2x^2$

$$V = 2\pi \int_0^{3/2} (2 - x)(3x - 2x^2)\,dx$$

$$= 2\pi \int_0^{3/2} (2x^3 - 7x^2 + 6x)\,dx$$

$$= 2\pi\left[\frac{x^4}{2} - \frac{7x^3}{3} + 3x^2\right]_0^{3/2} = \frac{45\pi}{16}$$



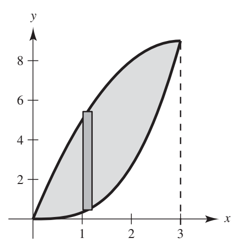
26. $p(x) = 3 - x,\ h(x) = (6x - x^2) - \dfrac{1}{3}x^3$

$$V = 2\pi\int_0^3 (3 - x)\left(6x - x^2 - \frac{x^3}{3}\right)dx$$

$$= 2\pi\int_0^3 \left(\frac{x^4}{3} - 9x^2 + 18x\right)dx$$

$$= 2\pi\left[\frac{x^5}{15} - 3x^3 + 9x^2\right]_0^3$$

$$= \frac{162\pi}{5}$$



27. The shell method would be easier:

$$V = 2\pi\int_0^4 \left[4 - (y - 2)^2\right] y\,dy$$

Using the disk method:

$$V = \pi\int_0^4 \left[\left(2 + \sqrt{4 - x}\right)^2 - \left(2 - \sqrt{4 - x}\right)^2\right]dx$$

$\left[\textbf{Note: } V = \dfrac{128\pi}{3}\right]$

28. The shell method is easier: $V = 2\pi\int_0^{\ln 4} x(4 - e^x)\,dx$

Using the disk method, $x = \ln(4 - y)$ and

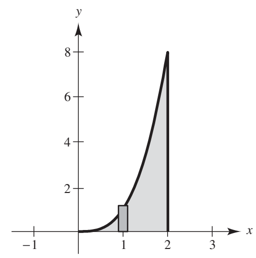
$$V = \pi\int_0^3 (\ln(4 - y))^2\,dy.$$

$\left[\textbf{Note: } V = \pi\left[8(\ln 2)^2 - 8\ln 2 + 3\right]\right]$

29. (a) **Disk**

$R(x) = x^3,\ r(x) = 0$

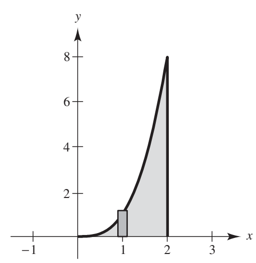
$$V = \pi\int_0^2 x^6\,dx = \pi\left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$$



(b) **Shell**

$p(x) = x,\ h(x) = x^3$

$$V = 2\pi\int_0^2 x^4\,dx = 2\pi\left[\frac{x^5}{5}\right]_0^2 = \frac{64\pi}{5}$$



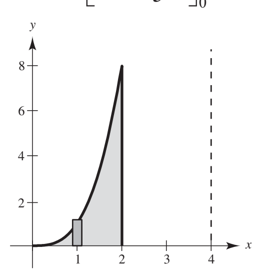
(c) **Shell**

$p(x) = 4 - x,\ h(x) = x^3$

$$V = 2\pi\int_0^2 (4 - x)x^3\,dx$$

$$= 2\pi\int_0^2 (4x^3 - x^4)\,dx$$

$$= 2\pi\left[x^4 - \frac{1}{5}x^5\right]_0^2 = \frac{96\pi}{5}$$



30. (a) **Disk**

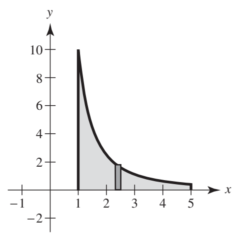
$$R(x) = \frac{10}{x^2},\ r(x) = 0$$

$$V = \pi\int_1^5 \left(\frac{10}{x^2}\right)^2 dx$$

$$= 100\pi\int_1^5 x^{-4}\,dx$$

$$= 100\pi\left[\frac{x^{-3}}{-3}\right]_1^5$$

$$= -\frac{100\pi}{3}\left(\frac{1}{125} - 1\right) = \frac{496}{15}\pi$$



(b) **Shell**

$$R(x) = x,\ r(x) = 0$$

$$V = 2\pi\int_1^5 x\left(\frac{10}{x^2}\right) dx$$

$$= 20\pi\int_1^5 \frac{1}{x}\,dx$$

$$= 20\pi\Big[\ln|x|\Big]_1^5 = 20\pi \ln 5$$

(c) **Disk**

$$R(x) = 10,\ r(x) = 10 - \frac{10}{x^2}$$

$$V = \pi\int_1^5 \left[10^2 - \left(10 - \frac{10}{x^2}\right)^2\right] dx$$

$$= \pi\left[\frac{100}{3x^3} - \frac{200}{x}\right]_1^5 = \frac{1904}{15}\pi$$

31. (a) **Shell**

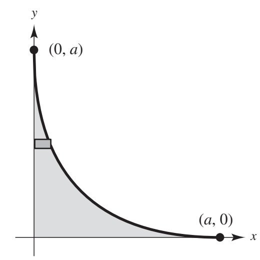
$$p(y) = y,\ h(y) = \left(a^{1/2} - y^{1/2}\right)^2$$

$$V = 2\pi\int_0^a y\left(a - 2a^{1/2}y^{1/2} + y\right) dy$$

$$= 2\pi\int_0^a \left(ay - 2a^{1/2}y^{3/2} + y^2\right) dy$$

$$= 2\pi\left[\frac{a}{2}y^2 - \frac{4a^{1/2}}{5}y^{5/2} + \frac{y^3}{3}\right]_0^a$$

$$= 2\pi\left(\frac{a^3}{2} - \frac{4a^3}{5} + \frac{a^3}{3}\right) = \frac{\pi a^3}{15}$$



(b) Same as part (a) by symmetry

(c) **Shell**

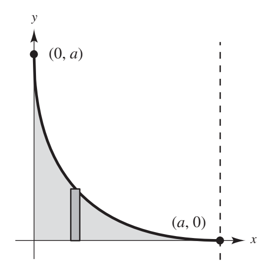
$$p(x) = a - x,\ h(x) = \left(a^{1/2} - x^{1/2}\right)^2$$

$$V = 2\pi\int_0^a (a - x)\left(a^{1/2} - x^{1/2}\right)^2 dx$$

$$= 2\pi\int_0^a \left(a^2 - 2a^{3/2}x^{1/2} + 2a^{1/2}x^{3/2} - x^2\right) dx$$

$$= 2\pi\left[a^2x - \frac{4}{3}a^{3/2}x^{3/2} + \frac{4}{5}a^{1/2}x^{5/2} - \frac{1}{3}x^3\right]_0^a$$

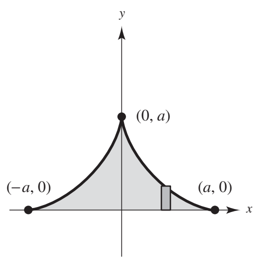
$$= \frac{4\pi a^3}{15}$$



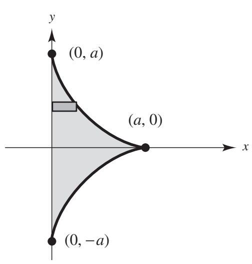
32. (a) **Disk**

$$R(x) = \left(a^{2/3} - x^{2/3}\right)^{3/2},\ r(x) = 0$$

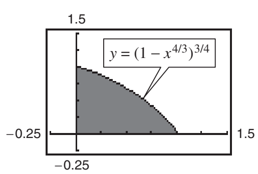
$$\begin{aligned} V &= \pi\int_{-a}^{a}\left(a^{2/3} - x^{2/3}\right)^3\,dx \\ &= 2\pi\int_0^a\left(a^2 - 3a^{4/3}x^{2/3} + 3a^{2/3}x^{4/3} - x^2\right)dx \\ &= 2\pi\left[a^2x - \frac{9}{5}a^{4/3}x^{5/3} + \frac{9}{7}a^{2/3}x^{7/3} - \frac{1}{3}x^3\right]_0^a \\ &= 2\pi\left(a^3 - \frac{9}{5}a^3 + \frac{9}{7}a^3 - \frac{1}{3}a^3\right) = \frac{32\pi a^3}{105} \end{aligned}$$



(b) Same as part (a) by symmetry



33. (a)

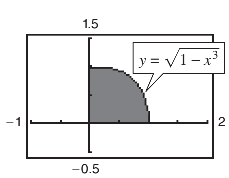


(b) $x^{4/3} + y^{4/3} = 1,\ x = 0,\ y = 0$

$$y = \left(1 - x^{4/3}\right)^{3/4}$$

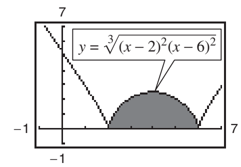
$$V = 2\pi\int_0^1 x\left(1 - x^{4/3}\right)^{3/4}\,dx \approx 1.5056$$

34. (a)



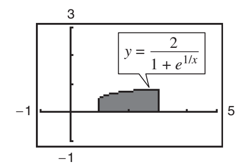
(b) $V = 2\pi\int_0^1 x\sqrt{1 - x^3}\,dx \approx 2.3222$

35. (a)



(b) $V = 2\pi\int_2^6 x\sqrt[3]{(x-2)^2(x-6)^2}\,dx \approx 187.249$

36. (a)



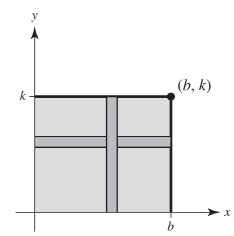
(b) $V = 2\pi\int_1^3 \dfrac{2x}{1 + e^{1/x}}\,dx \approx 19.0162$

37. (a) radius $= k$

height $= b$

(b) radius $= b$

height $= k$



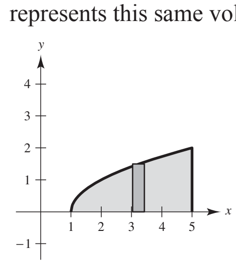
38. Rotating the region from $[0, 3]$ produces the same region as rotating the region from $[-3, 3]$, since the function is symmetric about the y-axis. Note that you will not be successful if you attempt to integrate from $[-3, 3]$.

39. $\pi\int_1^5 (x - 1)\,dx = \pi\int_1^5\left(\sqrt{x-1}\right)^2\,dx$

This integral represents the volume of the solid generated by revolving the region bounded by $y = \sqrt{x - 1}$, $y = 0$, and $x = 5$ about the x-axis by using the disk method.

$2\pi\int_0^2 y\left[5 - \left(y^2 + 1\right)\right]dy$

represents this same volume by using the shell method.



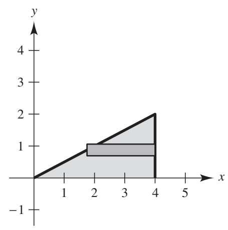
Disk method

40. $2\pi\int_0^4 x\left(\frac{x}{2}\right) dx$

This integral represents the volume of the solid generated by revolving the region bounded by $y = x/2$, $y = 0$, and $x = 4$ about the y-axis by using the shell method.

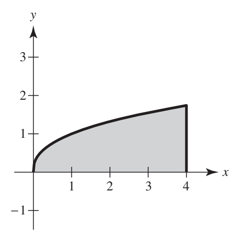
$$\pi\int_0^2 \left[16 - (2y)^2\right] dy = \pi\int_0^2 \left[(4)^2 - (2y)^2\right] dy$$

represents this same volume by using the disk method.



Disk method

41.



(a) Around x-axis: $V = \pi\int_0^4 \left(x^{2/5}\right)^2 dx = \left[\pi\frac{5}{9}x^{9/5}\right]_0^4$

$$= \frac{5}{9}\pi(4)^{9/5} \approx 6.7365\pi$$

(b) Around y-axis: $V = 2\pi\int_0^4 x\left(x^{2/5}\right) dx$

$$= \left[2\pi\frac{5}{12}x^{12/5}\right]_0^4 \approx 23.2147\pi$$

(c) Around $x = 4$:

$$V = 2\pi\int_0^4 (4 - x)x^{2/5}\, dx \approx 16.5819\pi$$

So, $(a) < (c) < (b)$.

42. (a) The figure will be a circle of radius AB and center A.

(b) The figure will be a circular cylinder of radius AB.

(c) Disk method: $V = \pi\int_0^3 \left[g(y)\right]^2 dy$

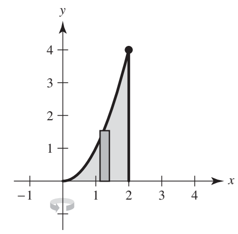
Shell method: $V = 2\pi\int_0^{2.45} x\, f(x)\, dx$

43. $2\pi\int_0^2 x^3\, dx = 2\pi\int_0^2 x\left(x^2\right) dx$

(a) Plane region bounded by

$y = x^2$, $y = 0$, $x = 0$, $x = 2$

(b) Revolved about the y-axis

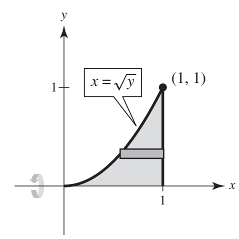


Other answers possible.

44. $2\pi\int_0^1 \left(y - y^{3/2}\right) dy = 2\pi\int_0^1 y\left(1 - \sqrt{y}\right) dy$

(a) Plane region bounded by $x = \sqrt{y}$, $x = 1$, $y = 0$

(b) Revolved about the x-axis



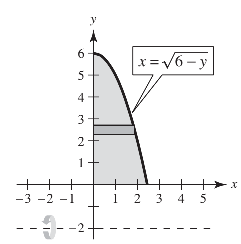
Other answers possible.

45. $2\pi\int_0^6 (y + 2)\sqrt{6 - y}\, dy$

(a) Plane region bounded by

$x = \sqrt{6 - y}$, $x = 0$, $y = 0$

(b) Revolved around line $y = -2$

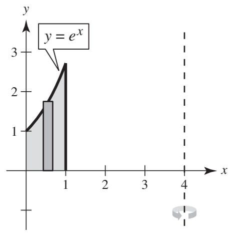


Other answers possible.

46. $2\pi\int_0^1 (4 - x)e^x\,dx$

(a) Plane region bounded by
$y = e^x, y = 0, x = 0, x = 1$

(b) Revolved about the line $x = 4$



47. $p(x) = x, h(x) = 2 - \frac{1}{2}x^2$

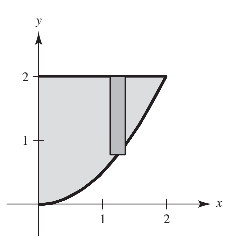
$$V = 2\pi\int_0^2 x\left(2 - \frac{1}{2}x^2\right) dx$$
$$= 2\pi\int_0^2 \left(2x - \frac{1}{2}x^3\right) dx$$
$$= 2\pi\left[x^2 - \frac{1}{8}x^4\right]_0^2 = 4\pi \quad \text{(total volume)}$$

Now find x_0 such that:

$$\pi = 2\pi\int_0^{x_0} \left(2x - \frac{1}{2}x^3\right) dx$$
$$1 = 2\left[x^2 - \frac{1}{8}x^4\right]_0^{x_0}$$
$$1 = 2x_0^2 - \frac{1}{4}x_0^4$$
$$x_0^4 - 8x_0^2 + 4 = 0$$
$$x_0^2 = 4 \pm 2\sqrt{3} \quad \text{(Quadratic Formula)}$$

Take $x_0 = \sqrt{4 - 2\sqrt{3}} \approx 0.73205$, because the other root is too large.

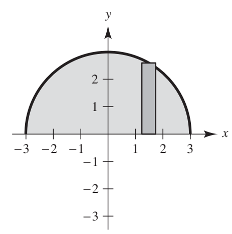
Diameter: $2\sqrt{4 - 2\sqrt{3}} \approx 1.464$



48. Total volume of the hemisphere is
$\frac{1}{2}\left(\frac{4}{3}\right)\pi r^3 = \frac{2}{3}\pi(3)^3 = 18\pi$. By the Shell Method,
$p(x) = x, h(x) = \sqrt{9 - x^2}$. Find x_0 such that:

$$6\pi = 2\pi\int_0^{x_0} x\sqrt{9 - x^2}\,dx$$
$$6 = -\int_0^{x_0} \left(9 - x^2\right)^{1/2}(-2x)\,dx$$
$$= \left[-\frac{2}{3}\left(9 - x^2\right)^{3/2}\right]_0^{x_0} = 18 - \frac{2}{3}\left(9 - x_0^2\right)^{3/2}$$
$$\left(9 - x_0^2\right)^{3/2} = 18$$
$$x_0 = \sqrt{9 - 18^{2/3}} \approx 1.460$$

Diameter: $2\sqrt{9 - 18^{2/3}} \approx 2.920$



49. $V = 4\pi\int_{-1}^{1} (2 - x)\sqrt{1 - x^2}\,dx$

$$= 8\pi\int_{-1}^{1} \sqrt{1 - x^2}\,dx - 4\pi\int_{-1}^{1} x\sqrt{1 - x^2}\,dx$$
$$= 8\pi\left(\frac{\pi}{2}\right) + 2\pi\int_{-1}^{1} x\left(1 - x^2\right)^{1/2}(-2)\,dx$$
$$= 4\pi^2 + \left[2\pi\left(\frac{2}{3}\right)\left(1 - x^2\right)^{3/2}\right]_{-1}^{1} = 4\pi^2$$

50. $V = 4\pi\int_{-r}^{r} (R - x)\sqrt{r^2 - x^2}\,dx$

$$= 4\pi R\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx - 4\pi\int_{-r}^{r} x\sqrt{r^2 - x^2}\,dx$$
$$= 4\pi R\left(\frac{\pi r^2}{2}\right) + \left[2\pi\left(\frac{2}{3}\right)\left(r^2 - x^2\right)^{3/2}\right]_{-r}^{r}$$
$$= 2\pi^2 r^2 R$$

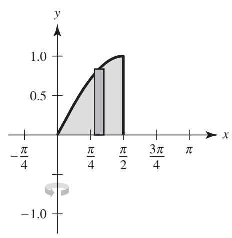
51. (a) $\frac{d}{dx}[\sin x - x\cos x + C] = \cos x + x\sin x - \cos x$

$= x\sin x$

So, $\int x\sin x\,dx = \sin x - x\cos x + C.$

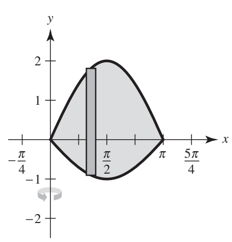
(b) (i) $p(x) = x, h(x) = \sin x$

$$V = 2\pi\int_0^{\pi/2} x\sin x\,dx$$
$$= 2\pi[\sin x - x\cos x]_0^{\pi/2}$$
$$= 2\pi[(1 - 0) - 0] = 2\pi$$



(ii) $p(x) = x, h(x) = 2\sin x - (-\sin x) = 3\sin x$

$$V = 2\pi\int_0^{\pi} x(3\sin x)\,dx$$
$$= 6\pi\int_0^{\pi} x\sin x\,dx$$
$$= 6\pi[\sin x - x\cos x]_0^{\pi}$$
$$= 6\pi(\pi) = 6\pi^2$$



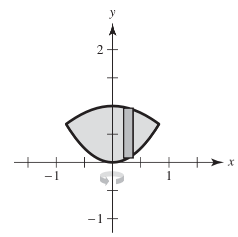
52. (a) $\frac{d}{dx}[\cos x + x\sin x + C] = -\sin x + \sin x + x\cos x$

$= x\cos x$

Hence, $\int x\cos x\,dx = \cos x + x\sin x + C.$

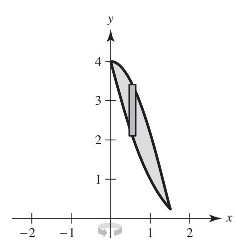
(b) (i) $x^2 = \cos x \Rightarrow x \approx \pm 0.8241$

$$V \approx 2(2\pi)\int_0^{0.8241} x\left[\cos x - x^2\right]dx$$
$$= 4\pi\left[\cos x + x\sin x - \frac{x^4}{4}\right]_0^{0.8241} \approx 2.1205$$



(ii) $4\cos x = (x - 2)^2 \Rightarrow x = 0, 1.5110$

$$V \approx 2\pi\int_0^{1.511} x\left[4\cos x - (x - 2)^2\right]dx$$
$$= 2\pi\int_0^{1.511}\left[4\cos x + 4x\sin x - \frac{(x - 2)^3}{3}\right]_0^{1.511}$$
$$= 6.2993$$

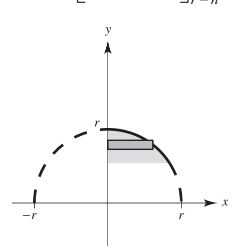


53. Disk Method

$R(y) = \sqrt{r^2 - y^2}$

$r(y) = 0$

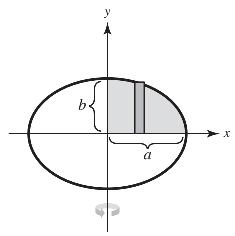
$$V = \pi\int_{r-h}^{r}\left(r^2 - y^2\right)dy$$
$$= \pi\left[r^2y - \frac{y^3}{3}\right]_{r-h}^{r} = \frac{1}{3}\pi h^2(3r - h)$$



54. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$



$$p(x) = x,\ h(x) = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$V = 2(2\pi)\int_0^a xb\sqrt{1 - \frac{x^2}{a^2}}\,dx$$

$$= \frac{4\pi b}{a}\int_0^a \sqrt{a^2 - x^2}\,x\,dx$$

$$= \frac{4\pi b}{a}\left[\left(\frac{-(a^2 - x^2)^{3/2}}{3}\right)\right]_0^a$$

$$= \frac{4\pi b}{3a}a^3 = \frac{4}{3}\pi a^2 b$$

If the region is revolved about the x-axis, then by symmetry the volume would be $V = \frac{4}{3}\pi ab^2$.

Note: If $a = b$, then volume is that of a sphere.

55. (a) Area of region $= \int_0^b \left[ab^n - ax^n\right]dx$

$$= \left[ab^n x - a\frac{x^{n+1}}{n+1}\right]_0^b$$

$$= ab^{n+1} - a\frac{b^{n+1}}{n+1}$$

$$= ab^{n+1}\left(1 - \frac{1}{n+1}\right)$$

$$= ab^{n+1}\left(\frac{n}{n+1}\right)$$

$$R_1(n) = \frac{ab^{n+1}\left[n/(n+1)\right]}{(ab^n)b} = \frac{n}{n+1}$$

(b) $\lim_{n\to\infty} R_1(n) = \lim_{n\to\infty} \frac{n}{n+1} = 1$

$\lim_{n\to\infty}(ab^n)b = \infty$

(c) **Disk Method:**

$$V = 2\pi\int_0^b x\left(ab^n - ax^n\right)dx$$

$$= 2\pi a\int_0^b \left(xb^n - x^{n+1}\right)dx$$

$$= 2\pi a\left[\frac{b^n}{2}x^2 - \frac{x^{n+2}}{n+2}\right]_0^b$$

$$= 2\pi a\left[\frac{b^{n+2}}{2} - \frac{b^{n+2}}{n+2}\right] = \pi ab^{n+2}\left(\frac{n}{n+2}\right)$$

$$R_2(n) = \frac{\pi ab^{n+2}\left[n/(n+2)\right]}{(\pi b^2)(ab^n)} = \left(\frac{n}{n+2}\right)$$

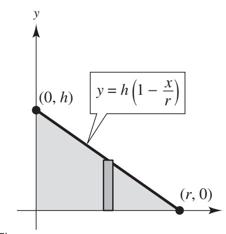
(d) $\lim_{n\to\infty} R_2(n) = \lim_{n\to\infty}\left(\frac{n}{n+2}\right) = 1$

$\lim_{n\to\infty}(\pi b^2)(ab^n) = \infty$

(e) As $n \to \infty$, the graph approaches the line $x = b$.

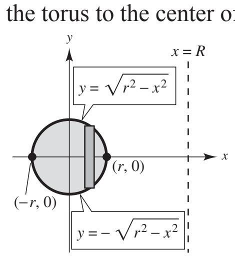
56. (a) $2\pi\int_0^r hx\left(1 - \frac{x}{r}\right)dx$ (ii)

is the volume of a right circular cone with the radius of the base as r and height h.



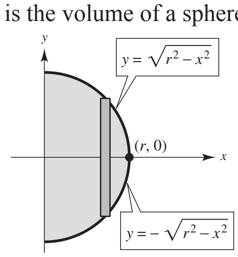
(b) $2\pi\int_{-r}^r (R - x)\left(2\sqrt{r^2 - x^2}\right)dx$ (v)

is the volume of a torus with the radius of its circular cross section as r and the distance from the axis of the torus to the center of its cross section as R.



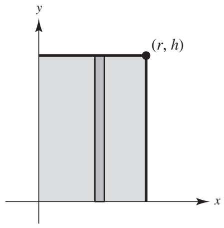
(c) $2\pi\int_0^r 2x\sqrt{r^2 - x^2}\,dx$ (iii)

is the volume of a sphere with radius r.



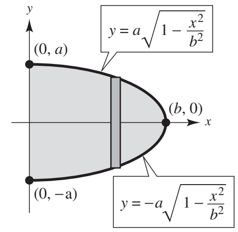
(d) $2\pi\int_0^r hx\,dx$ (i)

is the volume of a right circular cylinder with a radius of r and a height of h.



(e) $2\pi\int_0^b 2ax\sqrt{1-\left(x^2/b^2\right)}\,dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.



57. Top line: $y - 50 = \dfrac{40-50}{20-0}(x-0) = -\dfrac{1}{2}x \Rightarrow y = -\dfrac{1}{2}x + 50$

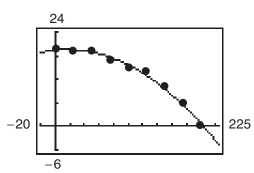
Bottom line: $y - 40 = \dfrac{0-40}{40-20}(x-20) = -2(x-20) \Rightarrow y = -2x + 80$

$$V = 2\pi\int_0^{20} x\left(-\frac{1}{2}x + 50\right)dx + 2\pi\int_{20}^{40} x(-2x+80)\,dx$$

$$= 2\pi\int_0^{20}\left(-\frac{1}{2}x^2 + 50x\right)dx + 2\pi\int_{20}^{40}\left(-2x^2 + 80x\right)dx$$

$$= 2\pi\left[-\frac{x^3}{6} + 25x^2\right]_0^{20} + 2\pi\left[-\frac{2x^3}{3} + 40x^2\right]_{20}^{40} = 2\pi\left(\frac{26{,}000}{3}\right) + 2\pi\left(\frac{32{,}000}{3}\right) \approx 121{,}475 \text{ ft}^3$$

58. (a) $d = -0.000561x^2 + 0.0189x + 19.39$



(b) $V \approx 2\pi\int_0^{200} xd(x)\,dx \approx 2\pi(213{,}800) = 1{,}343{,}345 \text{ ft}^3$

(c) Number of gallons $\approx V(7.48) = 10{,}048{,}221$ gal

59. $V_1 = \pi\int_{1/4}^{c}\frac{1}{x^2}\,dx = \pi\left[-\frac{1}{x}\right]_{1/4}^{c} = \pi\left[-\frac{1}{c} + 4\right] = \frac{4c-1}{c}\pi$

$V_2 = 2\pi\int_{1/4}^{c} x\left(\frac{1}{x}\right)dx = \left[2\pi x\right]_{1/4}^{c} = 2\pi\left(c - \frac{1}{4}\right)$

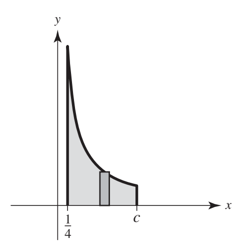
$$V_1 = V_2 \Rightarrow \frac{4c-1}{c}\pi = 2\pi\left(c - \frac{1}{4}\right)$$

$$4c - 1 = 2c\left(c - \frac{1}{4}\right)$$

$$4c^2 - 9c + 2 = 0$$

$$(4c-1)(c-2) = 0$$

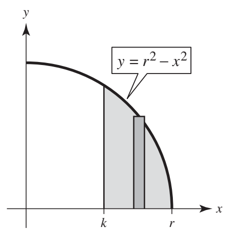
$$c = 2 \quad \left(c = \frac{1}{4} \text{ yields no volume.}\right)$$



60. (a) $p(x) = x,\ h(x) = r^2 - x^2$

Shell method:

$$V = 2\pi \int_k^r x\left(r^2 - x^2\right) dx$$
$$= -\pi \int_k^r \left(r^2 - x^2\right)(-2x)\, dx$$
$$= -\pi \left[\frac{\left(r^2 - x^2\right)^2}{2}\right]_k^r$$
$$= -\pi \left[0 - \frac{\left(r^2 - k^2\right)^2}{2}\right] = \frac{\pi}{2}\left(r^2 - k^2\right)^2$$

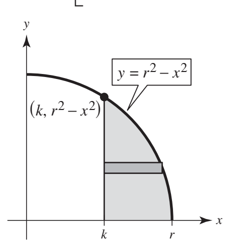


(b) $y = r^2 - x^2$

$x = \sqrt{r^2 - y}$

Disk method:

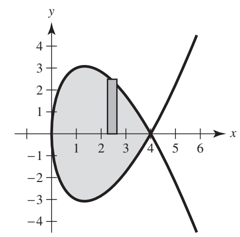
$$V = \pi \int_0^{r^2 - k^2} \left[\left(\sqrt{r^2 - y}\right)^2 - k^2\right] dy$$
$$= \pi \int_0^{r^2 - k^2} \left[r^2 - y - k^2\right] dy$$
$$= \pi \left[\left(r^2 - k^2\right)y - \frac{y^2}{2}\right]_0^{r^2 - k^2}$$
$$= \pi \left[\left(r^2 - k^2\right)^2 - \frac{\left(r^2 - k^2\right)^2}{2}\right] = \frac{\pi}{2}\left(r^2 - k^2\right)^2$$



61. $y^2 = x(4 - x)^2, \quad 0 \le x \le 4$

$y_1 = \sqrt{x(4 - x)^2} = (4 - x)\sqrt{x}$

$y_2 = -\sqrt{x(4 - x)^2} = -(4 - x)\sqrt{x}$



(a) $V = \pi \int_0^4 x(4 - x)^2\, dx$

$$= \pi \int_0^4 \left(x^3 - 8x^2 + 16x\right) dx$$
$$= \pi \left[\frac{x^4}{4} - \frac{8x^3}{3} + 8x^2\right]_0^4 = \frac{64\pi}{3}$$

(b) $V = 4\pi \int_0^4 x(4 - x)\sqrt{x}\, dx$

$$= 4\pi \int_0^4 \left(4x^{3/2} - x^{5/2}\right) dx$$
$$= 4\pi \left[\frac{8}{5}x^{5/2} - \frac{2}{7}x^{7/2}\right]_0^4 = \frac{2048\pi}{35}$$

(c) $V = 4\pi \int_0^4 (4 - x)(4 - x)\sqrt{x}\, dx$

$$= 4\pi \int_0^4 \left(16\sqrt{x} - 8x^{3/2} + x^{5/2}\right) dx$$
$$= 4\pi \left[\frac{32}{3}x^{3/2} - \frac{16}{5}x^{5/2} + \frac{2}{7}x^{7/2}\right]_0^4 = \frac{8192\pi}{105}$$

Section 7.4 Arc Length and Surfaces of Revolution

1. The graph of a function f is rectifiable between $(a, f(a))$ and $(b, f(b))$ if f' is continuous on $[a, b]$.

2. For a smooth curve given by $y = f(x)$ on $[a, b]$, the arc length of f between a and b is

$$s = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

3. The function is $f(x) = 2x^2$ $\left(\text{or } f(x) = 2x^2 + C\right)$, since $f'(x) = 4x$.

4. A surface of revolution is generated by revolving the graph of a continuous function about a line.

5. (a) $d = \sqrt{(5-2)^2 + (3-1)^2} = \sqrt{9+4} = \sqrt{13}$

(b) $m = \dfrac{3-1}{5-2} = \dfrac{2}{3}$

Line $y - 1 = \dfrac{2}{3}(x-2)$ or $y = \dfrac{2}{3}x - \dfrac{1}{3}$

$y' = \dfrac{2}{3}$

$s = \int_2^5 \sqrt{1 + \left(\dfrac{2}{3}\right)^2}\, dx = \left[\dfrac{\sqrt{13}}{3}x\right]_2^5 = \sqrt{13}$

6. (a) $d = \sqrt{[4-(-2)]^2 + (-6-2)^2} = \sqrt{36+64} = 10$

(b) $m = \dfrac{2-(-6)}{-2-4} = \dfrac{8}{-6} = -\dfrac{4}{3}$

Line: $y - 2 = -\dfrac{4}{3}(x+2)$ or $y = -\dfrac{4}{3}x - \dfrac{2}{3}$

$y' = -\dfrac{4}{3}$

$s = \int_{-2}^4 \sqrt{1 + \left(-\dfrac{4}{3}\right)^2}\, dx = \int_{-2}^4 \dfrac{5}{3}\, dx = \left[\dfrac{5}{3}x\right]_{-2}^4 = 10$

7. $y = \dfrac{2}{3}(x^2+1)^{3/2}$

$y' = (x^2+1)^{1/2}(2x), \quad 0 \le x \le 1$

$1 + (y')^2 = 1 + 4x^2(x^2+1)$

$= 4x^4 + 4x^2 + 1 = (2x^2+1)^2$

$s = \int_0^1 \sqrt{1 + (y')^2}\, dx$

$= \int_0^1 (2x^2+1)\, dx = \left[\dfrac{2x^3}{3} + x\right]_0^1 = \dfrac{5}{3}$

8. $y = \dfrac{x^4}{8} + \dfrac{1}{4x^2}$

$y' = \dfrac{1}{2}x^3 - \dfrac{1}{2x^3}, \quad 1 \le x \le 3$

$1 + (y')^2 = \left(\dfrac{1}{2}x^3 + \dfrac{1}{2x^3}\right)^2, \quad [1, 3]$

$s = \int_a^b \sqrt{1 + (y')^2}\, dx$

$= \int_1^3 \left(\dfrac{1}{2}x^3 + \dfrac{1}{2x^3}\right) dx$

$= \left[\dfrac{1}{8}x^4 - \dfrac{1}{4x^2}\right]_1^3$

$= \dfrac{92}{9} \approx 10.222$

9. $y = \frac{2}{3}x^{3/2} + 1$

$y' = x^{1/2}, \quad 0 \le x \le 1$

$s = \int_0^1 \sqrt{1+x}\, dx$

$= \left[\frac{2}{3}(1+x)^{3/2}\right]_0^1 = \frac{2}{3}\left(\sqrt{8} - 1\right) \approx 1.219$

10. $y = 2x^{3/2} + 3$

$y' = 3x^{1/2}, \quad 0 \le x \le 9$

$s = \int_0^9 \sqrt{1+9x}\, dx$

$= \left[\frac{2}{27}(1+9x)^{3/2}\right]_0^9 = \frac{2}{27}\left(82^{3/2} - 1\right) \approx 54.929$

11. $y = \dfrac{3}{2}x^{2/3}$

$y' = \dfrac{1}{x^{1/3}}, \quad 1 \le x \le 8$

$s = \int_1^8 \sqrt{1 + \left(\dfrac{1}{x^{1/3}}\right)^2}\, dx$

$= \int_1^8 \sqrt{\dfrac{x^{2/3}+1}{x^{2/3}}}\, dx$

$= \dfrac{3}{2}\int_1^8 \sqrt{x^{2/3}+1}\left(\dfrac{2}{3x^{1/3}}\right) dx$

$= \dfrac{3}{2}\left[\dfrac{2}{3}\left(x^{2/3}+1\right)^{3/2}\right]_1^8$

$= 5\sqrt{5} - 2\sqrt{2} \approx 8.352$

12. $y = \dfrac{3}{2}x^{2/3} + 4$

$y' = x^{-1/3}, \quad 1 \le x \le 27$

$s = \int_1^{27} \sqrt{1 + \left(\dfrac{1}{x^{1/3}}\right)^2}\, dx$

$= \int_1^{27} \sqrt{\dfrac{x^{2/3}+1}{x^{2/3}}}\, dx$

$= \dfrac{3}{2}\int_1^{27} \sqrt{x^{2/3}+1}\left(\dfrac{2}{3x^{1/3}}\right) dx$

$= \left[\dfrac{3}{2} \cdot \dfrac{2}{3}\left(x^{2/3}+1\right)^{3/2}\right]_1^{27}$

$= 10^{3/2} - 2^{3/2} \approx 28.794$

13.

$$y = \frac{x^5}{10} + \frac{1}{6x^3}, \quad 2 \le x \le 5$$

$$y' = \frac{x^4}{2} - \frac{1}{2x^4} = \frac{1}{2}\left(x^4 - \frac{1}{x^4}\right)$$

$$1 + (y')^2 = 1 + \frac{1}{4}\left(x^4 - \frac{1}{x^4}\right)^2 = 1 + \frac{1}{4}\left(x^8 - 2 + \frac{1}{x^8}\right)$$

$$= \frac{1}{4}\left(x^8 + 2 + \frac{1}{x^8}\right) = \frac{1}{4}\left(x^4 + \frac{1}{x^4}\right)^2$$

$$s = \int_2^5 \sqrt{1 + (y')^2}\,dx = \int_2^5 \frac{1}{2}\left(x^4 + \frac{1}{x^4}\right)dx$$

$$= \frac{1}{2}\left[\frac{x^5}{5} - \frac{1}{3x^3}\right]_2^5 = \frac{1}{2}\left[\left(625 - \frac{1}{375}\right) - \left(\frac{32}{5} - \frac{1}{24}\right)\right]$$

$$= \frac{618{,}639}{2000} \approx 309.320$$

14.

$$y = \frac{x^7}{14} + \frac{1}{10x^5}, [1, 2]$$

$$y' = \frac{x^6}{2} - \frac{1}{2x^6} = \frac{1}{2}\left(x^6 - \frac{1}{x^6}\right)$$

$$1 + (y')^2 = 1 + \frac{1}{4}\left(x^{12} - 2 + \frac{1}{x^{12}}\right)$$

$$= \frac{1}{4}\left(x^{12} + 2 + \frac{1}{x^{12}}\right)$$

$$= \left[\frac{1}{2}\left(x^6 + \frac{1}{x^6}\right)\right]^2$$

$$s = \int_1^2 \sqrt{1 + (y')^2}\,dx$$

$$= \int_1^2 \frac{1}{2}\left(x^6 + \frac{1}{x^6}\right)dx$$

$$= \left[\frac{1}{2}\left(\frac{x^7}{7} - \frac{1}{5x^5}\right)\right]_1^2$$

$$= \frac{1}{2}\left(\frac{128}{7} - \frac{1}{160}\right) - \frac{1}{2}\left(\frac{1}{7} - \frac{1}{5}\right)$$

$$= \frac{20{,}537}{2240} \approx 9.168$$

15.

$$y = \ln(\sin x), \quad \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

$$y' = \frac{1}{\sin x}\cos x = \cot x$$

$$1 + (y')^2 = 1 + \cot^2 x = \csc^2 x$$

$$s = \int_{\pi/4}^{3\pi/4} \csc x\,dx$$

$$= \Big[\ln\left|\csc x - \cot x\right|\Big]_{\pi/4}^{3\pi/4}$$

$$= \ln\left(\sqrt{2} + 1\right) - \ln\left(\sqrt{2} - 1\right) \approx 1.763$$

16.

$$y = \ln(\cos x), \quad \left[0, \frac{\pi}{3}\right]$$

$$y' = \frac{-\sin x}{\cos x} = -\tan x$$

$$1 + (y')^2 = 1 + \tan^2 x = \sec^2 x$$

$$s = \int_0^{\pi/3} \sqrt{\sec^2 x}\,dx$$

$$= \int_0^{\pi/3} \sec x\,dx$$

$$= \ln\left|\sec x + \tan x\right|\Big]_0^{\pi/3}$$

$$= \ln\left(2 + \sqrt{3}\right) \approx 1.3170$$

17.

$$y = \frac{1}{2}\left(e^x + e^{-x}\right)$$

$$y' = \frac{1}{2}\left(e^x - e^{-x}\right), \quad [0, 2]$$

$$1 + (y')^2 = \left[\frac{1}{2}\left(e^x + e^{-x}\right)\right]^2, \quad [0, 2]$$

$$s = \int_0^2 \sqrt{\left[\frac{1}{2}\left(e^x + e^{-x}\right)\right]^2}\,dx$$

$$= \frac{1}{2}\int_0^2 \left(e^x + e^{-x}\right)dx$$

$$= \frac{1}{2}\left[e^x - e^{-x}\right]_0^2 = \frac{1}{2}\left(e^2 - \frac{1}{e^2}\right) \approx 3.627$$

18. $y = \ln\left(\dfrac{e^x + 1}{e^x - 1}\right) = \ln(e^x + 1) - \ln(e^x - 1), \ [\ln 6, \ln 8]$

$$y' = \frac{e^x}{e^x + 1} - \frac{e^x}{e^x - 1} = \frac{2e^x}{1 - e^{2x}}$$

$$1 + (y')^2 = 1 + \frac{4e^{2x}}{1 - 2e^{2x} + e^{4x}} = \frac{1 + 2e^{2x} + e^{4x}}{\left(1 - e^{2x}\right)^2} = \frac{\left(1 + e^{2x}\right)^2}{\left(1 - e^{2x}\right)^2}$$

$$s = \int_a^b \sqrt{1 + \left(y^1\right)^2}\, dx = \int_{\ln 6}^{\ln 8} \frac{1 + e^{2x}}{e^{2x} - 1}\, dx = \int_{\ln 6}^{\ln 8} \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx = \int_{\ln 6}^{\ln 8} \coth x\, dx$$

$$= \left[\ln(\sinh x)\right]_{\ln 6}^{\ln 8} = \ln\left(\frac{63}{16}\right) - \ln\left(\frac{35}{12}\right)$$

$$= \ln\left(\frac{27}{20}\right) \approx 0.3001$$

19. $x = \dfrac{1}{3}\left(y^2 + 2\right)^{3/2}, \quad 0 \le y \le 4$

$$\frac{dx}{dy} = y\left(y^2 + 2\right)^{1/2}$$

$$s = \int_0^4 \sqrt{1 + y^2\left(y^2 + 2\right)}\, dy$$

$$= \int_0^4 \sqrt{y^4 + 2y^2 + 1}\, dy$$

$$= \int_0^4 \left(y^2 + 1\right) dy$$

$$= \left[\frac{y^3}{3} + y\right]_0^4 = \frac{64}{3} + 4 = \frac{76}{3}$$

20. $x = \dfrac{1}{3}\sqrt{y}(y - 3), \quad 1 \le y \le 4$

$$x = \frac{1}{3}\left(y^{3/2} - 3y^{1/2}\right)$$

$$\frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4}y + \frac{1}{4}y^{-1} - \frac{1}{2}$$

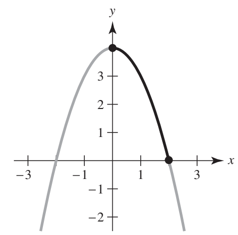
$$= \frac{1}{4}\left(y + 2 + y^{-1}\right) = \frac{1}{4}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2$$

$$s = \int_1^4 \frac{1}{2}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right) dy$$

$$= \left[\frac{1}{2}\left(\frac{3}{2}y^{3/2} + 2y^{1/2}\right)\right]_1^4$$

$$= \frac{1}{2}\left(\frac{16}{3} + 4\right) - \frac{1}{2}\left(\frac{2}{3} + 2\right) = \frac{10}{3}$$

21. (a) $y = 4 - x^2, \quad [0, 2]$



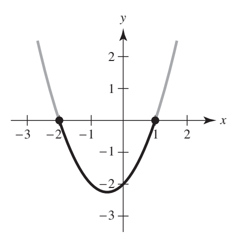
(b) $y' = -2x$

$$1 + (y')^2 = 1 + 4x^2$$

$$L = \int_0^2 \sqrt{1 + 4x^2}\, dx$$

(c) $L \approx 4.647$

22. (a) $y = x^2 + x - 2, \quad [-2, 1]$



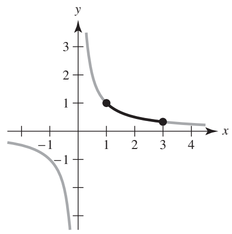
(b) $y' = 2x + 1$

$$1 + (y')^2 = 1 + 4x^2 + 4x + 1$$

$$L = \int_{-2}^1 \sqrt{2 + 4x + 4x^2}\, dx$$

(c) $L \approx 5.653$

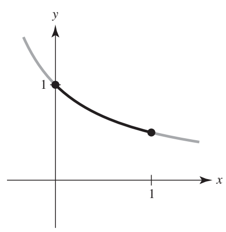
23. (a) $y = \dfrac{1}{x}, \quad [1, 3]$



(b) $$y' = -\frac{1}{x^2}$$
$$1 + (y')^2 = 1 + \frac{1}{x^4}$$
$$L = \int_1^3 \sqrt{1 + \frac{1}{x^4}}\, dx$$

(c) $L \approx 2.147$

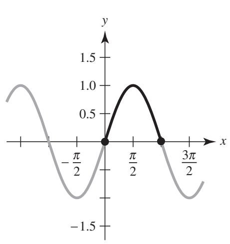
24. (a) $y = \dfrac{1}{1 + x}, \quad [0, 1]$



(b) $$y' = -\frac{1}{(1 + x)^2}$$
$$1 + (y')^2 = 1 + \frac{1}{(1 + x)^4}$$
$$L = \int_0^1 \sqrt{1 + \frac{1}{(1 + x)^4}}\, dx$$

(c) $L \approx 1.132$

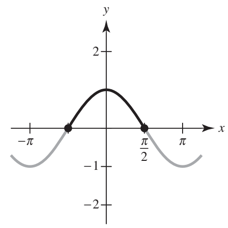
25. (a) $y = \sin x, \quad [0, \pi]$



(b) $$y' = \cos x$$
$$1 + (y')^2 = 1 + \cos^2 x$$
$$L = \int_0^\pi \sqrt{1 + \cos^2 x}\, dx$$

(c) $L \approx 3.820$

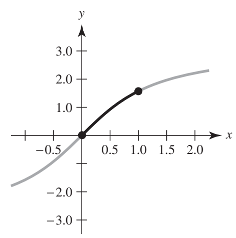
26. (a) $y = \cos x, \quad \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$



(b) $$y' = -\sin x$$
$$1 + (y')^2 = 1 + \sin^2 x$$
$$L = \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin^2 x}\, dx$$

(c) 3.820

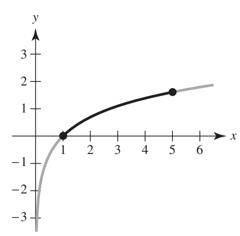
27. (a) $y = 2 \arctan x, \quad [0, 1]$



(b) $$y' = \frac{2}{1 + x^2}$$
$$L = \int_0^1 \sqrt{1 + \frac{4}{(1 + x^2)^2}}\, dx$$

(c) $L \approx 1.871$

28. (a) $y = \ln x, \quad [1, 5]$



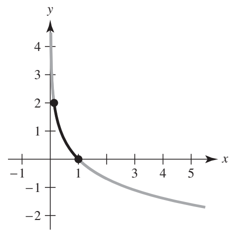
(b) $$y' = \frac{1}{x}$$
$$1 + (y')^2 = 1 + \frac{1}{x^2}$$
$$L = \int_1^5 \sqrt{1 + \frac{1}{x^2}}\, dx$$

(c) $L \approx 4.367$

29. (a) $x = e^{-y}, \quad 0 \le y \le 2$

$y = -\ln x$

$1 \ge x \ge e^{-2} \approx 0.135$



(b) $y' = -\dfrac{1}{x}$

$1 + (y')^2 = 1 + \dfrac{1}{x^2}$

$L = \displaystyle\int_{e^{-2}}^{1} \sqrt{1 + \frac{1}{x^2}}\, dx$

(c) $L \approx 2.221$

Alternatively, you can do all the computations with respect to y.

(a) $x = e^{-y}, \quad 0 \le y \le 2$

(b) $\dfrac{dx}{dy} = -e^{-y}$

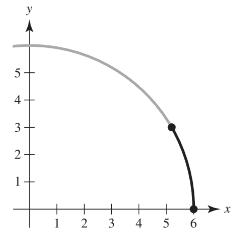
$1 + \left(\dfrac{dx}{dy}\right)^2 = 1 + e^{-2y}$

$L = \displaystyle\int_0^2 \sqrt{1 + e^{-2y}}\, dy$

(c) $L \approx 2.221$

30. (a) $x = \sqrt{36 - y^2}, \quad 0 \le y \le 3$

$y = \sqrt{36 - x^2}, \quad 3\sqrt{3} \le x \le 6$



(b) $\dfrac{dx}{dy} = \dfrac{1}{2}\left(36 - y^2\right)^{-1/2}(-2y) = \dfrac{-y}{\sqrt{36 - y^2}}$

$L = \displaystyle\int_0^3 \sqrt{1 + \frac{y^2}{36 - y^2}}\, dy = \int_0^3 \frac{6}{\sqrt{36 - y^2}}\, dy$

(c) $L \approx 3.142 \quad (\pi)$

31. $y = x^3, \quad [0, 4]$

(a) $d = \sqrt{(4 - 0)^2 + (64 - 0)^2} \approx 64.125$

(b) $d = \sqrt{(1 - 0)^2 + (1 - 0)^2} + \sqrt{(2 - 1)^2 + (8 - 1)^2} + \sqrt{(3 - 2)^2 + (27 - 8)^2} + \sqrt{(4 - 3)^2 + (64 - 27)^2} \approx 64.525$

(c) 64.672

32. $f(x) = \left(x^2 - 4\right)^2, \quad [0, 4]$

(a) $d = \sqrt{(4 - 0)^2 + (144 - 16)^2} \approx 128.062$

(b) $d = \sqrt{(1 - 0)^2 + (9 - 16)^2} + \sqrt{(2 - 1)^2 + (0 - 9)^2} + \sqrt{(3 - 2)^2 + (25 - 0)^2} + \sqrt{(4 - 3)^2 + (144 - 25)^2} \approx 160.151$

(c) 160.287

33.
$$y = 20\cosh\frac{x}{20}, \quad -20 \le x \le 20$$
$$y' = \sinh\frac{x}{20}$$
$$1 + (y')^2 = 1 + \sinh^2\frac{x}{20} = \cosh^2\frac{x}{20}$$
$$L = \int_{-20}^{20}\cosh\frac{x}{20}\,dx = 2\int_0^{20}\cosh\frac{x}{20}\,dx = \left[2(20)\sinh\frac{x}{20}\right]_0^{20} = 40\sinh(1) \approx 47.008 \text{ m}$$

34.
$$y = 31 - 10\left(e^{x/20} + e^{-x/20}\right)$$
$$y' = -\frac{1}{2}\left(e^{x/20} - e^{-x/20}\right)$$
$$1 + (y')^2 = 1 + \frac{1}{4}\left(e^{x/10} - 2 + e^{-x/10}\right) = \left[\frac{1}{2}\left(e^{x/20} + e^{-x/20}\right)\right]^2$$
$$s = \int_{-20}^{20}\sqrt{\left[\frac{1}{2}\left(e^{x/20} + e^{-x/20}\right)\right]^2}\,dx = \frac{1}{2}\int_{-20}^{20}\left(e^{x/20} + e^{-x/20}\right)dx = \left[10\left(e^{x/20} - e^{-x/20}\right)\right]_{-20}^{20} = 20\left(e - \frac{1}{e}\right) \approx 47 \text{ ft}$$

So, there are $100(47) = 4700$ square feet of roofing on the barn.

35. $y = 693.8597 - 68.7672\cosh 0.0100333x$

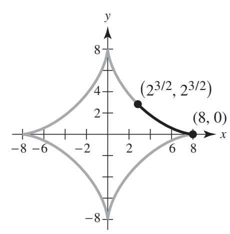
$y' = -0.6899619478\sinh 0.0100333x$

$$s = \int_{-299.2239}^{299.2239}\sqrt{1 + (-0.6899619478\sinh 0.0100333x)^2}\,dx \approx 1480$$

(Use a graphing utility.)

36. $x^{2/3} + y^{2/3} = 4$

$$y^{2/3} = 4 - x^{2/3}$$
$$y = \left(4 - x^{2/3}\right)^{3/2}$$
$$y' = \frac{3}{2}\left(4 - x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = \frac{-\left(4 - x^{2/3}\right)^{1/2}}{x^{1/3}}$$



In order to avoid division by 0, compute the arc length for $2^{3/2} \le x \le 8$, and multiply the answer by 8, as indicated in the figure.

$$1 + (y')^2 = 1 + \frac{4 - x^{2/3}}{x^{2/3}}, \quad 2^{3/2} \le x \le 8$$
$$= \frac{4}{x^{2/3}}$$
$$s = 8\int_{2^{3/2}}^{8}\sqrt{\frac{4}{x^{2/3}}}\,dx$$
$$= 16\int_{2^{3/2}}^{8} x^{-1/3}\,dx$$
$$= 16\left[\frac{3}{2}x^{2/3}\right]_{2^{3/2}}^{8}$$
$$= 24(4 - 2) = 48$$

37.

$$y = \sqrt{9 - x^2}$$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$1 + (y')^2 = \frac{9}{9 - x^2}$$

$$s = \int_0^2 \sqrt{\frac{9}{9 - x^2}}\,dx = \int_0^2 \frac{3}{\sqrt{9 - x^2}}\,dx$$

$$= \left[3 \arcsin \frac{x}{3}\right]_0^2 = 3\left(\arcsin \frac{2}{3} - \arcsin 0\right)$$

$$= 3 \arcsin \frac{2}{3} \approx 2.1892$$

38.

$$y = \sqrt{25 - x^2}$$

$$y' = \frac{-x}{\sqrt{25 - x^2}}$$

$$1 + (y')^2 = \frac{25}{25 - x^2}$$

$$s = \int_{-3}^4 \sqrt{\frac{25}{25 - x^2}}\,dx = \int_{-3}^4 \frac{5}{\sqrt{25 - x^2}}\,dx$$

$$= \left[5 \arcsin \frac{x}{5}\right]_{-3}^4 = 5\left[\arcsin \frac{4}{5} - \arcsin\left(-\frac{3}{5}\right)\right]$$

$$\approx 7.8540$$

$$\frac{1}{4}\left[2\pi(5)\right] \approx 7.8540 = s$$

39.

$$y = \frac{x^3}{3}$$

$$y' = x^2, \quad [0, 3]$$

$$S = 2\pi\int_0^3 \frac{x^3}{3}\sqrt{1 + x^4}\,dx$$

$$= \frac{\pi}{6}\int_0^3 \left(1 + x^4\right)^{1/2}\left(4x^3\right)dx$$

$$= \left[\frac{\pi}{9}\left(1 + x^4\right)^{3/2}\right]_0^3$$

$$= \frac{\pi}{9}\left(82\sqrt{82} - 1\right) \approx 258.85$$

40.

$$y = 2\sqrt{x}$$

$$y' = \frac{1}{\sqrt{x}}, \quad [4, 9]$$

$$S = 2\pi\int_4^9 2\sqrt{x}\sqrt{1 + \frac{1}{x}}\,dx$$

$$= 4\pi\int_4^9 \sqrt{x + 1}\,dx$$

$$= \left[\frac{8}{3}\pi(x + 1)^{3/2}\right]_4^9$$

$$= \frac{8\pi}{3}\left(10^{3/2} - 5^{3/2}\right) \approx 171.258$$

41.

$$y = \frac{x^3}{6} + \frac{1}{2x}$$

$$y' = \frac{x^2}{2} - \frac{1}{2x^2}$$

$$1 + (y')^2 = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2, \quad [1, 2]$$

$$S = 2\pi\int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)dx$$

$$= 2\pi\int_1^2 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3}\right)dx$$

$$= 2\pi\left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2}\right]_1^2 = \frac{47\pi}{16}$$

42.

$$y = 3x$$

$$y' = 3$$

$$1 + (y')^2 = 10, \quad [0, 3]$$

$$S = 2\pi\int_0^3 3x\sqrt{10}\,dx$$

$$= 6\pi\sqrt{10}\left[\frac{x^2}{2}\right]_0^3$$

$$= 27\sqrt{10}\pi$$

43.

$$y = \sqrt{4 - x^2}$$

$$y' = \frac{1}{2}\left(4 - x^2\right)^{-1/2}(-2x) = \frac{-x}{\sqrt{4 - x^2}}, \ -1 \le x \le 1$$

$$1 + (y')^2 = 1 + \frac{x^2}{4 - x^2} = \frac{4}{4 - x^2}$$

$$S = 2\pi\int_{-1}^1 \sqrt{4 - x^2} \cdot \sqrt{\frac{4}{4 - x^2}}\,dx$$

$$= 4\pi\int_{-1}^1 dx = 4\pi\left[x\right]_{-1}^1 = 8\pi$$

44.

$$y = \sqrt{9 - x^2}, \quad -2 \le x \le 2$$

$$y' = \frac{1}{2}\left(9 - x^2\right)^{-1/2}(-2x) = \frac{-x}{\sqrt{9 - x^2}}$$

$$1 + (y')^2 = 1 + \frac{x^2}{\left(9 - x^2\right)} = \frac{9}{9 - x^2}$$

$$S = 2\pi\int_{-2}^2 \sqrt{9 - x^2}\,\frac{3}{\sqrt{9 - x^2}}\,dx = 2\pi\int_{-2}^2 3\,dx$$

$$= 2\pi\left[3x\right]_{-2}^2 = 24\pi$$

45. $y = \sqrt[3]{x} + 2$

$$y' = \frac{1}{3x^{2/3}}, \quad [1, 8]$$

$$S = 2\pi\int_1^8 x\sqrt{1 + \frac{1}{9x^{4/3}}}\,dx$$

$$= \frac{2\pi}{3}\int_1^8 x^{1/3}\sqrt{9x^{4/3} + 1}\,dx$$

$$= \frac{\pi}{18}\int_1^8 \left(9x^{4/3} + 1\right)^{1/2}\left(12x^{1/3}\right)dx$$

$$= \left[\frac{\pi}{27}\left(9x^{4/3} + 1\right)^{3/2}\right]_1^8$$

$$= \frac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right) \approx 199.48$$

46. $y = 9 - x^2, \quad [0, 3]$

$$y' = -2x$$

$$S = 2\pi\int_0^3 x\sqrt{1 + 4x^2}\,dx$$

$$= \frac{\pi}{4}\int_0^3 \left(1 + 4x^2\right)^{1/2}(8x)\,dx$$

$$= \left[\frac{\pi}{6}\left(1 + 4x^2\right)^{3/2}\right]_0^3 = \frac{\pi}{6}\left(37^{3/2} - 1\right) \approx 117.319$$

47. $y = 1 - \frac{x^2}{4}$

$$y' = -\frac{x}{2}, \quad 0 \le x \le 2$$

$$1 + (y')^2 = 1 + \frac{x^2}{4} = \frac{4 + x^2}{4}$$

$$S = 2\pi\int_0^2 x\sqrt{\frac{4 + x^2}{4}}\,dx$$

$$= \pi\int_0^2 x\sqrt{4 + x^2}\,dx$$

$$= \frac{1}{2}\pi\int_0^2 \left(4 + x^2\right)^{1/2}(2x)\,dx$$

$$= \frac{1}{2}\pi\left[\frac{2}{3}\left(4 + x^2\right)^{3/2}\right]_0^2$$

$$= \frac{\pi}{3}\left(8^{3/2} - 4^{3/2}\right)$$

$$= \frac{\pi}{3}\left(16\sqrt{2} - 8\right) \approx 15.318$$

48. $y = \frac{x}{2} + 3$

$$y' = \frac{1}{2}$$

$$1 + (y')^2 = \frac{5}{4}, 1 \le x \le 5$$

$$S = 2\pi\int_1^5 x\sqrt{\frac{5}{4}}\,dx$$

$$= \sqrt{5}\pi\left[\frac{x^2}{2}\right]_1^5$$

$$= \sqrt{5}\pi\left(\frac{25}{2} - \frac{1}{2}\right) = 12\sqrt{5}\pi$$

49. $y = \sin x$

$$y' = \cos x, \quad [0, \pi]$$

$$S = 2\pi\int_0^\pi \sin x\sqrt{1 + \cos^2 x}\,dx \approx 14.4236$$

50. $y = \ln x$

$$y' = \frac{1}{x}$$

$$1 + (y')^2 = \frac{x^2 + 1}{x^2}, \quad [1, e]$$

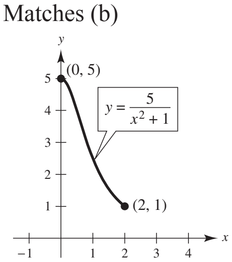
$$S = 2\pi\int_1^e x\sqrt{\frac{x^2 + 1}{x^2}}\,dx = 2\pi\int_1^e \sqrt{x^2 + 1}\,dx$$

$$\approx 22.943$$

51. $\displaystyle\int_0^2 \sqrt{1 + \left[\frac{d}{dx}\left(\frac{5}{x^2 + 1}\right)\right]^2}\,dx$

$s \approx 5$

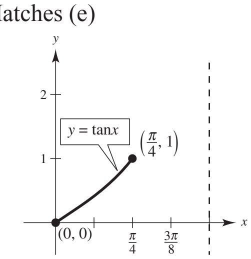
Matches (b)



52. $\displaystyle\int_0^{\pi/4} \sqrt{1 + \left[\frac{d}{dx}(\tan x)\right]^2}\,dx$

$s \approx 1$

Matches (e)



53. $f(x) = \frac{1}{4}e^x + e^{-x}, [a, b]$

Integral: $\int_a^b \left(\frac{1}{4}e^x + e^{-x}\right) dx = \left[\frac{e^x}{4} - e^{-x}\right]_a^b = \frac{e^b}{4} - e^{-b} + \frac{e^a}{4} - e^{-a}$

Arc length:

$$f'(x) = \frac{1}{4}e^x - e^{-x}$$

$$1 + f'(x)^2 = 1 + \frac{1}{16}e^{2x} - \frac{1}{2} + e^{-2x}$$

$$= \frac{1}{16}e^{2x} + \frac{1}{2} + e^{-2x}$$

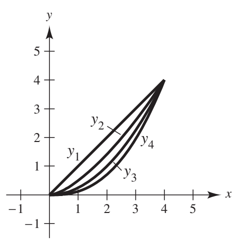
$$= \left(\frac{1}{4}e^x + e^{-x}\right)^2$$

$$s = \int_a^b \left(\frac{1}{4}e^x + e^{-x}\right) dx = \frac{e^b}{4} - e^{-b} + \frac{e^a}{4} - e^{-a}$$

They have the same value.

54. The surface of revolution given by f_1 will be larger. $r(x)$ is larger for f_1.

55. (a)



(b) y_1, y_2, y_3, y_4

(c) $y_1' = 1, \quad s_1 = \int_0^4 \sqrt{2}\, dx \approx 5.657$

$y_2' = \frac{3}{4}x^{1/2}, \quad s_2 = \int_0^4 \sqrt{1 + \frac{9x}{16}}\, dx \approx 5.759$

$y_3' = \frac{1}{2}x, \quad s_3 = \int_0^4 \sqrt{1 + \frac{x^2}{4}}\, dx \approx 5.916$

$y_4' = \frac{5}{16}x^{3/2}, \quad s_4 = \int_0^4 \sqrt{1 + \frac{25}{256}x^3}\, dx \approx 6.063$

56. (a) Area of circle with radius L: $A = \pi L^2$

Area of sector with central angle θ (in radians):

$$S = \frac{\theta}{2\pi}A = \frac{\theta}{2\pi}(\pi L^2) = \frac{1}{2}L^2\theta$$

(b) Let s be the arc length of the sector, which is the circumference of the base of the cone. Here, $s = L\theta = 2\pi r$, and you have

$$S = \frac{1}{2}L^2\theta = \frac{1}{2}L^2\left(\frac{s}{L}\right) = \frac{1}{2}Ls = \frac{1}{2}L(2\pi r) = \pi rL.$$

(c) The lateral surface area of the frustum is the difference of the large cone and the small one.

$$S = \pi r_2(L + L_1) - \pi r_1 L_1$$

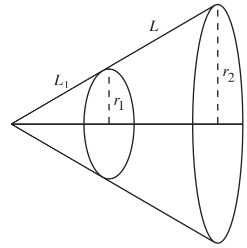
$$= \pi r_2 L + \pi L_1(r_2 - r_1)$$

By similar triangles,

$\frac{L + L_1}{r_2} = \frac{L_1}{r_1} \Rightarrow Lr_1 = L_1(r_2 - r_1)$. So,

$$S = \pi r_2 L + \pi L_1(r_2 - r_1) = \pi r_2 L + \pi L r_1$$

$$= \pi L(r_1 + r_2).$$



57. $y = \frac{3x}{4}, \quad y' = \frac{3}{4}$

$$1 + (y')^2 = 1 + \frac{9}{16} = 25/16$$

$$S = 2\pi\int_0^4 x\sqrt{\frac{25}{16}}\,dx = \frac{5\pi}{2}\left[\frac{x^2}{2}\right]_0^4 = 20\pi$$

58. $y = \frac{hx}{r}$

$$y' = \frac{h}{r}$$

$$1 + (y')^2 = \frac{r^2 + h^2}{r^2}$$

$$S = 2\pi\int_0^r x\sqrt{\frac{r^2 + h^2}{r^2}}\,dx$$

$$= \left[\frac{2\pi\sqrt{r^2 + h^2}}{r}\left(\frac{x^2}{2}\right)\right]_0^r = \pi r\sqrt{r^2 + h^2}$$

59. $y = \sqrt{9 - x^2}$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$\sqrt{1 + (y')^2} = \frac{3}{\sqrt{9 - x^2}}$$

$$S = 2\pi\int_0^2 \frac{3x}{\sqrt{9 - x^2}}\,dx$$

$$= -3\pi\int_0^2 \frac{-2x}{\sqrt{9 - x^2}}\,dx$$

$$= \left[-6\pi\sqrt{9 - x^2}\right]_0^2$$

$$= 6\pi\left(3 - \sqrt{5}\right) \approx 14.40$$

See figure in Exercise 60.

60. From Exercise 58 you have:

$$S = 2\pi\int_0^a \frac{rx}{\sqrt{r^2 - x^2}}\,dx$$

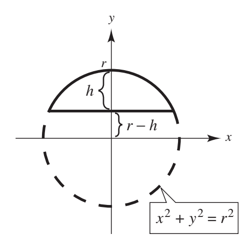
$$= -r\pi\int_0^a \frac{-2x\,dx}{\sqrt{r^2 - x^2}}$$

$$= \left[-2r\pi\sqrt{r^2 - x^2}\right]_0^a$$

$$= 2r^2\pi - 2r\pi\sqrt{r^2 - a^2}$$

$$= 2r\pi\left(r - \sqrt{r^2 - a^2}\right)$$

$$= 2\pi rh \text{ (where } h \text{ is the height of the zone)}$$



61. (a) Approximate the volume by summing six disks of thickness 3 and circumference C_i equal to the average of the given circumferences:

$$V \approx \sum_{i=1}^{6} \pi r_i^2(3) = \sum_{i=1}^{6} \pi\left(\frac{C_i}{2\pi}\right)^2(3) = \frac{3}{4\pi}\sum_{i=1}^{6} C_i^2$$

$$= \frac{3}{4\pi}\left[\left(\frac{50 + 65.5}{2}\right)^2 + \left(\frac{65.5 + 70}{2}\right)^2 + \left(\frac{70 + 66}{2}\right)^2 + \left(\frac{66 + 58}{2}\right)^2 + \left(\frac{58 + 51}{2}\right)^2 + \left(\frac{51 + 48}{2}\right)^2\right]$$

$$= \frac{3}{4\pi}\left[57.75^2 + 67.75^2 + 68^2 + 62^2 + 54.5^2 + 49.5^2\right] = \frac{3}{4\pi}(21813.625) = 5207.62 \text{ in.}^3$$

(b) The lateral surface area of a frustum of a right circular cone is $\pi s(R + r)$. For the first frustum:

$$S_1 \approx \pi\left[3^2 + \left(\frac{65.5 - 50}{2\pi}\right)^2\right]^{1/2}\left[\frac{50}{2\pi} + \frac{65.5}{2\pi}\right]$$

$$= \left(\frac{50 + 65.5}{2}\right)\left[9 + \left(\frac{65.5 - 50}{2\pi}\right)^2\right]^{1/2}.$$

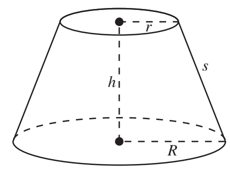
Adding the six frustums together:

$$S \approx \left(\frac{50 + 65.5}{2}\right)\left[9 + \left(\frac{15.5}{2\pi}\right)^2\right]^{1/2} + \left(\frac{65.5 + 70}{2}\right)\left[9 + \left(\frac{4.5}{2\pi}\right)^2\right]^{1/2}$$

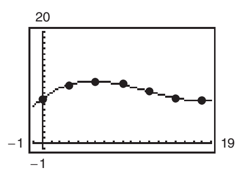
$$+ \left(\frac{70 + 66}{2}\right)\left[9 + \left(\frac{4}{2\pi}\right)^2\right]^{1/2} + \left(\frac{66 + 58}{2}\right)\left[9 + \left(\frac{8}{2\pi}\right)^2\right]^{1/2}$$

$$+ \left(\frac{58 + 51}{2}\right)\left[9 + \left(\frac{7}{2\pi}\right)^2\right]^{1/2} + \left(\frac{51 + 48}{2}\right)\left[9 + \left(\frac{3}{2\pi}\right)^2\right]^{1/2}$$

$$\approx 224.30 + 208.96 + 208.54 + 202.06 + 174.41 + 150.37 = 1168.64$$



(c) $r = 0.00401y^3 - 0.1416y^2 + 1.232y + 7.943$



(d) $V = \int_0^{18} \pi r^2 dy \approx 5275.9 \text{ in.}^3$

$S = \int_0^{18} 2\pi r(y)\sqrt{1 + r'(y)^2}\, dy \approx 1179.5 \text{ in.}^2$

62. (a) $y = f(x) = 0.0000001953x^4 - 0.0001804x^3 + 0.0496x^2 - 4.8323x + 536.9270$

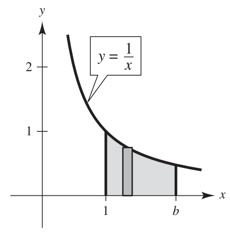
(b) Area $= \int_0^{400} f(x)\, dx \approx 131{,}734.5 \text{ ft}^2 \approx 3.0$ acres $\left(1 \text{ acre} = 43{,}560 \text{ ft}^2\right)$

(Answers will vary.)

(c) $L = \int_0^{400} \sqrt{1 + f'(x)^2}\, dx \approx 794.9$ ft

(Answers will vary.)

63. (a) $V = \pi\int_1^b \frac{1}{x^2}\,dx = \left[-\frac{\pi}{x}\right]_1^b = \pi\left(1 - \frac{1}{b}\right)$



(b) $S = 2\pi\int_1^b \frac{1}{x}\sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\,dx$

$= 2\pi\int_1^b \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx$

$= 2\pi\int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\,dx$

(c) $\lim_{b\to\infty} V = \lim_{b\to\infty} \pi\left(1 - \frac{1}{b}\right) = \pi$

(d) Because

$\frac{\sqrt{x^4 + 1}}{x^3} > \frac{\sqrt{x^4}}{x^3} = \frac{1}{x} > 0$ on $[1, b]$,

you have

$\int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\,dx > \int_1^b \frac{1}{x}\,dx = \left[\ln x\right]_1^b = \ln b$

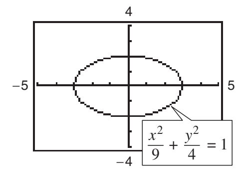
and $\lim_{b\to\infty} \ln b \to \infty$. So,

$\lim_{b\to\infty} 2\pi\int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\,dx = \infty.$

64. (a) $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Ellipse: $y_1 = 2\sqrt{1 - \frac{x^2}{9}}$

$y_2 = -2\sqrt{1 - \frac{x^2}{9}}$



(b) $y = 2\sqrt{1 - \frac{x^2}{9}}, \quad 0 \le x \le 3$

$y' = 2\left(\frac{1}{2}\right)\left(1 - \frac{x^2}{9}\right)^{-1/2}\left(\frac{-2x}{9}\right)$

$= \frac{-2x}{9\sqrt{1 - (x^2/9)}} = \frac{-2x}{3\sqrt{9 - x^2}}$

$L = \int_0^3 \sqrt{1 + \frac{4x^2}{81 - 9x^2}}\,dx$

(c) You cannot evaluate this definite integral, because the integrand is not defined at $x = 3$. Also, the integrand does not have an elementary antiderivative.

65. $y = \frac{1}{3}\left(x^{3/2} - 3x^{1/2} + 2\right)$

When $x = 0$, $y = \frac{2}{3}$. So, the fleeing object has traveled $\frac{2}{3}$ unit when it is caught.

$y' = \frac{1}{3}\left(\frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}\right) = \left(\frac{1}{2}\right)\frac{x - 1}{x^{1/2}}$

$1 + (y')^2 = 1 + \frac{(x - 1)^2}{4x} = \frac{(x + 1)^2}{4x}$

$s = \int_0^1 \frac{x + 1}{2x^{1/2}}\,dx = \frac{1}{2}\int_0^1 \left(x^{1/2} + x^{-1/2}\right)dx$

$= \frac{1}{2}\left[\frac{2}{3}x^{3/2} + 2x^{1/2}\right]_0^1 = \frac{4}{3} = 2\left(\frac{2}{3}\right)$

The pursuer has traveled twice the distance that the fleeing object has traveled when it is caught.

66.
$$y = \frac{1}{3}x^{1/2} - x^{3/2}$$
$$y' = \frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{6}\left(x^{-1/2} - 9x^{1/2}\right)$$
$$1 + (y')^2 = 1 + \frac{1}{36}\left(x^{-1} - 18 + 81x\right) = \frac{1}{36}\left(x^{-1/2} + 9x^{1/2}\right)^2$$
$$S = 2\pi\int_0^{1/3}\left(\frac{1}{3}x^{1/2} - x^{3/2}\right)\sqrt{\frac{1}{36}\left(x^{-1/2} + 9x^{1/2}\right)^2}\,dx = \frac{2\pi}{6}\int_0^{1/3}\left(\frac{1}{3}x^{1/2} - x^{3/2}\right)\left(x^{-1/2} + 9x^{1/2}\right)dx$$
$$= \frac{\pi}{3}\int_0^{1/3}\left(\frac{1}{3} + 2x - 9x^2\right)dx = \frac{\pi}{3}\left[\frac{1}{3}x + x^2 - 3x^3\right]_0^{1/3} = \frac{\pi}{27}\text{ ft}^2 \approx 0.1164\text{ ft}^2 \approx 16.8\text{ in.}^2$$

Amount of glass needed: $V = \frac{\pi}{27}\left(\frac{0.015}{12}\right) \approx 0.00015\text{ ft}^3 \approx 0.25\text{ in.}^3$

67. $x^{2/3} + y^{2/3} = 4$
$$y^{2/3} = 4 - x^{2/3}$$
$$y = \left(4 - x^{2/3}\right)^{3/2}, \quad 0 \le x \le 8$$
$$y' = \frac{3}{2}\left(4 - x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = \frac{-\left(4 - x^{2/3}\right)^{1/2}}{x^{1/3}}$$
$$1 + (y')^2 = 1 + \frac{4 - x^{2/3}}{x^{2/3}} = \frac{4}{x^{2/3}}$$
$$S = 2\pi\int_0^8\left(4 - x^{2/3}\right)^{3/2}\sqrt{\frac{4}{x^{2/3}}}\,dx = 4\pi\int_0^8\frac{\left(4 - x^{2/3}\right)^{3/2}}{x^{1/3}}\,dx = \left[-\frac{12\pi}{5}\left(4 - x^{2/3}\right)^{5/2}\right]_0^8 = \frac{384\pi}{5}$$

[Surface area of portion above the x-axis]

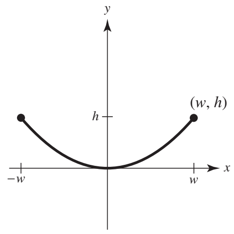
68.
$$y^2 = \frac{1}{12}x(4 - x)^2, \quad 0 \le x \le 4$$
$$y = \frac{(4 - x)\sqrt{x}}{\sqrt{12}}$$
$$y' = \frac{(4 - 3x)\sqrt{3}}{12\sqrt{x}}$$
$$1 + (y')^2 = 1 + \frac{(4 - 3x)^2}{48x}$$
$$= \frac{48x + 16 - 24x + 9x^2}{48x} = \frac{(4 + 3x)^2}{48x}, \quad x \ne 0$$
$$S = 2\pi\int_0^4\frac{(4 - x)\sqrt{x}}{\sqrt{12}}\cdot\frac{(4 + 3x)}{\sqrt{48x}}\,dx$$
$$= 2\pi\int_0^4\frac{(4 - x)(4 + 3x)}{24}\,dx$$
$$= \frac{\pi}{12}\int_0^4\left(16 + 8x - 3x^2\right)dx = \frac{\pi}{12}\left[16x + 4x^2 - x^3\right]_0^4 = \frac{\pi}{12}(64 + 64 - 64) = \frac{16\pi}{3}$$

69. $y = kx^2,\ y' = 2kx$

$1 + (y')^2 = 1 + 4k^2x^2$

$h = kw^2 \Rightarrow k = \dfrac{h}{w^2} \Rightarrow 1 + (y') = 1 + \dfrac{4h^2}{w^4}x^2$

By symmetry, $C = 2\int_0^w \sqrt{1 + \dfrac{4h^2}{w^4}x^2}\,dx$.



70. $C = 2\int_0^w \sqrt{1 + \dfrac{4h^2}{w^4}x^2}\,dx$

$= 2\int_0^{700} \sqrt{1 + \dfrac{4(155)^2x^2}{700^4}}\,dx = 1444.5 \text{ m}$

71. $y = f(x) = \cosh x$

$y' = \sinh x$

$1 + (y')^2 = 1 + \sinh^2 x = \cosh^2 x$

$\text{Area} = \int_0^t \cosh x\,dx = [\sinh x]_0^t = \sinh t$

$\text{Arc length} = \int_0^t \sqrt{1 + (y')^2}\,dx$

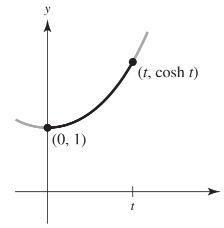
$= \int_0^t \cosh x\,dx = \sinh x\Big]_0^t$

$= \sinh t.$

Another curve with this property is $g(x) = 1$.

$\text{Area} = \int_0^t dx = t$

$\text{Arc length} = t$



72. Let (x_0, y_0) be the point on the graph of $y^2 = x^3$ where the tangent line makes an angle of 45° with the x-axis.

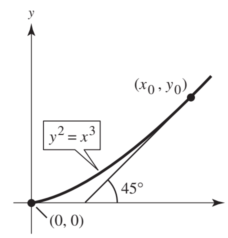
$y = x^{3/2}$

$y' = \frac{3}{2}x^{1/2} = 1$

$x_0 = \frac{4}{9}$

$L = \int_0^{4/9} \sqrt{1 + \frac{9}{4}x}\,dx$

$= \frac{8}{27}\left(2\sqrt{2} - 1\right)$



Section 7.5 Work

1. Work is done by a force when it moves an object.

2. Work done by a constant force is $W = FD$. You need integration for work done by a variable force, $W = \int_a^b F(x)\,dx$.

3. Hooke's Law says that the force needed to extend or compress a spring by some distance d is proportional to that distance, $F = kd$.

4. You can write $\Delta w = (F)(\Delta x)$ or $\Delta W = (\Delta F)(x)$.

5. $W = Fd = 1200(40) = 48{,}000$ ft-lb

6. $W = Fd = 3000(6) = 18{,}000$ ft-lb

7. $W = Fd = 112(8) = 896$ joules (Newton-meters)

8. $W = Fd = 7\left(\dfrac{1}{4}\right) = \dfrac{7}{4}$ mi-tons

9. $F(x) = kx$

$5 = k(3)$

$k = \dfrac{5}{3}$

$F(x) = \dfrac{5}{3}x$

$W = \int_0^7 F(x)\,dx = \int_0^7 \dfrac{5}{3}x\,dx = \left[\dfrac{5}{6}x^2\right]_0^7 = \dfrac{245}{6}$ in.-lb

≈ 40.833 in.-lb ≈ 3.403 ft-lb

10. $F(x) = kx$

$250 = k(30) \Rightarrow k = \dfrac{25}{3}$

$W = \int_{20}^{50} F(x)\,dx$

$= \int_{20}^{50} \dfrac{25}{3}x\,dx = \dfrac{25x^2}{6}\Big]_{20}^{50}$

$= 8750$ n-cm

$= 87.5$ joules or Nm

11. $F(x) = kx$

$20 = k(9)$

$k = \frac{20}{9}$

$W = \int_0^{12} \frac{20}{9}x\,dx = \left[\frac{10}{9}x^2\right]_0^{12} = 160$ in.-lb $= \frac{40}{3}$ ft-lb

12. $F(x) = kx$

$15 = k(1) = k$

$W = 2\int_0^4 15x\,dx = \left[15x^2\right]_0^4 = 240$ ft-lb

13. $W = 18 = \int_0^{1/3} kx\,dx = \left[\frac{kx^2}{2}\right]_0^{1/3} = \frac{k}{18} \Rightarrow k = 324$

$W = \int_{1/3}^{7/12} 324x\,dx = \left[162x^2\right]_{1/3}^{7/12} = 37.125$ ft-lb

[**Note:** 4 inches $= \frac{1}{3}$ foot]

14. $W = 6 = \int_0^{1/2} kx\,dx = \left[\frac{kx^2}{2}\right]_0^{1/2} = \frac{k}{8} \Rightarrow k = 48$

$W = \int_{1/2}^{3/4} 48x\,dx = \left[24x^2\right]_{1/2}^{3/4} = 24\left(\frac{9}{16} - \frac{1}{4}\right) = \frac{15}{2}$ joules

15. Assume that Earth has a radius of 4000 miles.

$F(x) = \frac{k}{x^2}$

$5 = \frac{k}{(4000)^2}$

$k = 80{,}000{,}000$

$F(x) = \frac{80{,}000{,}000}{x^2}$

(a) $W = \int_{4000}^{4100} \frac{80{,}000{,}000}{x^2}\,dx = \left[\frac{-80{,}000{,}000}{x}\right]_{4000}^{4100}$

≈ 487.8 mi-tons $\approx 5.15 \times 10^9$ ft-lb

(b) $W = \int_{4000}^{4300} \frac{80{,}000{,}000}{x^2}\,dx$

≈ 1395.3 mi-ton $\approx 1.47 \times 10^{10}$ ft-ton

16. $W = \int_{4000}^{h} \frac{80{,}000{,}000}{x^2}\,dx$

$= \left[-\frac{80{,}000{,}000}{x}\right]_{4000}^{h}$

$= \frac{-80{,}000{,}000}{h} + 20{,}000$

$\lim_{h\to\infty} W = 20{,}000$ mi-ton $\approx 2.1 \times 10^{11}$ ft-lb

17. Assume that Earth has a radius of 4000 miles.

$F(x) = \frac{k}{x^2}$

$10 = \frac{k}{(4000)^2}$

$k = 160{,}000{,}000$

$F(x) = \frac{160{,}000{,}000}{x^2}$

(a) $W = \int_{4000}^{15{,}000} \frac{160{,}000{,}000}{x^2}\,dx = \left[-\frac{160{,}000{,}000}{x}\right]_{4000}^{15{,}000} \approx -10{,}666.667 + 40{,}000$

$= 29{,}333.333$ mi-ton

$\approx 2.93 \times 10^4$ mi-ton

$\approx 3.10 \times 10^{11}$ ft-lb

(b) $W = \int_{4000}^{26{,}000} \frac{160{,}000{,}000}{x^2}\,dx = \left[-\frac{160{,}000{,}000}{x}\right]_{4000}^{26{,}000} \approx -6{,}153.846 + 40{,}000$

$= 33{,}846.154$ mi-ton

$\approx 3.38 \times 10^4$ mi-ton

$\approx 3.57 \times 10^{11}$ ft-lb

18. Weight on surface of moon: $\frac{1}{6}(12) = 2$ tons

Weight varies inversely as the square of distance from the center of the moon. Therefore:

$$F(x) = \frac{k}{x^2}$$

$$2 = \frac{k}{(1100)^2}$$

$$k = 2.42 \times 10^6$$

$$W = \int_{1100}^{1150} \frac{2.42 \times 10^6}{x^2}\,dx = \left[\frac{-2.42 \times 10^6}{x}\right]_{1100}^{1150} = 2.42 \times 10^6\left(\frac{1}{1100} - \frac{1}{1150}\right)$$

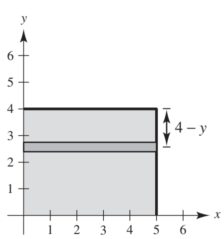
$$\approx 95.652 \text{ mi-ton} \approx 1.01 \times 10^9 \text{ ft-lb}$$

19. Weight of each layer: $62.4(20)\,\Delta y$

Distance: $4 - y$

(a) $W = \int_2^4 62.4(20)(4 - y)\,dy = \left[4992y - 624y^2\right]_2^4 = 2496$ ft-lb

(b) $W = \int_0^4 62.4(20)(4 - y)\,dy = \left[4992y - 624y^2\right]_0^4 = 9984$ ft-lb



20. The bottom half had to be pumped a greater distance than the top half.

21. Volume of disk: $\pi(2)^2\,\Delta y = 4\pi\,\Delta y$

Weight of disk of water: $9800(4\pi)\,\Delta y$

Distance the disk of water is moved: $5 - y$

$$W = \int_0^4 (5 - y)(9800)4\pi\,dy = 39{,}200\pi\int_0^4 (5 - y)\,dy$$

$$= 39{,}200\pi\left[5y - \frac{y^2}{2}\right]_0^4$$

$$= 39{,}200\pi(12) = 470{,}400\pi \text{ newton–meters}$$

22. Volume of disk: $4\pi\,\Delta y$

Weight of disk: $9800(4\pi)\,\Delta y$

Distance the disk of water is moved: y

$$W = \int_{10}^{12} y(9800)(4\pi)\,dy = 39{,}200\pi\left[\frac{y^2}{2}\right]_{10}^{12}$$

$$= 39{,}200\pi(22)$$

$$= 862{,}400\pi \text{ newton–meters}$$

23. Volume of disk: $\pi\left(\frac{2}{3}y\right)^2\,\Delta y$

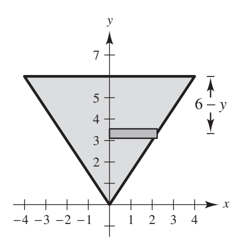
Weight of disk: $62.4\pi\left(\frac{2}{3}y\right)^2\,\Delta y$

Distance: $6 - y$

$$W = \frac{4(62.4)\pi}{9}\int_0^6 (6 - y)y^2\,dy$$

$$= \frac{4}{9}(62.4)\pi\left[2y^3 - \frac{1}{4}y^4\right]_0^6$$

$$= 2995.2\pi \text{ ft-lb}$$



24. Volume of disk: $\pi\left(\frac{2}{3}y\right)^2 \Delta y$

Weight of disk: $62.4\pi\left(\frac{2}{3}y\right)^2 \Delta y$

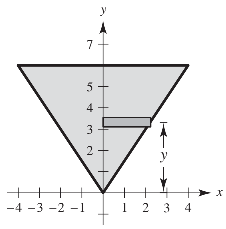
Distance: y

(a) $W = \frac{4}{9}(62.4)\pi\int_0^2 y^3 dy$

$= \left[\frac{4}{9}(62.4)\pi\left(\frac{1}{4}y^4\right)\right]_0^2 \approx 110.9\pi \text{ ft} \cdot \text{lb}$

(b) $W = \frac{4}{9}(62.4)\pi\int_4^6 y^3 dy$

$= \left[\frac{4}{9}(62.4)\pi\left(\frac{1}{4}y^4\right)\right]_4^6 \approx 7210.7\pi$ ft-lb



25. Volume of disk: $\pi\left(\sqrt{36 - y^2}\right)^2 \Delta y$

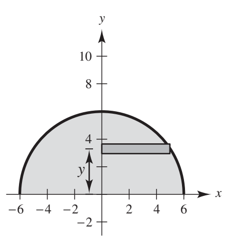
Weight of disk: $62.4\pi\left(36 - y^2\right) \Delta y$

Distance: y

$W = 62.4\pi\int_0^6 y\left(36 - y^2\right) dy$

$= 62.4\pi\int_0^6 \left(36y - y^3\right) dy = 62.4\pi\left[18y^2 - \frac{1}{4}y^4\right]_0^6$

$= 20{,}217.6\pi$ ft-lb



26. Volume of each layer: $\frac{y+3}{3}(3)\,\Delta y = (y + 3)\,\Delta y$

Weight of each layer: $53.1(y + 3)\,\Delta y$

Distance: $6 - y$

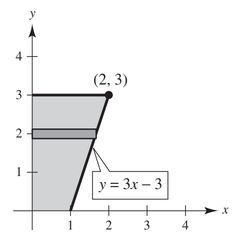
$W = \int_0^3 53.1(6 - y)(y + 3)\,dy$

$= 53.1\int_0^3 \left(18 + 3y - y^2\right) dy$

$= 53.1\left[18y + \frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3$

$= 53.1\left(\frac{117}{2}\right)$

$= 3106.35$ ft-lb



27. Volume of layer: $V = lwh = 4(2)\sqrt{(9/4) - y^2}\,\Delta y$

Weight of layer: $W = 42(8)\sqrt{(9/4) - y^2}\,\Delta y$

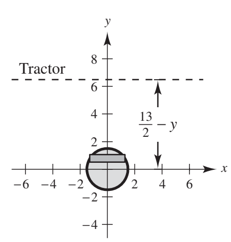
Distance: $\frac{13}{2} - y$

$W = \int_{-1.5}^{1.5} 42(8)\sqrt{\frac{9}{4} - y^2}\left(\frac{13}{2} - y\right) dy$

$= 336\left[\frac{13}{2}\int_{-1.5}^{1.5}\sqrt{\frac{9}{4} - y^2}\,dy - \int_{-1.5}^{1.5}\sqrt{\frac{9}{4} - y^2}\,y\,dy\right]$

The second integral is zero because the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius $\frac{3}{2}$. So, the work is

$W = 336\left(\frac{13}{2}\right)\pi\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right) = 2457\pi$ ft-lb.



28. Volume of layer: $V = 12(2)\sqrt{(25/4) - y^2}\,\Delta y$

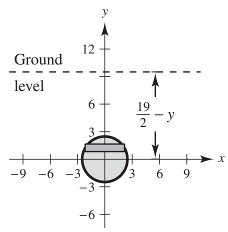
Weight of layer: $W = 42(24)\sqrt{(25/4) - y^2}\,\Delta y$

Distance: $\frac{19}{2} - y$

$$W = \int_{-2.5}^{2.5} 42(24)\sqrt{\tfrac{25}{4} - y^2}\left(\tfrac{19}{2} - y\right)dy = 1008\left[\tfrac{19}{2}\int_{-2.5}^{2.5}\sqrt{\tfrac{25}{4} - y^2}\,dy + \int_{-2.5}^{2.5}\sqrt{\tfrac{25}{4} - y^2}(-y)\,dy\right]$$

The second integral is zero because the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius $\frac{5}{2}$. So, the work is

$$W = 1008\left(\tfrac{19}{2}\right)\pi\left(\tfrac{5}{2}\right)^2\left(\tfrac{1}{2}\right) = 29{,}925\pi \text{ ft-lb} \approx 94{,}012.16 \text{ ft-lb.}$$



29. Weight of section of chain: $3\,\Delta y$

Distance: $20 - y$. $\Delta W = (\text{force increment})(\text{distance}) = (3\,\Delta y)(20 - y)$

$$W = \int_0^{20}(20 - y)3\,dy = 3\left[20y - \frac{y^2}{2}\right]_0^{20} = 3\left[400 - \frac{400}{2}\right] = 600 \text{ ft-lb}$$

30. The lower $\frac{2}{3}(20)$ feet of chain are raised with a constant force

$$W_1 = 3\left(\frac{2}{3}(20)\right)\left(\frac{20}{3}\right) = \frac{800}{3} \text{ ft-lb}$$

The top $\frac{1}{3}(20)$ feet are raised with a variable force.

Weight of section: $3\,\Delta y$

Distance: $\frac{1}{3}(20) - y$

$$W_2 = \int_0^{20/3} 3\left(\frac{20}{3} - y\right)dy = 3\left[\frac{20}{3}y - \frac{y^2}{2}\right]_0^{20/3} = \frac{200}{3} \text{ ft-lb}$$

$$W = W_1 + W_2 = \frac{800}{3} + \frac{200}{3} = \frac{1000}{3} \text{ ft-lb}$$

31. The lower 10 feet of fence are raised 10 feet with a constant force.

$W_1 = 3(10)(10) = 300$ ft-lb

The top 10 feet are raised with a variable force.

Weight of section: $3\,\Delta y$

Distance: $10 - y$

$$W_2 = \int_0^{10} 3(10 - y)\,dy = 3\left[10y - \frac{y^2}{2}\right]_0^{10} = 150 \text{ ft-lb}$$

$W = W_1 + W_2 = 300 + 150 = 450$ ft-lb

32. From Exercise 29, the work required to lift the chain is 600 ft-lb.

The work required to lift the 500-pound load is $500(20) = 10{,}000$ ft-lb.

The total is $600 + 10{,}000 = 10{,}600$ ft-lb.

33. Weight of section of chain: $3\,\Delta y$

Distance: $15 - 2y$

$$W = 3\int_0^{7.5}(15 - 2y)\,dy = \left[-\tfrac{3}{4}(15 - 2y)^2\right]_0^{7.5} = \tfrac{3}{4}(15)^2 = 168.75 \text{ ft-lb}$$

34. $W = 3\int_0^6 (12 - 2y)\,dy = \left[-\frac{3}{4}(12 - 2y)^2\right]_0^6$

$= \frac{3}{4}(12)^2 = 108$ ft-lb

35. No. Something can require a lot of effort but take no work. There is no work because there is no change in distance.

36. Yes. The work is again $W = FD = 50(4) = 200$ ft-lb.

37. $W = \int_a^b G\frac{m_1 m_2}{x^2}\,dx$

$= G\,m_1 m_2 \int_a^b x^{-2}\,dx$

$= G\,m_1 m_2\left[-\frac{1}{x}\right]_a^b$

$= G m_1 m_2\left(\frac{1}{a} - \frac{1}{b}\right)$

38. $F = G\frac{m_1 m_2}{d^2}$

If the distance between the particles is multiplied by n, then the force is

$F_1 = G\frac{m_1 m_2}{(nd)^2} = \frac{1}{n^2}F_1.$

39. $F(x) = \frac{k}{(2 - x)^2}$

$W = \int_{-2}^{1} \frac{k}{(2 - x)^2}\,dx = \left[\frac{k}{2 - x}\right]_{-2}^{1} = k\left(1 - \frac{1}{4}\right) = \frac{3k}{4}$ (units of work)

40. Because the work equals the area under the force function, you have $(c) < (d) < (a) < (b)$.

41. (a) $W = \int_0^9 6\,dx = 54$ ft-lb

(b) $W = \int_0^7 20\,dx + \int_7^9 (-10x + 90)\,dx = 140 + 20$

$= 160$ ft-lb

(c) $W = \int_0^9 \frac{1}{27}x^2\,dx = \left.\frac{x^3}{81}\right]_0^9 = 9$ ft-lb

(d) $W = \int_0^9 \sqrt{x}\,dx = \left.\frac{2}{3}x^{3/2}\right]_0^9 = \frac{2}{3}(27) = 18$ ft-lb

42. (a) $W = 60(3) = 180$ ft-lb

(b) $W = 80(1) + 40(1) = 120$ ft-lb

(c) $W = 60(0) = 0$ ft-lb

$c < b < a$

43. $p = \frac{k}{V}$

$1000 = \frac{k}{2}$

$k = 2000$

$W = \int_2^3 \frac{2000}{V}\,dV$

$= \left[2000 \ln|V|\right]_2^3 = 2000 \ln\left(\frac{3}{2}\right) \approx 810.93$ ft-lb

44. $p = \frac{k}{V}$

$2500 = \frac{k}{1} \Rightarrow k = 2500$

$W = \int_1^3 \frac{2500}{V}\,dV = \left[2500 \ln V\right]_1^3 = 2500 \ln 3$

≈ 2746.53 ft-lb

45. $W = \int_0^5 1000\left[1.8 - \ln(x + 1)\right]dx \approx 3249.44$ ft-lb

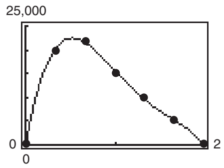
46. $W = \int_0^4 \left(\frac{e^{x^2} - 1}{100}\right)dx \approx 11{,}494$ ft-lb

47. $W = \int_0^5 100x\sqrt{125 - x^3}\,dx \approx 10{,}330.3$ ft-lb

48. $W = \int_0^2 1000 \sinh x\,dx \approx 2762.2$ ft-lb

49. (a) $W \approx \frac{2-0}{3(6)}\left[0 + 4(20{,}000) + 2(22{,}000) + 4(15{,}000) + 2(10{,}000) + 4(5000) + 0\right] \approx 24888.889$ ft-lb

(b) $F(x) = -16{,}261.36x^4 + 85{,}295.45x^3 - 157{,}738.64x^2 + 104{,}386.36x - 32.4675$



(c) $F(x)$ is a maximum when $x \approx 0.524$ feet.

(d) $W = \int_0^2 F(x)\,dx \approx 25{,}180.5$ ft-lb

Section 7.6 Moments, Centers of Mass, and Centroids

1. Weight is a force that is dependent on gravity. Mass is a measure of a body's resistance to change in motion and is independent of the gravitational system in which the body is located. The weight (or force) of an object is its mass times the acceleration due to gravity, $F = mg$.

2. Yes. $M_0 = 5(-3) + 2(-1) + 1(1) + 5(2) + 1(6) = 0$

3. A planar lamina is a flat plate of material of constant density. The center of mass of a planar lamina is its balancing point.

4. The Theorem of Pappus allows you to find the volume of a solid of revolution. See Theorem 7.1

5. $\bar{x} = \frac{7(-5) + 3(0) + 5(3)}{7 + 3 + 5} = \frac{-20}{15} = -\frac{4}{3}$

6. $\bar{x} = \frac{0.1(1) + 0.2(2) + 0.2(3) + 0.5(4)}{0.1 + 0.2 + 0.2 + 0.5} = \frac{3.1}{1} = 3.1$

7. $\bar{x} = \frac{1(6) + 3(10) + 2(3) + 9(2) + 5(4)}{1 + 3 + 2 + 9 + 5} = \frac{80}{20} = 4$

8. $\bar{x} = \frac{8(-2) + 5(6) + 5(0) + 12(3) + 2(-5)}{8 + 5 + 5 + 12 + 2} = \frac{40}{32} = \frac{5}{4}$

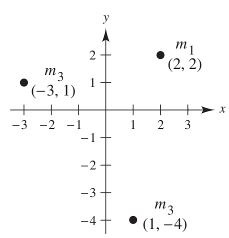
9.
$$48x = 72(L - x) = 72(10 - x)$$
$$48x = 720 - 72x$$
$$120x = 720$$
$$x = 6 \text{ ft}$$

10. $200x = 600(5 - x)$ (person is on the left)
$$200x = 3000 - 600x$$
$$800x = 3000$$
$$x = \tfrac{15}{4} = 3\tfrac{3}{4} \text{ ft}$$

11. $\bar{x} = \frac{5(2) + 1(-3) + 3(1)}{5 + 1 + 3} = \frac{10}{9}$

$\bar{y} = \frac{5(2) + 1(1) + 3(-4)}{5 + 1 + 3} = -\frac{1}{9}$

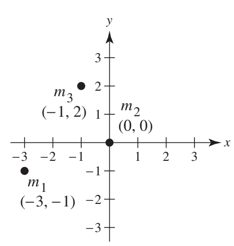
$(\bar{x}, \bar{y}) = \left(\frac{10}{9}, -\frac{1}{9}\right)$



12. $\bar{x} = \frac{8(-3) + 1(0) + 4(-1)}{8 + 1 + 4} = -\frac{28}{13}$

$\bar{y} = \frac{8(-1) + 1(0) + 4(2)}{8 + 1 + 4} = \frac{0}{13} = 0$

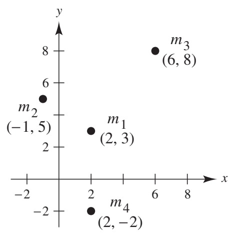
$(\bar{x}, \bar{y}) = \left(-\frac{28}{13}, 0\right)$



13. $\bar{x} = \dfrac{12(2) + 6(-1) + (9/2)(6) + 15(2)}{12 + 6 + (9/2) + 15} = \dfrac{75}{37.5} = 2$

$\bar{y} = \dfrac{12(3) + 6(5) + (9/2)(8) + 15(-2)}{12 + 6 + (9/2) + 15} = \dfrac{72}{37.5} = \dfrac{48}{25}$

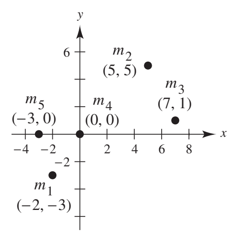
$(\bar{x}, \bar{y}) = \left(2, \dfrac{48}{25}\right)$



14. $\bar{x} = \dfrac{3(-2) + 4(5) + 2(7) + 1(0) + 6(-3)}{3 + 4 + 2 + 1 + 6} = \dfrac{5}{8}$

$\bar{y} = \dfrac{3(-3) + 4(5) + 2(1) + 1(0) + 6(0)}{3 + 4 + 2 + 1 + 6} = \dfrac{13}{16}$

$(\bar{x}, \bar{y}) = \left(\dfrac{5}{8}, \dfrac{13}{16}\right)$



15. $m = \rho\int_0^2 \dfrac{x}{2}\,dx = \left[\rho\dfrac{x^2}{4}\right]_0^2 = \rho$

$M_x = \rho\int_0^2 \dfrac{1}{2}\left(\dfrac{x}{2}\right)^2 dx$

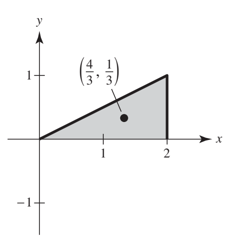
$= \dfrac{\rho}{8}\left[\dfrac{x^3}{3}\right]_0^2 = \dfrac{\rho}{3}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{\rho/3}{\rho} = \dfrac{1}{3}$

$M_y = \rho\int_0^2 x\left(\dfrac{x}{2}\right) dx = \dfrac{\rho}{2}\left[\dfrac{x^3}{3}\right]_0^2 = \dfrac{4}{3}\rho$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{4/3\rho}{\rho} = \dfrac{4}{3}$

$(\bar{x}, \bar{y}) = \left(\dfrac{4}{3}, \dfrac{1}{3}\right)$



16. $m = \rho\int_0^6 (6 - x)\,dx = \rho\left[6x - \dfrac{x^2}{2}\right]_0^6 = 18\rho$

$M_x = \rho\int_0^6 \dfrac{1}{2}(6 - x)^2\,dx = \dfrac{\rho}{2}\int_0^6 (36 - 12x + x^2)\,dx$

$= \dfrac{\rho}{2}\left[36x - 6x^2 + \dfrac{x^3}{3}\right]_0^6$

$= \dfrac{\rho}{2}[72] = 36\rho$

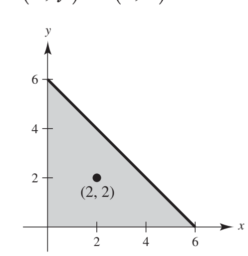
$\bar{y} = \dfrac{M_x}{m} = \dfrac{36\rho}{18\rho} = 2$

$M_y = \rho\int_0^6 x(6 - x)\,dx = \rho\int_0^6 (6x - x^2)\,dx$

$= \rho\left[3x^2 - \dfrac{x^3}{3}\right]_0^6 = 36\rho$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{36\rho}{18\rho} = 2$

$(\bar{x}, \bar{y}) = (2, 2)$



17. $m = \rho\int_0^4 \sqrt{x}\,dx = \left[\frac{2\rho}{3}x^{3/2}\right]_0^4 = \frac{16\rho}{3}$

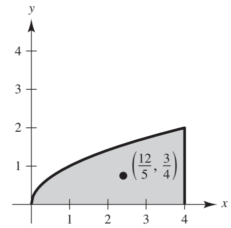
$M_x = \rho\int_0^4 \frac{\sqrt{x}}{2}\left(\sqrt{x}\right)dx = \left[\rho\frac{x^2}{4}\right]_0^4 = 4\rho$

$\bar{y} = \frac{M_x}{m} = 4\rho\left(\frac{3}{16\rho}\right) = \frac{3}{4}$

$M_y = \rho\int_0^4 x\sqrt{x}\,dx = \left[\rho\frac{2}{5}x^{5/2}\right]_0^4 = \frac{64\rho}{5}$

$\bar{x} = \frac{M_y}{m} = \frac{64\rho}{5}\left(\frac{3}{16\rho}\right) = \frac{12}{5}$

$(\bar{x}, \bar{y}) = \left(\frac{12}{5}, \frac{3}{4}\right)$



18. $m = \rho\int_0^2 \frac{x^2}{3}\,dx = \rho\left[\frac{x^3}{9}\right]_0^2 = \frac{8}{9}\rho$

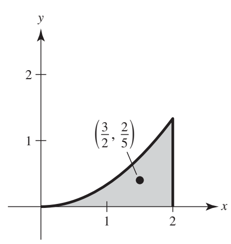
$M_x = \rho\int_0^2 \frac{1}{2}\left(\frac{x^2}{3}\right)^2 dx = \frac{\rho}{18}\left[\frac{x^5}{5}\right]_0^2 = \frac{16}{45}\rho$

$M_y = \rho\int_0^2 x\left(\frac{x^2}{3}\right)dx = \frac{\rho}{3}\left[\frac{x^4}{4}\right]_0^2 = \frac{4}{3}\rho$

$\bar{x} = \frac{M_y}{m} = \frac{4/3\rho}{(8/9)\rho} = \frac{3}{2}$

$\bar{y} = \frac{M_x}{m} = \frac{16/45\rho}{8/9\rho} = \frac{2}{5}$

$(\bar{x}, \bar{y}) = \left(\frac{3}{2}, \frac{2}{5}\right)$



19. $m = \rho\int_0^1 \left(x^2 - x^3\right)dx = \rho\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\rho}{12}$

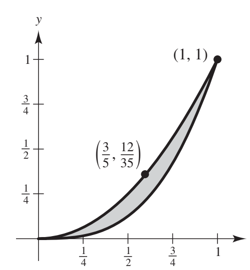
$M_x = \rho\int_0^1 \frac{\left(x^2 + x^3\right)}{2}\left(x^2 - x^3\right)dx = \frac{\rho}{2}\int_0^1 \left(x^4 - x^6\right)dx = \frac{\rho}{2}\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 = \frac{\rho}{35}$

$\bar{y} = \frac{M_x}{m} = \frac{\rho}{35}\left(\frac{12}{\rho}\right) = \frac{12}{35}$

$M_y = \rho\int_0^1 x\left(x^2 - x^3\right)dx = \rho\int_0^1 \left(x^3 - x^4\right)dx = \rho\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = \frac{\rho}{20}$

$\bar{x} = \frac{M_y}{m} = \frac{\rho}{20}\left(\frac{12}{\rho}\right) = \frac{3}{5}$

$(\bar{x}, \bar{y}) = \left(\frac{3}{5}, \frac{12}{35}\right)$



20. $m = \rho\int_0^4\left(\sqrt{x} - \frac{x}{2}\right)dx = \rho\left[\frac{2}{3}x^{3/2} - \frac{x^2}{4}\right]_0^4 = \rho\left[\frac{16}{3} - 4\right] = \frac{4}{3}\rho$

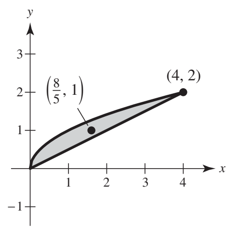
$M_x = \rho\int_0^4 \frac{1}{2}\left(\sqrt{x} + \frac{x}{2}\right)\left(\sqrt{x} - \frac{x}{2}\right)dx = \frac{1}{2}\rho\int_0^4\left(x - \frac{x^2}{4}\right)dx = \frac{\rho}{2}\left[\frac{x^2}{2} - \frac{x^3}{12}\right]_0^4 = \frac{\rho}{2}\left[8 - \frac{16}{3}\right] = \frac{4}{3}\rho$

$\bar{y} = \frac{M_x}{m} = \frac{4/3\rho}{4/3\rho} = 1$

$M_y = \rho\int_0^4 x\left(\sqrt{x} - \frac{x}{2}\right)dx = \rho\left[\frac{2}{5}x^{5/2} - \frac{x^3}{6}\right]_0^4 = \rho\left[\frac{64}{5} - \frac{32}{3}\right] = \frac{32}{15}\rho$

$\bar{x} = \frac{M_y}{m} = \frac{32/15\rho}{4/3\rho} = \frac{8}{5}$

$(\bar{x}, \bar{y}) = (8/5, 1)$



21. $m = \rho\int_0^3\left[\left(-x^2 + 4x + 2\right) - (x + 2)\right]dx = -\rho\left[\frac{x^3}{3} + \frac{3x^2}{2}\right]_0^3 = \frac{9\rho}{2}$

$M_x = \rho\int_0^3\left[\frac{\left(-x^2 + 4x + 2\right) + (x + 2)}{2}\right]\left[\left(-x^2 + 4x + 2\right) - (x + 2)\right]dx$

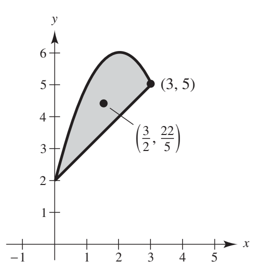
$= \frac{\rho}{2}\int_0^3\left(-x^2 + 5x + 4\right)\left(-x^2 + 3x\right)dx = \frac{\rho}{2}\int_0^3\left(x^4 - 8x^3 + 11x^2 + 12x\right)dx = \frac{\rho}{2}\left[\frac{x^5}{5} - 2x^4 + \frac{11x^3}{3} + 6x^2\right]_0^3 = \frac{99\rho}{5}$

$\bar{y} = \frac{M_x}{m} = \frac{99\rho}{5}\left(\frac{2}{9\rho}\right) = \frac{22}{5}$

$M_y = \rho\int_0^3 x\left[\left(-x^2 + 4x - 2\right) - (x + 2)\right]dx = \rho\int_0^3\left(-x^3 + 3x^2\right)dx = \rho\left[-\frac{x^4}{4} + x^3\right]_0^3 = \frac{27\rho}{4}$

$\bar{x} = \frac{M_y}{m} = \frac{27\rho}{4}\left(\frac{2}{9\rho}\right) = \frac{3}{2}$

$(\bar{x}, \bar{y}) = \left(\frac{3}{2}, \frac{22}{5}\right)$



22. $m = \rho\int_0^9\left[\left(\sqrt{x}+1\right)-\left(\frac{1}{3}x+1\right)\right]dx = \rho\int_0^9\left(\sqrt{x}-\frac{1}{3}x\right)dx = \rho\left[\frac{2}{3}x^{3/2}-\frac{x^2}{6}\right]_0^9 = \rho\left(18-\frac{27}{2}\right) = \frac{9}{2}\rho$

$$M_x = \rho\int_0^9 \frac{\sqrt{x}+1+(1/3)x+1}{2}\left(\sqrt{x}+1-\frac{1}{3}x-1\right)dx = \frac{\rho}{2}\int_0^9\left(\sqrt{x}+\frac{1}{3}x+2\right)\left(\sqrt{x}-\frac{1}{3}x\right)dx$$

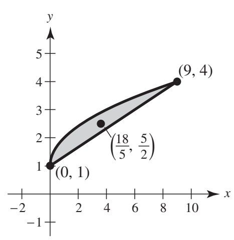
$$= \frac{\rho}{2}\int_0^9\left(x-\frac{1}{3}x^{3/2}+\frac{1}{3}x^{3/2}-\frac{1}{9}x^2+2\sqrt{x}-\frac{2}{3}x\right)dx = \frac{\rho}{2}\int_0^9\left(\frac{1}{3}x-\frac{1}{9}x^2+2\sqrt{x}\right)dx$$

$$= \frac{\rho}{2}\left[\frac{x^2}{6}-\frac{x^3}{27}+\frac{4}{3}x^{3/2}\right]_0^9 = \frac{\rho}{2}\left[\frac{27}{2}-27+36\right] = \frac{45}{4}\rho$$

$$M_y = \rho\int_0^9 x\left[\sqrt{x}+1-\frac{1}{3}x-1\right]dx = \rho\int_0^9\left(x^{3/2}-\frac{1}{3}x^2\right)dx = \rho\left[\frac{2}{5}x^{5/2}-\frac{1}{9}x^3\right]_0^9 = \rho\left[\frac{486}{5}-81\right] = \frac{81}{5}\rho$$

$$\bar{x} = \frac{M_y}{m} = \frac{(81/5)\rho}{(9/2)\rho} = \frac{18}{5};\ \bar{y} = \frac{M_x}{m} = \frac{(45/4)\rho}{(9/2)\rho} = \frac{5}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{18}{5}, \frac{5}{2}\right)$$



23. $m = \rho\int_0^8 x^{2/3}\,dx = \rho\left[\frac{3}{5}x^{5/3}\right]_0^8 = \frac{96\rho}{5}$

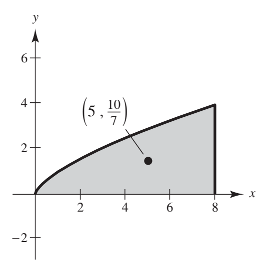
$$M_x = \rho\int_0^8 \frac{x^{2/3}}{2}\left(x^{2/3}\right)dx = \frac{\rho}{2}\left[\frac{3}{7}x^{7/3}\right]_0^8 = \frac{192\rho}{7}$$

$$\bar{y} = \frac{M_x}{m} = \frac{192\rho}{7}\left(\frac{5}{96\rho}\right) = \frac{10}{7}$$

$$M_y = \rho\int_0^8 x\left(x^{2/3}\right)dx = \rho\left[\frac{3}{8}x^{8/3}\right]_0^8 = 96\rho$$

$$\bar{x} = \frac{M_y}{m} = 96\rho\left(\frac{5}{96\rho}\right) = 5$$

$$(\bar{x}, \bar{y}) = \left(5, \frac{10}{7}\right)$$



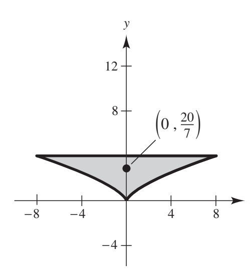
24. $m = 2\rho\int_0^8\left(4-x^{2/3}\right)dx = 2\rho\left[4x-\frac{3}{5}x^{5/3}\right]_0^8 = \frac{128\rho}{5}$

By symmetry, M_y and $\bar{x} = 0$.

$$M_x = 2\rho\int_0^8\left(\frac{4+x^{2/3}}{2}\right)\left(4-x^{2/3}\right)dx = \rho\left[16x-\frac{3}{7}x^{7/3}\right]_0^8 = \frac{512\rho}{7}$$

$$\bar{y} = \frac{512\rho}{7}\left(\frac{5}{128\rho}\right) = \frac{20}{7}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{20}{7}\right)$$



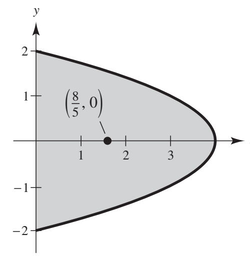
25. $m = 2\rho\int_0^2 (4 - y^2)\,dy = 2\rho\left[4y - \frac{y^3}{3}\right]_0^2 = \frac{32\rho}{3}$

$M_y = 2\rho\int_0^2 \left(\frac{4 - y^2}{2}\right)(4 - y^2)\,dy = \rho\left[16y - \frac{8}{3}y^3 + \frac{y^5}{5}\right]_0^2 = \frac{256\rho}{15}$

$\bar{x} = \frac{M_y}{m} = \frac{256\rho}{15}\left(\frac{3}{32\rho}\right) = \frac{8}{5}$

By symmetry, M_x and $\bar{y} = 0$.

$(\bar{x}, \bar{y}) = \left(\frac{8}{5}, 0\right)$



26. $m = \rho\int_0^3 (3y - y^2)\,dy = \rho\left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^2 = \frac{9}{2}\rho$

$M_y = \rho\int_0^3 \frac{1}{2}(3y - y^2)\,dy = \frac{\rho}{2}\int_0^3 (9y^2 - 6y^3 + y^4)\,dy$

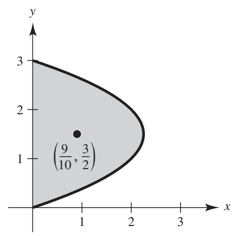
$= \frac{\rho}{2}\left[3y^3 - \frac{3y^4}{2} + \frac{y^5}{5}\right]_0^3 = \frac{81}{20}\rho$

$\bar{x} = \frac{M_y}{m} = \frac{81/20\rho}{9/2\rho} = \frac{9}{10}$

$M_x = \rho\int_0^3 y(3y - y^2)\,dy = \rho\left[y^3 - \frac{y^4}{4}\right]_0^3 = \frac{27}{4}\rho$

$\bar{y} = \frac{M_x}{m} = \frac{27/4\rho}{9/2\rho} = \frac{3}{2}$

$(\bar{x}, \bar{y}) = \left(\frac{9}{10}, \frac{3}{2}\right)$



27. $m = \rho\int_0^3 \left[(2y - y^2) - (-y)\right]dy = \rho\left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3 = \frac{9\rho}{2}$

$M_y = \rho\int_0^3 \frac{\left[(2y - y^2) + (-y)\right]}{2}\left[(2y - y^2) - (-y)\right]dy = \frac{\rho}{2}\int_0^3 (y - y^2)(3y - y^2)\,dy$

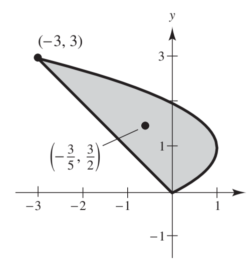
$= \frac{\rho}{2}\int_0^3 (y^4 - 4y^3 + 3y^2)\,dy = \frac{\rho}{2}\left[\frac{y^5}{5} - y^4 + y^3\right]_0^3 = -\frac{27\rho}{10}$

$\bar{x} = \frac{M_y}{m} = -\frac{27\rho}{10}\left(\frac{2}{9\rho}\right) = -\frac{3}{5}$

$M_x = \rho\int_0^3 y\left[(2y - y^2) - (-y)\right]dy = \rho\int_0^3 (3y^2 - y^3)\,dy = \rho\left[y^3 - \frac{y^4}{4}\right]_0^3 = \frac{27\rho}{4}$

$\bar{y} = \frac{M_x}{m} = \frac{27\rho}{4}\left(\frac{2}{9\rho}\right) = \frac{3}{2}$

$(\bar{x}, \bar{y}) = \left(-\frac{3}{5}, \frac{3}{2}\right)$



28. $m = \rho\int_{-1}^{2}\left[(y+2) - y^2\right]dy = \rho\left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2} = \frac{9\rho}{2}$

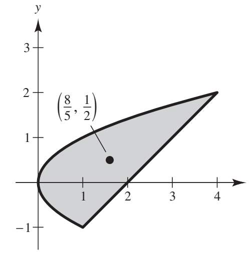
$$M_y = \rho\int_{-1}^{2}\frac{\left[(y+2) + y^2\right]}{2}\left[(y+2) - y^2\right]dy = \frac{\rho}{2}\int_{-1}^{2}\left[(y+2)^2 - y^4\right]dy = \frac{\rho}{2}\left[\frac{(y+2)^3}{3} - \frac{y^5}{5}\right]_{-1}^{2} = \frac{36\rho}{5}$$

$$\bar{x} = \frac{M_y}{m} = \frac{36\rho}{5}\left(\frac{2}{9\rho}\right) = \frac{8}{5}$$

$$M_x = \rho\int_{-1}^{2}y\left[(y+2) - y^2\right]dy = \rho\int_{-1}^{2}\left(2y + y^2 - y^3\right)dy = \rho\left[y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right]_{-1}^{2} = \frac{9\rho}{4}$$

$$\bar{y} = \frac{M_x}{m} = \frac{9\rho}{4}\left(\frac{2}{9\rho}\right) = \frac{1}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8}{5}, \frac{1}{2}\right)$$



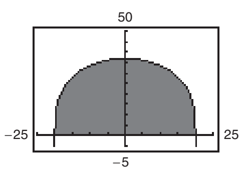
29. $m = \rho\int_{-20}^{20}5\sqrt[3]{400 - x^2}\,dx \approx 1239.76\rho$

$$M_x = \rho\int_{-20}^{20}\frac{5\sqrt[3]{400 - x^2}}{2}\left(5\sqrt[3]{400 - x^2}\right)dx$$

$$= \frac{25\rho}{2}\int_{-20}^{20}\left(400 - x^2\right)^{2/3}dx \approx 20064.27$$

$$\bar{y} = \frac{M_x}{m} \approx 16.18$$

$\bar{x} = 0$ by symmetry. Therefore, the centroid is $(0, 16.2)$.



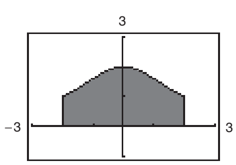
30. $m = \rho\int_{-2}^{2}\frac{8}{x^2+4}\,dx \approx 6.2832\rho$

$$M_x = \rho\int_{-2}^{2}\frac{1}{2}\left(\frac{8}{x^2+4}\right)\left(\frac{8}{x^2+4}\right)dx$$

$$= 32\rho\int_{-2}^{2}\frac{1}{\left(x^2+4\right)^2}\,dx \approx 5.14149\rho$$

$$\bar{y} = \frac{M_x}{m} \approx 0.8$$

$\bar{x} = 0$ by symmetry. Therefore, the centroid is $(0, 0.8)$.



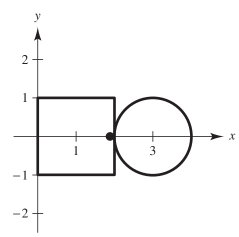
31. Centroids of the given regions: $(1, 0)$ and $(3, 0)$

Area: $A = 4 + \pi$

$$\bar{x} = \frac{4(1) + \pi(3)}{4 + \pi} = \frac{4 + 3\pi}{4 + \pi}$$

$$\bar{y} = \frac{4(0) + \pi(0)}{4 + \pi} = 0$$

$$(\bar{x}, \bar{y}) = \left(\frac{4 + 3\pi}{4 + \pi}, 0\right) \approx (1.88, 0)$$



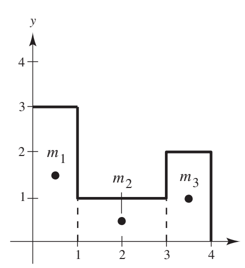
32. Centroids of the given regions: $\left(\frac{1}{2}, \frac{3}{2}\right)$, $\left(2, \frac{1}{2}\right)$, and $\left(\frac{7}{2}, 1\right)$

Area: $A = 3 + 2 + 2 = 7$

$$\bar{x} = \frac{3(1/2) + 2(2) + 2(7/2)}{7} = \frac{25/2}{7} = \frac{25}{14}$$

$$\bar{y} = \frac{3(3/2) + 2(1/2) + 2(1)}{7} = \frac{15/2}{7} = \frac{15}{14}$$

$$(\bar{x}, \bar{y}) = \left(\frac{25}{14}, \frac{15}{14}\right)$$



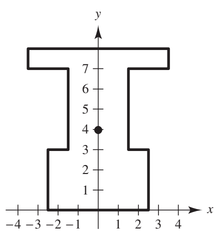
33. Centroids of the given regions: $\left(0, \frac{3}{2}\right)$, $(0, 5)$, and $\left(0, \frac{15}{2}\right)$

Area: $A = 15 + 12 + 7 = 34$

$$\bar{x} = \frac{15(0) + 12(0) + 7(0)}{34} = 0$$

$$\bar{y} = \frac{15(3/2) + 12(5) + 7(15/2)}{34} = \frac{135}{34}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{135}{34}\right) \approx (0, 3.97)$$



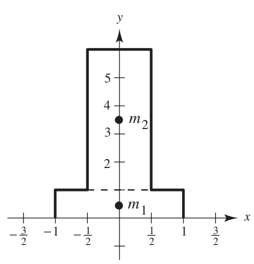
34. $m_1 = \frac{7}{8}(2) = \frac{7}{4}$, $P_1 = \left(0, \frac{7}{16}\right)$

$m_2 = \frac{7}{8}\left(6 - \frac{7}{8}\right) = \frac{287}{64}$, $P_2 = \left(0, \frac{55}{16}\right)$

By symmetry, $\bar{x} = 0$.

$$\bar{y} = \frac{(7/4)(7/16) + (287/64)(55/16)}{(7/4) + (287/64)} = \frac{16{,}569}{6384} = \frac{789}{304}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{789}{304}\right) \approx (0, 2.595)$$



35. Centroids of the given regions: $(1, 0)$ and $(3, 0)$

Mass: $4 + 2\pi$

$$\bar{x} = \frac{4(1) + 2\pi(3)}{4 + 2\pi} = \frac{2 + 3\pi}{2 + \pi}$$

$$\bar{y} = 0$$

$$(\bar{x}, \bar{y}) = \left(\frac{2 + 3\pi}{2 + \pi}, 0\right) \approx (2.22, 0)$$

36. Centroids of the given regions: $(3, 0)$ and $(1, 0)$

Mass: $8 + \pi$

$$\bar{y} = 0$$

$$\bar{x} = \frac{8(1) + \pi(3)}{8 + \pi} = \frac{8 + 3\pi}{8 + \pi}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8 + 3\pi}{8 + \pi}, 0\right) \approx (1.56, 0)$$

37. $r = 5$ is distance between center of circle and y-axis.

$A \approx \pi(4)^2 = 16\pi$ is the area of circle. So,

$V = 2\pi rA = 2\pi(5)(16\pi) = 160\pi^2 \approx 1579.14.$

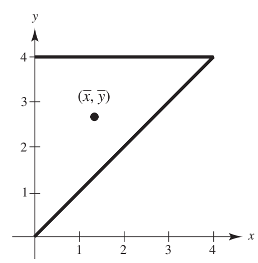
38. $V = 2\pi rA = 2\pi(3)(4\pi) = 24\pi^2$

39. $A = \frac{1}{2}(4)(4) = 8$

$$\bar{y} = \left(\frac{1}{8}\right)\frac{1}{2}\int_0^4 (4 + x)(4 - x)\,dx = \frac{1}{16}\left[16x - \frac{x^3}{3}\right]_0^4 = \frac{8}{3}$$

$$r = \bar{y} = \frac{8}{3}$$

$$V = 2\pi rA = 2\pi\left(\frac{8}{3}\right)(8) = \frac{128\pi}{3} \approx 134.04$$



40. $A = \int_2^6 2\sqrt{x-2}\,dx = \frac{4}{3}(x-2)^{3/2}\Big]_2^6 = \frac{32}{3}$

$M_y = \int_2^6 (x)2\sqrt{x-2}\,dx = 2\int_2^6 x\sqrt{x-2}\,dx$

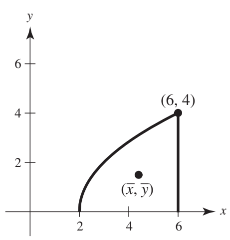
Let $u = x - 2$, $x = u + 2$, $du = dx$:

$$M_y = 2\int_0^4 (u+2)\sqrt{u}\,du$$
$$= 2\int_0^4 \left(u^{3/2} + 2u^{1/2}\right)du$$
$$= 2\left[\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2}\right]_0^4$$
$$= 2\left(\frac{64}{5} + \frac{32}{3}\right) = \frac{704}{15}$$

$$\bar{x} = \frac{M_y}{A} = \frac{704/15}{32/3} = \frac{22}{5}$$

$$r = \bar{x} = \frac{22}{5}$$

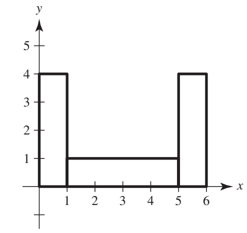
$$V = 2\pi rA = 2\pi\left(\frac{22}{5}\right)\left(\frac{32}{3}\right) = \frac{1408\pi}{15} \approx 294.89$$



41. The center of mass is translated horizontally k units as well.

42. The center of a rectangle is its balancing point. Answers will vary.

43. Answers will vary. *Sample answer*: Use three rectangles with width 1 and length 4 and place them as follows.



$(\bar{x}, \bar{y}) = (3, 1.5)$

44. (a) Yes. The region is shifted upward two units.

$(\bar{x}, \bar{y}) = (1.2, 1.4 + 2) = (1.2, 3.4)$

(b) Yes. The region is shifted to the right two units.

$(\bar{x}, \bar{y}) = (1.2 + 2, 1.4) = (3.2, 1.4)$

(c) Yes. The region is reflected in the x-axis.

$(\bar{x}, \bar{y}) = (1.2, -1.4)$

45. $A = \frac{1}{2}(2a)c = ac$

$$\frac{1}{A} = \frac{1}{ac}$$

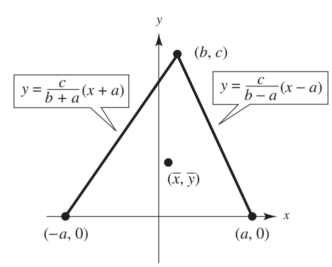
$$\bar{x} = \left(\frac{1}{ac}\right)\frac{1}{2}\int_0^c \left[\left(\frac{b-a}{c}y + a\right)^2 - \left(\frac{b+a}{c}y - a\right)^2\right]dy$$

$$= \frac{1}{2ac}\int_0^c \left[\frac{4ab}{c}y - \frac{4ab}{c^2}y^2\right]dy = \frac{1}{2ac}\left[\frac{2ab}{c}y^2 - \frac{4ab}{3c^2}y^3\right]_0^c = \frac{1}{2ac}\left(\frac{2}{3}abc\right) = \frac{b}{3}$$

$$\bar{y} = \frac{1}{ac}\int_0^c y\left[\left(\frac{b-a}{c}y + a\right) - \left(\frac{b+a}{c}y - a\right)\right]dy$$

$$= \frac{1}{ac}\int_0^c y\left(-\frac{2a}{c}y + 2a\right)dy = \frac{2}{c}\int_0^c \left(y - \frac{y^2}{c}\right)dy = \frac{2}{c}\left[\frac{y^2}{2} - \frac{y^3}{3c}\right]_0^c = \frac{c}{3}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{c}{3}\right)$$



From elementary geometry, $(b/3, c/3)$ is the point of intersection of the medians.

46. $A = bh = ac$

$$\frac{1}{A} = \frac{1}{ac}$$

$$\bar{x} = \frac{1}{ac}\frac{1}{2}\int_0^c \left[\left(\frac{b}{c}y + a\right)^2 - \left(\frac{b}{c}y\right)^2\right] dy$$

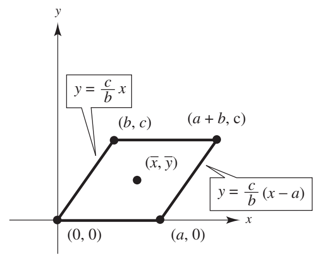
$$= \frac{1}{2ac}\int_0^c \left(\frac{2ab}{c}y + a^2\right) dy$$

$$= \frac{1}{2ac}\left[\frac{ab}{c}y^2 + a^2y\right]_0^c$$

$$= \frac{1}{2ac}\left[abc + a^2c\right] = \frac{1}{2}(b + a)$$

$$\bar{y} = \frac{1}{ac}\int_0^c y\left[\left(\frac{b}{c}y + a\right) - \left(\frac{b}{c}y\right)\right] dy = \left[\frac{1}{c}\frac{y^2}{2}\right]_0^c = \frac{c}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b + a}{2}, \frac{c}{2}\right)$$



This is the point of intersection of the diagonals.

47. $A = \frac{c}{2}(a + b)$

$$\frac{1}{A} = \frac{2}{c(a + b)}$$

$$\bar{x} = \frac{2}{c(a + b)}\int_0^c x\left(\frac{b - a}{c}x + a\right) dx = \frac{2}{c(a + b)}\int_0^c \left(\frac{b - a}{c}x^2 + ax\right) dx = \frac{2}{c(a + b)}\left[\frac{b - a}{c}\frac{x^3}{3} + \frac{ax^2}{2}\right]_0^c$$

$$= \frac{2}{c(a + b)}\left[\frac{(b - a)c^2}{3} + \frac{ac^2}{2}\right] = \frac{2}{c(a + b)}\left[\frac{2bc^2 - 2ac^2 + 3ac^2}{6}\right] = \frac{c(2b + a)}{3(a + b)} = \frac{(a + 2b)c}{3(a + b)}$$

$$\bar{y} = \frac{2}{c(a + b)}\frac{1}{2}\int_0^c \left(\frac{b - a}{c}x + a\right)^2 dx = \frac{1}{c(a + b)}\int_0^c \left[\left(\frac{b - a}{c}\right)^2 x^2 + \frac{2a(b - a)}{c}x + a^2\right] dx$$

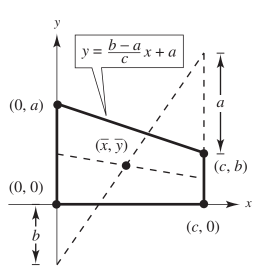
$$= \frac{1}{c(a + b)}\left[\left(\frac{b - a}{c}\right)^2\frac{x^3}{3} + \frac{2a(b - a)}{c}\frac{x^2}{2} + a^2x\right]_0^c = \frac{1}{c(a + b)}\left[\frac{(b - a)^2 c}{3} + ac(b - a) + a^2c\right]$$

$$= \frac{1}{3c(a + b)}\left[(b^2 - 2ab + a^2)c + 3ac(b - a) + 3a^2c\right]$$

$$= \frac{1}{3(a + b)}\left[b^2 - 2ab + a^2 + 3ab - 3a^2 + 3a^2\right] = \frac{a^2 + ab + b^2}{3(a + b)}$$

So, $(\bar{x}, \bar{y}) = \left(\frac{(a + 2b)c}{3(a + b)}, \frac{a^2 + ab + b^2}{3(a + b)}\right)$.

The one line passes through $\left(0, \frac{a}{2}\right)$ and $\left(c, \frac{b}{2}\right)$. Its equation is $y = \frac{b - a}{2c}x + \frac{a}{2}$. The other line passes through $(0, -b)$ and $(c, a + b)$. Its equation is $y = \frac{a + 2b}{c}x - b$. $(\bar{x}, \bar{y})$ is the point of intersection of these two lines.



48. $\bar{x} = 0$ by symmetry.

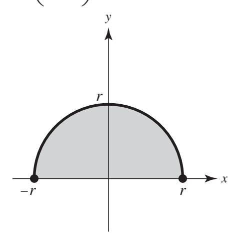
$$A = \frac{1}{2}\pi r^2$$

$$\frac{1}{A} = \frac{2}{\pi r^2}$$

$$\bar{y} = \frac{2}{\pi r^2}\frac{1}{2}\int_{-r}^{r}\left(\sqrt{r^2 - x^2}\right)^2\,dx$$

$$= \frac{1}{\pi r^2}\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r} = \frac{1}{\pi r^2}\left(\frac{4r^3}{3}\right) = \frac{4r}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{4r}{3\pi}\right)$$



49. $\bar{x} = 0$ by symmetry.

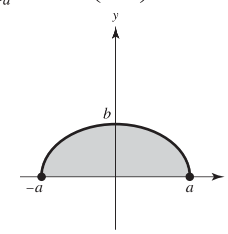
$$A = \frac{1}{2}\pi ab$$

$$\frac{1}{A} = \frac{2}{\pi ab}$$

$$\bar{y} = \frac{2}{\pi ab}\frac{1}{2}\int_{-a}^{a}\left(\frac{b}{a}\sqrt{a^2 - x^2}\right)^2\,dx$$

$$= \frac{1}{\pi ab}\left(\frac{b^2}{a^2}\right)\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{b}{\pi a^3}\left(\frac{4a^3}{3}\right) = \frac{4b}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{4b}{3\pi}\right)$$



50. $$A = \int_0^1\left[1 - \left(2x - x^2\right)\right]dx = \frac{1}{3}$$

$$\frac{1}{A} = 3$$

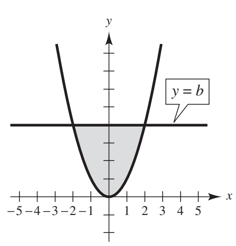
$$\bar{x} = 3\int_0^1 x\left[1 - \left(2x - x^2\right)\right]dx = 3\int_0^1\left[x - 2x^2 + x^3\right]dx = 3\left[\frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4}\right]_0^1 = \frac{1}{4}$$

$$\bar{y} = 3\int_0^1\frac{\left[1 + \left(2x - x^2\right)\right]}{2}\left[1 - \left(2x - x^2\right)\right]dx = \frac{3}{2}\int_0^1\left[1 - \left(2x - x^2\right)^2\right]dx$$

$$= \frac{3}{2}\int_0^1\left(1 - 4x^2 + 4x^3 - x^4\right)dx = \frac{3}{2}\left[x - \frac{4}{3}x^3 + x^4 - \frac{x^5}{5}\right]_0^1 = \frac{7}{10}$$

$$(\bar{x}, \bar{y}) = \left(\frac{1}{4}, \frac{7}{10}\right)$$

51. (a)



(b) $\bar{x} = 0$ by symmetry.

(c) $M_y = \int_{-\sqrt{b}}^{\sqrt{b}} x\left(b - x^2\right)dx = 0$ because $bx - x^3$ is odd.

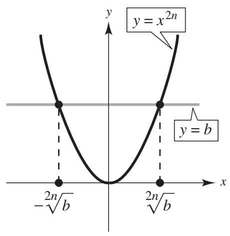
(d) $\bar{y} > \dfrac{b}{2}$ because there is more area above $y = \dfrac{b}{2}$ than below.

(e) $M_x = \int_{-\sqrt{b}}^{\sqrt{b}} \frac{(b + x^2)(b - x^2)}{2}\,dx = \int_{-\sqrt{b}}^{\sqrt{b}} \frac{b^2 - x^4}{2}\,dx = \frac{1}{2}\left[b^2x - \frac{x^5}{5}\right]_{-\sqrt{b}}^{\sqrt{b}} = b^2\sqrt{b} - \frac{b^2\sqrt{b}}{5} = \frac{4b^2\sqrt{b}}{5}$

$A = \int_{-\sqrt{b}}^{\sqrt{b}} (b - x^2)\,dx = \left[bx - \frac{x^3}{3}\right]_{-\sqrt{b}}^{\sqrt{b}} = \left(b\sqrt{b} - \frac{b\sqrt{b}}{3}\right)2 = 4\frac{b\sqrt{b}}{3}$

$\bar{y} = \frac{M_x}{A} = \frac{4b^2\sqrt{b}/5}{4b\sqrt{b}/3} = \frac{3}{5}b$

52. (a)



(b) $M_y = 0$ by symmetry.

$M_y = \int_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}} x(b - x^{2n})\,dx = 0$

because $bx - x^{2n+1}$ is an odd function.

(c) $\bar{y} > \frac{b}{2}$ because there is more area above $y = \frac{b}{2}$ than below.

(d) $M_x = \int_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}} \frac{(b + x^{2n})(b - x^{2n})}{2}\,dx = \int_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}} \frac{1}{2}(b^2 - x^{4n})\,dx$

$= \frac{1}{2}\left(b^2x - \frac{x^{4n+1}}{4n + 1}\right)\Bigg]_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}} = b^2b^{1/2n} - \frac{b^{(4n+1)/2n}}{4n + 1} = \frac{4n}{4n + 1}b^{(4n+1)/2n}$

$A = \int_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}} (b - x^{2n})\,dx = 2\left[bx - \frac{x^{2n+1}}{2n + 1}\right]_0^{\sqrt[2n]{b}} = 2\left[b \cdot b^{1/2n} - \frac{b^{(2n+1)/2n}}{2n + 1}\right] = \frac{4n}{2n + 1}b^{(2n+1)/2n}$

$\bar{y} = \frac{M_x}{A} = \frac{4nb^{(4n+1)/2n}/(4n + 1)}{4nb^{(24n+1)/2n}/(2n + 1)} = \frac{2n + 1}{4n + 1}b$

(e)

n	1	2	3	4
$\bar{y}$	$\frac{3}{5}b$	$\frac{5}{9}b$	$\frac{7}{13}b$	$\frac{9}{17}b$

(f) $\lim_{n\to\infty} \bar{y} = \lim_{n\to\infty} \frac{2n + 1}{4n + 1}b = \frac{1}{2}b$

(g) As $n \to \infty$, the figure gets narrower.

53. (a) $\bar{x} = 0$ by symmetry.

$$A = 2\int_0^{40} f(x)\,dx = \frac{2(40)}{3(4)}\left[30 + 4(29) + 2(26) + 4(20) + 0\right] = \frac{20}{3}(278) = \frac{5560}{3}$$

$$M_x = \int_{-40}^{40} \frac{f(x)^2}{2}\,dx = \frac{40}{3(4)}\left[30^2 + 4(29)^2 + 2(26)^2 + 4(20)^2 + 0\right] = \frac{10}{3}(7216) = \frac{72{,}160}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{72{,}160/3}{5560/3} = \frac{72{,}160}{5560} \approx 12.98$$

$(\bar{x}, \bar{y}) = (0, 12.98)$

(b) $y = \left(-1.02 \times 10^{-5}\right)x^4 - 0.0019x^2 + 29.28$ (Use nine data points.)

(c) $\bar{y} = \dfrac{M_x}{A} \approx \dfrac{23{,}697.68}{1843.54} \approx 12.85$

$(\bar{x}, \bar{y}) = (0, 12.85)$

54. Let $f(x)$ be the top curve, given by $l + d$. The bottom curve is $d(x)$.

x	0	0.5	1.0	1.5	2
f	2.0	1.93	1.73	1.32	0
d	0.50	0.48	0.43	0.33	0

(a) Area $= 2\int_0^2 \left[f(x) - d(x)\right] dx$

$$\approx 2\frac{2}{3(4)}\left[1.50 + 4(1.45) + 2(1.30) + 4(.99) + 0\right] = \frac{1}{3}[13.86] = 4.62$$

$$M_x = \int_{-2}^{2} \frac{f(x) + d(x)}{2}\left(f(x) - d(x)\right) dx$$

$$= \int_0^2 \left[f(x)^2 - d(x)^2\right] dx$$

$$= \frac{2}{3(4)}\left[3.75 + 4(3.4945) + 2(2.808) + 4(1.6335) + 0\right] = \frac{1}{6}[29.878] = 4.9797$$

$$\bar{y} = \frac{M_x}{A} = \frac{4.9797}{4.62} = 1.078$$

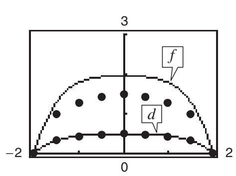
$(\bar{x}, \bar{y}) = (0, 1.078)$

(b) $f(x) = -0.1061x^4 - 0.06126x^2 + 1.9527$

$d(x) = -0.02648x^4 - 0.01497x^2 + .4862$

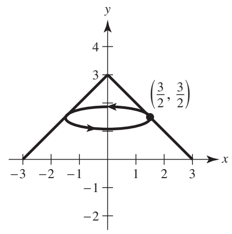
(c) $\bar{y} = \dfrac{M_x}{A} \approx \dfrac{4.9133}{4.59998} = 1.068$

$(\bar{x}, \bar{y}) = (0, 1.068)$

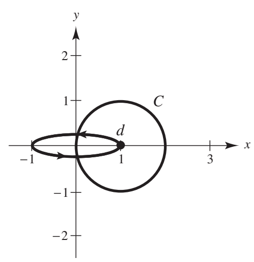


55. The centroid of the line joining $(0, 3)$ and $(3, 0)$ is $\left(\frac{3}{2}, \frac{3}{2}\right)$. The distance traveled by the centroid is $2\pi\left(\frac{3}{2}\right) = 3\pi$. The arc length of C is $3\sqrt{2}$. Therefore,

$S = (3\pi)(3\sqrt{2}) = 9\sqrt{2}\pi.$



56. The centroid of the circle is $(1, 0)$. The distance traveled by the centroid is 2π. The arc length of the circle is also 2π. Therefore, $S = (2\pi)(2\pi) = 4\pi^2$.



57. $A = \int_0^1 x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}$

$m = \rho A = \frac{\rho}{n+1}$

$M_x = \frac{\rho}{2}\int_0^1 (x^n)^2\,dx = \left[\frac{\rho}{2} \cdot \frac{x^{2n+1}}{2n+1}\right]_0^1 = \frac{\rho}{2(2n+1)}$

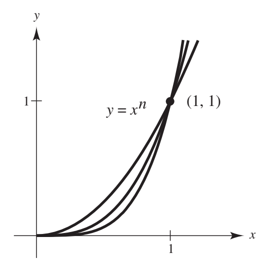
$M_y = \rho\int_0^1 x(x^n)\,dx = \left[\rho \cdot \frac{x^{n+2}}{n+2}\right]_0^1 = \frac{\rho}{n+2}$

$\bar{x} = \frac{M_y}{m} = \frac{n+1}{n+2}$

$\bar{y} = \frac{M_x}{m} = \frac{n+1}{2(2n+1)} = \frac{n+1}{4n+2}$

Centroid: $\left(\frac{n+1}{n+2}, \frac{n+1}{4n+2}\right)$

As $n \to \infty$, $(\bar{x}, \bar{y}) \to \left(1, \frac{1}{4}\right)$. The graph approaches the x-axis and the line $x = 1$ as $n \to \infty$.



58. $f(x) = x^n, g(x) = x^m, n > m$. Assume $\rho = 1$.

$m = \int_0^1 (x^m - x^n)\,dx = \left[\frac{x^{m+1}}{m+1} - \frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{m+1} - \frac{1}{n+1} = \frac{n-m}{(m+1)(n+1)}$

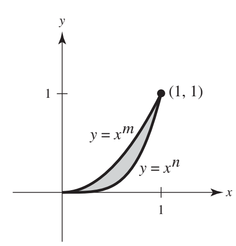
$M_x = \int_0^1 \frac{x^m + x^n}{2}(x^m - x^n)\,dx = \frac{1}{2}\left(\frac{1}{2m+1} - \frac{1}{2n+1}\right) = \frac{n-m}{(2m+1)(2n+1)}$

$M_y = \int_0^1 x(x^m - x^n)\,dx = \frac{1}{m+2} - \frac{1}{n+2} = \frac{n-m}{(m+2)(n+2)}$

$\bar{x} = \frac{M_y}{m} = \frac{(n-m)/(m+2)(n+2)}{(n-m)(m+1)(n+1)} = \frac{(m+1)(n+1)}{(m+2)(n+2)}$

$\bar{y} = \frac{M_x}{m} = \frac{(n-m)/(2m+1)(2n+1)}{(n-m)/(m+1)(n+1)} = \frac{(m+1)(n+1)}{(2m+1)(2n+1)}$

$(\bar{x}, \bar{y}) = \left(\frac{(m+1)(n+1)}{(m+2)(n+2)}, \frac{(m+1)(n+1)}{(2m+1)(2n+1)}\right)$



59. Let T be the shaded triangle with vertices $(-1, 4)$, $(1, 4)$, and $(0, 3)$. Let U be the large triangle with vertices $(-4, 4)$, $(4, 4)$, and $(0, 0)$. V consists of the region U minus the region T.

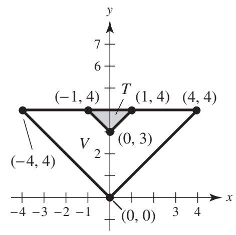
Centroid of T: $\left(0, \frac{11}{3}\right)$; Area $= 1$

Centroid of U: $\left(0, \frac{8}{3}\right)$; Area $= 16$

Area: $V = 16 - 1 = 15$

$\bar{x} = 0$ by symmetry.

$$15\bar{y} + 1\left(\frac{11}{3}\right) = 16\left(\frac{8}{3}\right)$$
$$15\bar{y} = \frac{117}{3}$$
$$\bar{y} = \frac{13}{5}$$
$$(\bar{x}, \bar{y}) = \left(0, \frac{13}{5}\right)$$



Section 7.7 Fluid Pressure and Fluid Force

1. Fluid pressure is the force per unit area over the surface of a body submerged in a fluid.

2. Yes, fluid pressure changes with depth. In fact, $P = wh$, where h is the depth.

3. $F = PA = [62.4(8)]3 = 1497.6$ lb

4. $F = PA = [62.4(8)]8 = 3993.6$ lb

5. $F = PA = [62.4(8)]10 = 4992$ lb

6. $F = PA = [62.4(8)]25 = 12{,}480$ lb

7. The weight-density of ethyl alcohol is $49.4\ \text{lb/ft}^3$.

$P = wh = (49.4)(5) = 247\ \text{lb/ft}^2$

Fluid force $= F = PA = (247)(9) = 2223$ lb

8. The weight density of ethyl alcohol is $49.4\ \text{lb/ft}^3$.

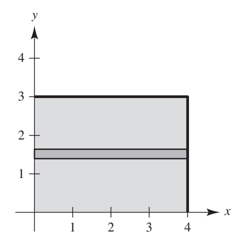
$P = wh = (49.4)(5) = 247\ \text{lb/ft}^2$

Fluid force $= F = PA = (247)(14) = 3458$ lb

9. $h(y) = 3 - y$

$L(y) = 4$

$$F = 62.4\int_0^3 (3 - y)(4)\,dy$$
$$= 249.6\int_0^3 (3 - y)\,dy$$
$$= 249.6\left[3y - \frac{y^2}{2}\right]_0^3 = 1123.2 \text{ lb}$$

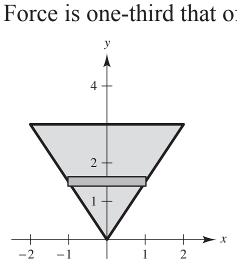


10. $h(y) = 3 - y$

$L(y) = \frac{4}{3}y$

$$F = 62.4\int_0^3 (3 - y)\left(\frac{4}{3}y\right) dy$$
$$= \frac{4}{3}(62.4)\int_0^3 \left(3y - y^2\right) dy$$
$$= \frac{4}{3}(62.4)\left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3 = 374.4 \text{ lb}$$

Force is one-third that of Exercise 7.



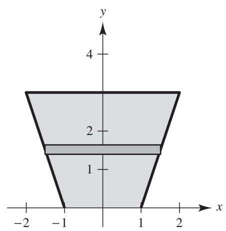
11. $h(y) = 3 - y$

$L(y) = 2\left(\dfrac{y}{3} + 1\right)$

$$F = 2(62.4)\int_0^3 (3 - y)\left(\frac{y}{3} + 1\right) dy$$

$$= 124.8\int_0^3 \left(3 - \frac{y^2}{3}\right) dy$$

$$= 124.8\left[3y - \frac{y^3}{9}\right]_0^3 = 748.8 \text{ lb}$$

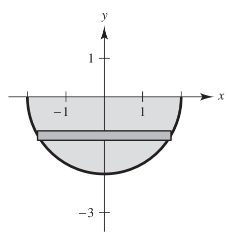


12. $h(y) = -y$

$L(y) = 2\sqrt{4 - y^2}$

$$F = 62.4\int_{-2}^0 (-y)(2)\sqrt{4 - y^2}\, dy$$

$$= \left[62.4\left(\tfrac{2}{3}\right)\left(4 - y^2\right)^{3/2}\right]_{-2}^0 = 332.8 \text{ lb}$$



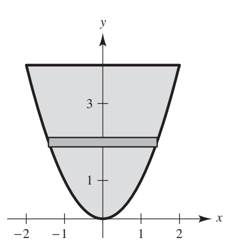
13. $h(y) = 4 - y$

$L(y) = 2\sqrt{y}$

$$F = 2(62.4)\int_0^4 (4 - y)\sqrt{y}\, dy$$

$$= 124.8\int_0^4 \left(4y^{1/2} - y^{3/2}\right) dy$$

$$= 124.8\left[\frac{8y^{3/2}}{3} - \frac{2y^{5/2}}{5}\right]_0^4 = 1064.96 \text{ lb}$$



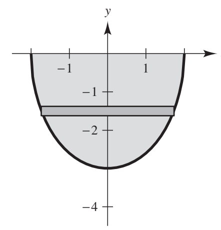
14. $h(y) = -y$

$L(y) = \frac{4}{3}\sqrt{9 - y^2}$

$$F = 62.4\int_{-3}^0 (-y)\tfrac{4}{3}\sqrt{9 - y^2}\, dy$$

$$= 62.4\left(\tfrac{2}{3}\right)\int_{-3}^0 \left(9 - y^2\right)^{1/2}(-2y)\, dy$$

$$= \left[62.4\left(\tfrac{4}{9}\right)\left(9 - y^2\right)^{3/2}\right]_{-3}^0 = 748.8 \text{ lb}$$

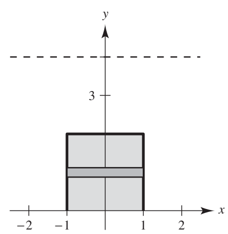


15. $h(y) = 4 - y$

$L(y) = 2$

$$F = 9800\int_0^2 2(4 - y)\, dy$$

$$= 9800\left[8y - y^2\right]_0^2 = 117{,}600 \text{ newtons}$$



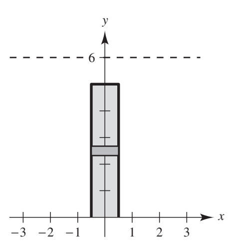
16. $h(y) = 6 - y$

$L(y) = 1$

$$F = 9800\int_0^5 1(6 - y)\, dy$$

$$= 9800\left[6y - \frac{y^2}{2}\right]_0^5$$

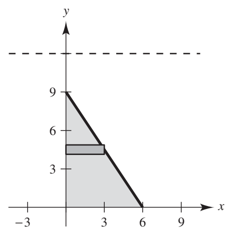
$= 171{,}500$ newtons



17. $h(y) = 12 - y$

$L(y) = 6 - \dfrac{2y}{3}$

$$F = 9800\int_0^9 (12 - y)\left(6 - \frac{2y}{3}\right) dy = 9800\left[72y - 7y^2 + \frac{2y^3}{9}\right]_0^9 = 2{,}381{,}400 \text{ newtons}$$



18. $h(y) = \left(1 + 3\sqrt{2}\right) - y$

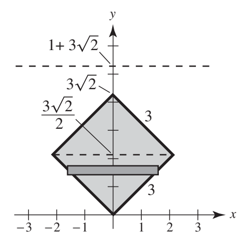
$L_1(y) = 2y$ (lower part)

$L_2(y) = 2\left(3\sqrt{2} - y\right)$ (upper part)

$$F = 2(9800)\left[\int_0^{3\sqrt{2}/2} \left(1 + 3\sqrt{2} - y\right)y\,dy + \int_{3\sqrt{2}/2}^{3\sqrt{2}} \left(1 + 3\sqrt{2} - y\right)\left(3\sqrt{2} - y\right) dy\right]$$

$$= 19{,}600\left[\left[\frac{y^2}{2} - 3\sqrt{2}y - \frac{y^3}{3}\right]_0^{3\sqrt{2}/2} + \left[3\sqrt{2}y + 18y + \frac{y^3}{3} - \frac{6\sqrt{2} + 1}{2}y\right]_{3\sqrt{2}/2}^{3\sqrt{2}}\right]$$

$$= 19{,}600\left[\frac{9\left(2\sqrt{2} + 1\right)}{4} + \frac{9\left(\sqrt{2} + 1\right)}{4}\right] = 44{,}100\left(3\sqrt{2} + 2\right) \text{ newtons}$$



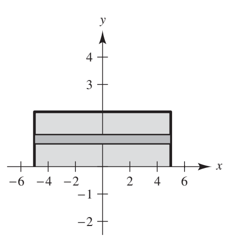
19. $h(y) = 2 - y$

$L(y) = 10$

$$F = 140.7\int_0^2 (2 - y)(10)\,dy$$

$$= 1407\int_0^2 (2 - y)\,dy$$

$$= 1407\left[2y - \frac{y^2}{2}\right]_0^2 = 2814 \text{ lb}$$



20. $h(y) = -y$

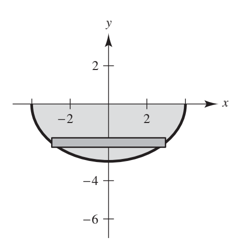
$L(y) = 2\left(\frac{4}{3}\sqrt{9 - y^2}\right)$

$$F = 140.7\int_{-3}^0 (-y)(2)\left(\frac{4}{3}\sqrt{9 - y^2}\right) dy$$

$$= \frac{(140.7)(4)}{3}\int_{-3}^0 \sqrt{9 - y^2}(-2y)\,dy$$

$$= \left[\frac{(140.7)(4)}{3}\left(\frac{2}{3}\right)\left(9 - y^2\right)^{3/2}\right]_{-3}^0$$

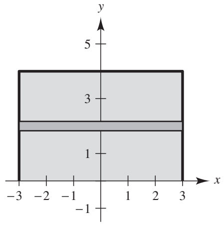
$$= 3376.8 \text{ lb}$$



21. $h(y) = 4 - y$

$L(y) = 6$

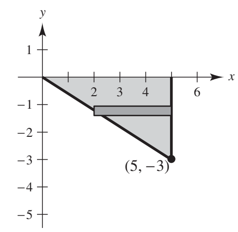
$$F = 140.7\int_0^4 (4 - y)(6)\,dy$$
$$= 844.2\int_0^4 (4 - y)\,dy$$
$$= 844.2\left[4y - \frac{y^2}{2}\right]_0^4 = 6753.6 \text{ lb}$$



22. $h(y) = -y$

$L(y) = 5 + \frac{5}{3}y$

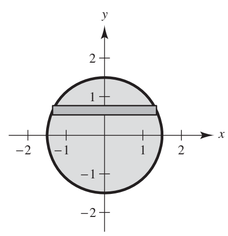
$$F = 140.7\int_{-3}^0 (-y)\left(5 + \tfrac{5}{3}y\right)dy$$
$$= 140.7\int_{-3}^0 \left(-5y - \tfrac{5}{3}y^2\right)dy$$
$$= 140.7\left[-\tfrac{5}{2}y^2 - \tfrac{5}{9}y^3\right]_{-3}^0$$
$$= 140.7\left[\tfrac{45}{2} - 15\right]$$
$$= 1055.25 \text{ lb}$$



23. $h(y) = -y$

$L(y) = 2\left(\frac{1}{2}\right)\sqrt{9 - 4y^2}$

$$F = 42\int_{-3/2}^0 (-y)\sqrt{9 - 4y^2}\,dy$$
$$= \frac{42}{8}\int_{-3/2}^0 \left(9 - 4y^2\right)^{1/2}(-8y)\,dy$$
$$= \left[\left(\tfrac{21}{4}\right)\left(\tfrac{2}{3}\right)\left(9 - 4y^2\right)^{3/2}\right]_{-3/2}^0$$
$$= 94.5 \text{ lb}$$



24. $h(y) = \frac{3}{2} - y$

$L(y) = 2\left(\frac{1}{2}\right)\sqrt{9 - 4y^2}$

$$F = 42\int_{-3/2}^{3/2}\left(\tfrac{3}{2} - y\right)\sqrt{9 - 4y^2}\,dy = 63\int_{-3/2}^{3/2}\sqrt{9 - 4y^2}\,dy + \frac{21}{4}\int_{-3/2}^{3/2}\sqrt{9 - 4y^2}(-8y)\,dy$$

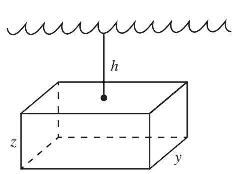
The second integral is zero because it is an odd function and the limits of integration are symmetric to the origin. The first integral is twice the area of a semicircle of radius $\frac{3}{2}$.

$$\left(\sqrt{9 - 4y^2} = 2\sqrt{(9/4) - y^2}\right)$$

So, the force is $63\left(\frac{9}{4}\pi\right) = 141.75\pi \approx 445.32$ lb.

25. You use horizontal representative rectangles because you are measuring total force against a region between two depths.

26. Consider a box with dimensions x, y, and z submerged a distance h.



The buoyant force is $w(h + z)\,xy - w(h)\,xy = wz\,xy$ pounds.

27. If the fluid force is one-half of 1123.2 lb, and the height of the water is b, then

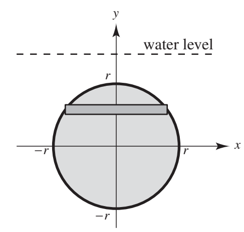
$$h(y) = b - y$$
$$L(y) = 4$$
$$F = 62.4\int_0^b (b - y)(4)\,dy = \frac{1}{2}(1123.2)$$
$$\int_0^b (b - y)\,dy = 2.25$$
$$\left[by - \frac{y^2}{2}\right]_0^b = 2.25$$
$$b^2 - \frac{b^2}{2} = 2.25$$
$$b^2 = 4.5 \Rightarrow b \approx 2.12 \text{ ft.}$$

The pressure increases with increasing depth.

28. The left window experiences the greater fluid force because there is more area farther down.

29. $h(y) = k - y$

$$L(y) = 2\sqrt{r^2 - y^2}$$
$$F = w\int_{-r}^{r} (k - y)\sqrt{r^2 - y^2}(2)\,dy = w\left[2k\int_{-r}^{r}\sqrt{r^2 - y^2}\,dy + \int_{-r}^{r}\sqrt{r^2 - y^2}(-2y)\,dy\right]$$



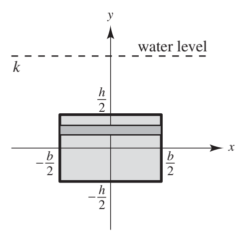
The second integral is zero because its integrand is odd and the limits of integration are symmetric to the origin. The first integral is the area of a semicircle with radius r.

$$F = w\left[(2k)\frac{\pi r^2}{2} + 0\right] = wk\pi r^2$$

30. (a) $F = wk\pi r^2 = (62.4)(7)(\pi 2^2) = 1747.2\pi$ lb (b) $F = wk\pi r^2 = (62.4)(5)(\pi 3^2) = 2808\pi$ lb

31. $h(y) = k - y$

$$L(y) = b$$
$$F = w\int_{-h/2}^{h/2} (k - y)b\,dy$$
$$= wb\left[ky - \frac{y^2}{2}\right]_{-h/2}^{h/2} = wb(hk) = wkhb$$



32. (a) $F = wkhb$

$$= (62.4)\left(\tfrac{11}{2}\right)(3)(5) = 5148 \text{ lb}$$

(b) $F = wkhb$

$$= (62.4)\left(\tfrac{17}{2}\right)(5)(10) = 26{,}520 \text{ lb}$$

33. From Exercise 31:

$$F = 64(15)(1)(1) = 960 \text{ lb}$$

34. From Exercise 29:

$$F = 64(15)\pi\left(\tfrac{1}{2}\right)^2 \approx 753.98 \text{ lb}$$

35. $h(y) = 4 - y$

$$F = 64.0\int_0^4 (4 - y)L(y)\,dy$$

Using the Midpoint Rule with $n = 4$, you have $\Delta x = \frac{4}{4} = 1$ and

$$F \approx 64.0\left[3.5\left(\frac{0 + 5}{2}\right) + 2.5\left(\frac{5 + 9}{2}\right) + 1.5\left(\frac{9 + 10.25}{2}\right) + 0.5\left(\frac{10.25 + 10.5}{2}\right)\right]$$
$$= 64.0(45.875) = 2936 \text{ lb}$$

36. $h(y) = 3 - y$

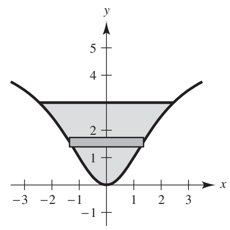
Solving $y = 5x^2/(x^2 + 4)$ for x, you obtain

$x = \sqrt{4y/(5 - y)}$.

$$L(y) = 2\sqrt{\frac{4y}{5 - y}}$$

$$F = 62.4(2)\int_0^3 (3 - y)\sqrt{\frac{4y}{5 - y}}\, dy$$

$$= 2(124.8)\int_0^3 (3 - y)\sqrt{\frac{y}{5 - y}}\, dy \approx 546.265 \text{ lb}$$

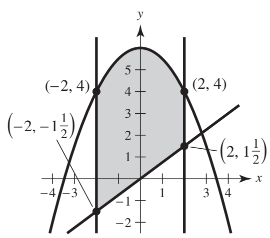


Review Exercises for Chapter 7

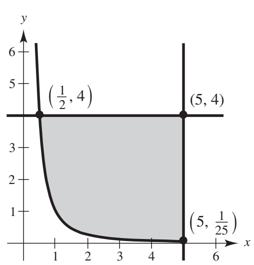
1. $A = \int_{-2}^{2} \left[\left(6 - \frac{x^2}{2}\right) - \frac{3}{4}x\right] dx$

$$= \left[6x - \frac{x^3}{6} - \frac{3x^2}{8}\right]_{-2}^{2}$$

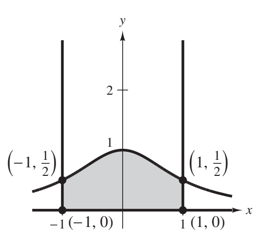
$$= \left(12 - \frac{4}{3} - \frac{3}{2}\right) - \left(-12 + \frac{4}{3} - \frac{3}{2}\right) = \frac{64}{3}$$



2. $A = \int_{1/2}^{5} \left(4 - \frac{1}{x^2}\right) dx = \left[4x + \frac{1}{x}\right]_{1/2}^{5} = \frac{81}{5}$



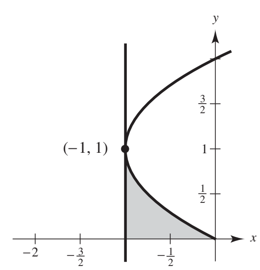
3. $A = \int_{-1}^{1} \frac{1}{x^2 + 1}\, dx = \left[\arctan x\right]_{-1}^{1} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$



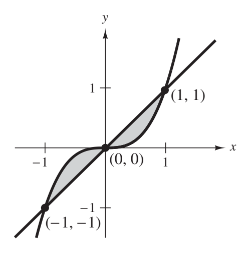
4. $A = \int_0^1 \left[\left(y^2 - 2y\right) - (-1)\right] dy$

$$= \int_0^1 \left(y^2 - 2y + 1\right) dy$$

$$= \int_0^1 (y - 1)^2\, dy = \left[\frac{(y - 1)^3}{3}\right]_0^1 = \frac{1}{3}$$

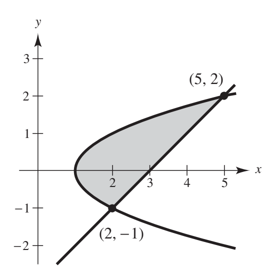


5. $A = 2\int_0^1 \left(x - x^3\right) dx = 2\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 = \frac{1}{2}$

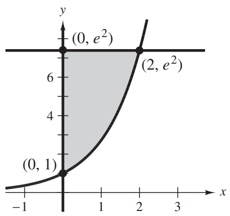


6. $A = \int_{-1}^{2} \left[(y + 3) - \left(y^2 + 1\right)\right] dy$

$$= \int_{-1}^{2} \left(2 + y - y^2\right) dy = \left[2y + \frac{1}{2}y^2 - \frac{1}{3}y^3\right]_{-1}^{2} = \frac{9}{2}$$



7. $A = \int_0^2 \left(e^2 - e^x\right) dx = \left[xe^2 - e^x\right]_0^2 = e^2 + 1$

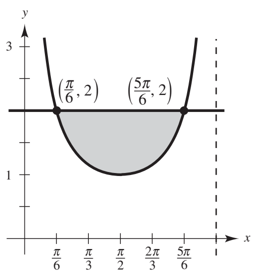


8. $A = 2\int_{\pi/6}^{\pi/2} (2 - \csc x)\, dx$

$= 2\Big[2x - \ln|\csc x - \cot x|\Big]_{\pi/6}^{\pi/2}$

$= 2\left([\pi - 0] - \left[\frac{\pi}{3} - \ln\left(2 - \sqrt{3}\right)\right]\right)$

$= 2\left[\frac{2\pi}{3} + \ln\left(2 - \sqrt{3}\right)\right] \approx 1.555$

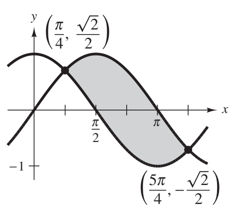


9. $A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx$

$= [-\cos x - \sin x]_{\pi/4}^{5\pi/4}$

$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$

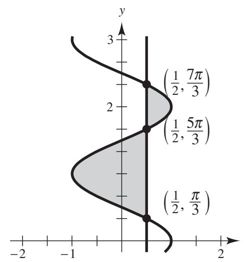
$= \frac{4}{\sqrt{2}} = 2\sqrt{2}$



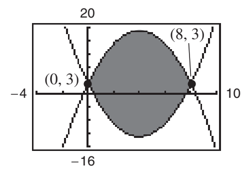
10. $A = \int_{\pi/3}^{5\pi/3} \left(\frac{1}{2} - \cos y\right) dy + \int_{5\pi/3}^{7\pi/3} \left(\cos y - \frac{1}{2}\right) dy$

$= \left[\frac{y}{2} - \sin y\right]_{\pi/3}^{5\pi/3} + \left[\sin y - \frac{y}{2}\right]_{5\pi/3}^{7\pi/3}$

$= \frac{\pi}{3} + 2\sqrt{3}$



11. (a)



(b) Points of intersection:

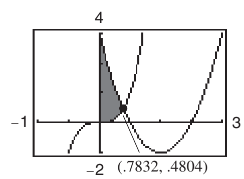
$x^2 - 8x + 3 = 3 + 8x - x^2$

$2x^2 - 16x = 0$ when $x = 0, 8$

$A = \int_0^8 \left[\left(3 + 8x - x^2\right) - \left(x^2 - 8x + 3\right)\right] dx$

≈ 170.6667

12. (a)



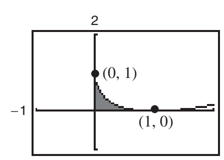
(b) Point of intersection:

$x^3 - x^2 + 4x - 3 = 0 \Rightarrow x \approx 0.783$

$A \approx \int_0^{0.783} \left(3 - 4x + x^2 - x^3\right) dx$

≈ 1.189

13. (a)

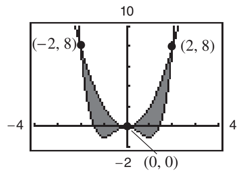


(b) $y = \left(1 - \sqrt{x}\right)^2$

$A = \int_0^1 \left(1 - \sqrt{x}\right)^2 dx$

≈ 0.1667

14. (a)



(b) Points of intersection:

$$x^4 - 2x^2 = 2x^2$$

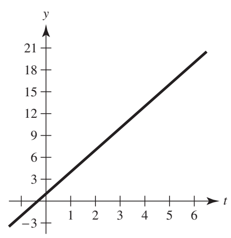
$$x^4 - 4x^2 = 0 \quad \text{when} \quad x = 0, \pm 2$$

$$A = 2\int_0^2 \left[2x^2 - \left(x^4 - 2x^2\right)\right] dx$$

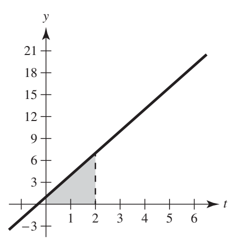
$$\approx 8.5333$$

15. $F(x) = \int_0^x (3t + 1)\,dt = \left[\dfrac{3t^2}{2} + t\right]_0^x = \dfrac{3x^2}{2} + x$

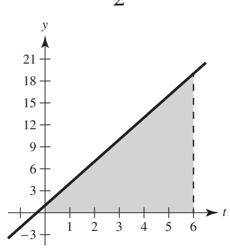
(a) $F(0) = 0$



(b) $F(2) = \dfrac{3(2)^2}{2} + 2 = 8$



(c) $F(6) = \dfrac{3(6^2)}{2} + 6 = 60$

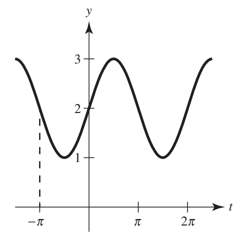


16. $F(x) = \int_{-\pi}^x (2 + \sin t)\,dt = \left[2t - \cos t\right]_{-\pi}^x$

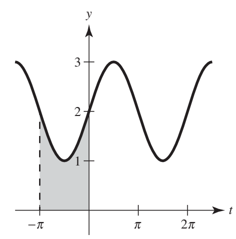
$$= (2x - \cos x) - (-2\pi + 1)$$

$$= 2x - \cos x + 2\pi - 1$$

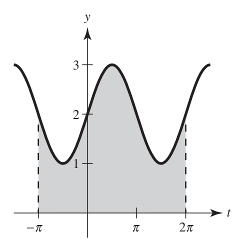
(a) $F(-\pi) = 2(-\pi) + 1 + 2\pi - 1 = 0$



(b) $F(0) = -\cos 0 + 2\pi - 1 = 2\pi - 2$



(c) $F(2\pi) = 2(2\pi) - \cos 2\pi + 2\pi - 1 = 6\pi - 2$



17. R_1 projects more revenue.

$$\int_0^5 \left[(2.98 + 0.65t) - (2.98 + 0.56t)\right] dt = \int_0^5 0.09t\,dt$$

$$= \left[\frac{0.09t^2}{2}\right]_0^5 = \$1.125 \text{ million}$$

18. R_2 projects more revenue.

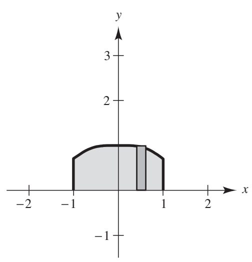
$$\int_0^5 \left[\left(4.87 + 0.61t + 0.07t^2\right) - \left(4.87 + 0.55t + 0.01t^2\right)\right] dt$$

$$= \int_0^5 \left(0.06t + 0.06t^2\right) dt$$

$$= 0.06\left[\frac{t^2}{2} + \frac{t^3}{3}\right]_0^5$$

$$= 0.06\left[\frac{25}{2} + \frac{125}{3}\right] = \$3.25 \text{ million}$$

19. $V = 2\pi\int_0^1\left[\frac{1}{\sqrt{1 + x^2}}\right]^2 dx = \left[2\pi \arctan x\right]_0^1 = 2\pi\left(\frac{\pi}{4} - 0\right) = \frac{\pi^2}{2}$

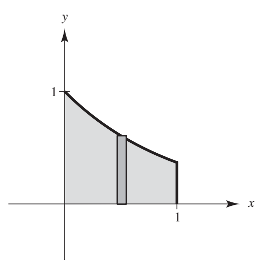


20. $V = \pi\int_0^1\left(e^{-x}\right)^2 dx$

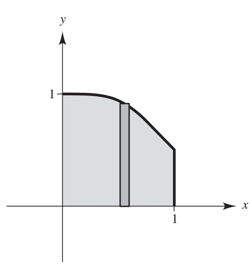
$$= \pi\int_0^1 e^{-2x} dx$$

$$= \left[-\frac{\pi}{2}e^{-2x}\right]_0^1$$

$$= \left(\frac{-\pi}{2e^2} + \frac{\pi}{2}\right) = \frac{\pi}{2}\left(1 - \frac{1}{e^2}\right)$$



21. $V = 2\pi\int_0^1 \frac{x}{x^4 + 1} dx = \pi\int_0^1 \frac{(2x)}{\left(x^2\right)^2 + 1} dx = \left[\pi \arctan\left(x^2\right)\right]_0^1 = \pi\left(\frac{\pi}{4} - 0\right) = \frac{\pi^2}{4}$



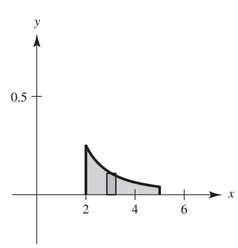
22. $V = 2\pi\int_2^5 x\left(\frac{1}{x^2}\right) dx$

$$= 2\pi\int_2^5 \frac{1}{x} dx$$

$$= \left[2\pi \ln|x|\right]_2^5$$

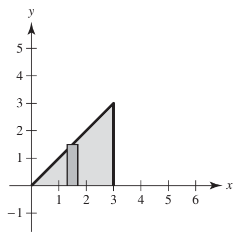
$$= 2\pi(\ln 5 - \ln 2)$$

$$= 2\pi \ln\left(\frac{5}{2}\right)$$



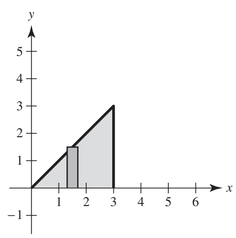
23. (a) **Disk**

$$V = \pi\int_0^3 x^2\,dx = \left[\frac{\pi x^3}{3}\right]_0^3 = 9\pi$$



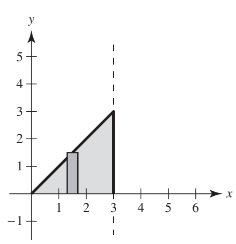
(b) **Shell**

$$V = 2\pi\int_0^3 x(x)\,dx = 2\pi\left[\frac{x^3}{3}\right]_0^3 = 18\pi$$



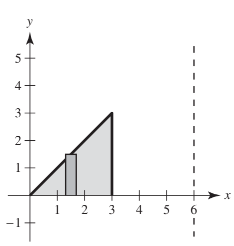
(c) **Shell**

$$V = 2\pi\int_0^3 (3 - x)x\,dx = 2\pi\left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = 9\pi$$



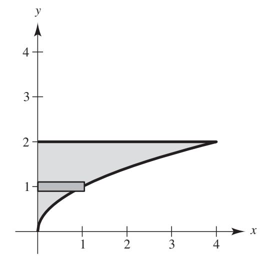
(d) **Shell**

$$V = 2\pi\int_0^3 (6 - x)x\,dx = 2\pi\left[3x^2 - \frac{x^3}{3}\right]_0^3 = 36\pi$$



24. (a) **Shell**

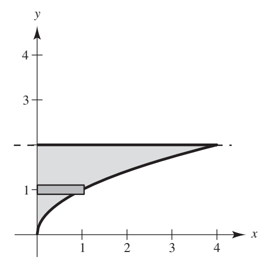
$$V = 2\pi\int_0^2 y^3\,dy = \left[\frac{\pi}{2}y^4\right]_0^2 = 8\pi$$



(b) **Shell**

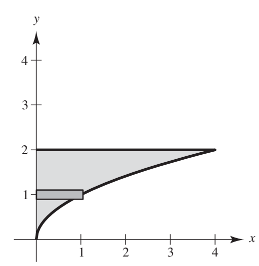
$$V = 2\pi\int_0^2 (2 - y)y^2\,dy$$

$$= 2\pi\int_0^2 \left(2y^2 - y^3\right)dy = 2\pi\left[\frac{2}{3}y^3 - \frac{1}{4}y^4\right]_0^2 = \frac{8\pi}{3}$$



(c) **Disk**

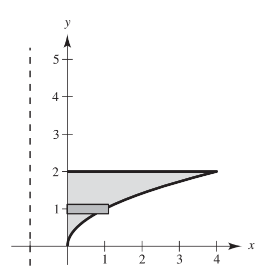
$$V = \pi\int_0^2 y^4\,dy = \left[\frac{\pi}{5}y^5\right]_0^2 = \frac{32\pi}{5}$$



(d) **Disk**

$$V = \pi\int_0^2\left[\left(y^2 + 1\right)^2 - 1^2\right]dy$$

$$= \pi\int_0^2 \left(y^4 + 2y^2\right)dy = \pi\left[\frac{1}{5}y^5 + \frac{2}{3}y^3\right]_0^2 = \frac{176\pi}{15}$$



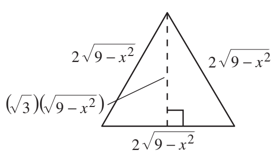
25. The volume of the spheroid is given by:

$$V = 4\pi\int_0^4 x\left(\tfrac{3}{4}\right)\sqrt{16 - x^2}\,dx$$
$$= \left[3\pi\left(-\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right)\left(16 - x^2\right)^{3/2}\right]_0^4$$
$$= 64\pi$$
$$\tfrac{1}{4}V = 16\pi$$

Disk:
$$\pi\int_{-3}^{y_0} \tfrac{16}{9}\left(9 - y^2\right)dy = 16\pi$$
$$\tfrac{1}{9}\int_{-3}^{y_0}\left(9 - y^2\right)dy = 1$$
$$\left[9y - \tfrac{1}{3}y^3\right]_{-3}^{y_0} = 9$$
$$\left(9y_0 - \tfrac{1}{3}y_0^3\right) - (-27 + 9) = 9$$
$$y_0^3 - 27y_0 - 27 = 0$$

By Newton's Method, $y_0 \approx -1.042$ and the depth of the gasoline is $3 - 1.042 = 1.958$ feet.

26.



$$A(x) = \tfrac{1}{2}bh = \tfrac{1}{2}\left(2\sqrt{9 - x^2}\right)\left(\sqrt{3}\sqrt{9 - x^2}\right)$$
$$= \sqrt{3}\left(9 - x^2\right)$$
$$V = \sqrt{3}\int_{-3}^{3}\left(9 - x^2\right)dx = \sqrt{3}\left[9x - \frac{x^3}{3}\right]_{-3}^{3}$$
$$= \sqrt{3}\left[(27 - 9) - (-27 + 9)\right] = 36\sqrt{3}$$

27.
$$f(x) = \tfrac{4}{5}x^{5/4},\ [0, 4]$$
$$f'(x) = x^{1/4}$$
$$1 + \left[f'(x)\right]^2 = 1 + \sqrt{x}$$
$$u = 1 + \sqrt{x}$$
$$x = (u - 1)^2$$
$$dx = 2(u - 1)\,du$$
$$s = \int_0^4 \sqrt{1 + \sqrt{x}}\,dx = 2\int_1^3 \sqrt{u}(u - 1)\,du$$
$$= 2\int_1^3\left(u^{3/2} - u^{1/2}\right)du$$
$$= 2\left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right]_1^3 = \tfrac{4}{15}\left[u^{3/2}(3u - 5)\right]_1^3$$
$$= \tfrac{8}{15}\left(1 + 6\sqrt{3}\right) = 6.076$$

28.
$$y = \frac{1}{3}x^{3/2} - 1,\quad [2, 6]$$
$$y' = \frac{1}{2}x^{1/2}$$
$$1 + (y')^2 = 1 + \frac{1}{4}x$$
$$s = \int_2^6 \sqrt{1 + \frac{1}{4}x}\,dx = \frac{1}{2}\int_2^6 \sqrt{4 + x}\,dx$$
$$= \frac{1}{2}\left[\frac{(4 + x)^{3/2}}{(3/2)}\right]_2^6$$
$$= \frac{1}{3}\left[(x + 4)^{3/2}\right]_2^6$$
$$= \frac{1}{3}\left(10^{3/2} - 6^{3/2}\right) \approx 5.642$$

29. $y = \dfrac{x^3}{18},\ 3 \le x \le 6$
$$y' = \frac{x^2}{6}$$
$$1 + (y')^2 = 1 + \frac{x^4}{36} = \frac{\left(36 + x^4\right)}{36}$$
$$S = 2\pi\int_3^6 \frac{x^3}{18}\sqrt{\frac{36 + x^4}{36}}\,dx$$
$$= \frac{\pi}{54}\int_3^6 \sqrt{36 + x^4}\,x^3\,dx$$
$$= \frac{1}{4}\cdot\frac{\pi}{54}\left[\frac{\left(36 + x^4\right)^{3/2}}{(3/2)}\right]_3^6$$
$$= \frac{\pi}{324}\left[\left(36 + x^4\right)^{3/2}\right]_3^6$$
$$= \frac{\pi}{324}\left(1332^{3/2} - 117^{3/2}\right)$$
$$\approx 459.098$$

30.
$$y = \sqrt{25 - x^2},\ -4 \le x \le 4$$
$$y' = \frac{-x}{\sqrt{25 - x^2}}$$
$$1 + (y')^2 = 1 + \frac{x^2}{25 - x^2} = \frac{25}{25 - x^2}$$
$$S = 2\pi\int_{-4}^4 \sqrt{25 - x^2}\sqrt{\frac{25}{25 - x^2}}\,dx$$
$$= 2\pi\int_{-4}^4 5\,dx$$
$$= 2\pi\left[5x\right]_{-4}^4$$
$$= 2\pi(40) = 80\pi$$

31. $y = \frac{x^2}{2} + 4, 0 \le x \le 2$

$y' = x$

$1 + (y')^2 = 1 + x^2$

$$S = 2\pi \int_0^2 x\sqrt{1 + x^2}\, dx$$

$$= \frac{2\pi}{3}\left[(x^2 + 1)^{3/2}\right]_0^2$$

$$= \frac{2\pi}{3}(5^{3/2} - 1) \approx 21.322$$

32. $y = \sqrt[3]{x} = x^{1/3}, 1 \le x \le 2$

$y' = \frac{1}{3}x^{-2/3}$

$1 + (y')^2 = 1 + \frac{1}{9x^{4/3}} = \frac{9x^{4/3} + 1}{9x^{4/3}}$

$$S = 2\pi \int_1^2 x\sqrt{\frac{9x^{4/3} + 1}{9x^{4/3}}}\, dx$$

$$= \frac{2\pi}{3}\int_1^2 x^{1/3}\sqrt{9x^{4/3} + 1}\, dx$$

$$= \frac{\pi}{27}\left[(9x^{4/3} + 1)^{3/2}\right]_1^2$$

$$= \frac{\pi}{27}\left[(9(2^{4/3}) + 1)^{3/2} - (10^{3/2})\right]$$

$$\approx 9.727$$

33. $F = kx$

$5 = k(1)$

$F = 5x$

$$W = \int_0^5 5x\, dx = \frac{5x^2}{2}\bigg]_0^5 = \frac{125}{2} \text{ in.-lb} \approx 5.21 \text{ ft-lb}$$

34. $F = kx$

$50 = k(1) \Rightarrow k = 50$

$$W = \int_0^{10} 50x\, dx = \left[25x^2\right]_0^{10} = 2500 \text{ in.-lb} \approx 208.3 \text{ ft-lb}$$

35. Assume that Earth has a radius of 4000 miles.

$F(x) = \frac{k}{x^2}$

$5 = \frac{k}{4000^2} \Rightarrow k = 80{,}000{,}000$

$F(x) = \frac{80{,}000{,}000}{x^2}$

$$W = \int_{4000}^{4200} \frac{80{,}000{,}000}{x^2}\, dx$$

$$= \left[\frac{-80{,}000{,}000}{x}\right]_{4000}^{4200}$$

$$= \frac{20{,}000}{21} \approx 952.38 \text{ mi-tons}$$

$$= 1.109 \times 10^{10} \text{ ft-lb}$$

(**Note:** One metric ton = 2205 pounds)

36. Volume of disk: $\pi\left(\frac{1}{3}\right)^2 \Delta y$ $\left[\text{diameter} = \frac{2}{3} \text{ ft}\right]$

Weight of disk: $62.4\pi\left(\frac{1}{3}\right)^2 \Delta y$

Distance: $190 - y$

$$W = \frac{62.4\pi}{9}\int_0^{165} (190 - y)\, dy$$

$$= \frac{62.4\pi}{9}\left[190y - \frac{y^2}{2}\right]_0^{165}$$

$$= \frac{62.4\pi}{9}\left[\frac{35{,}475}{2}\right] = 122{,}980\pi \text{ ft-lb}$$

$$\approx 193.2 \text{ foot-tons}$$

37. Weight of section of chain: $4\,\Delta x$

Distance moved: $10 - x$

$$W = 4\int_0^{10} (10 - x)\, dx = 4\left[10x - \frac{x^2}{2}\right]_0^{10}$$

$$= 200 \text{ ft-lb}$$

38. (a) Weight of section of cable: $5\,\Delta x$

Distance: $200 - x$

$$W = 5\int_0^{200} (200 - x)\, dx$$

$$= 5\left[200x - \frac{x^2}{2}\right]_0^{200}$$

$$= 100{,}000 \text{ ft-lb}$$

(b) Work to move 300 pounds 200 feet vertically:

$300(200) = 60{,}000$ ft-lb.

Total work: $100{,}000 + 60{,}000 = 160{,}000$ ft-lb

39. $\rho = \dfrac{k}{V}$

$500 = \dfrac{k}{1} \Rightarrow k = 500$

$$W = \int_1^4 \frac{500}{V}\,dV = \Big[500 \ln V\Big]_1^4 = 500 \ln 4 = 1000 \ln 2 \approx 693.15 \text{ ft-lb}$$

40. $\rho = \dfrac{k}{V}$

$800 = \dfrac{k}{2}$

$k = 1600$

$$W = \int_2^3 \frac{1600}{V}\,dV = \Big[1600 \ln|V|\Big]_2^3 = 1600 \ln\left(\frac{3}{2}\right) \approx 648.74 \text{ ft-lb}$$

41. $\bar{x} = \dfrac{8(-1) + 12(2) + 6(5) + 14(7)}{8 + 12 + 6 + 14} = \dfrac{144}{40} = \dfrac{18}{5} = 3.6$

42. $\bar{x} = \dfrac{3(2) + 2(-3) + 6(4) + 9(6)}{3 + 2 + 6 + 9} = \dfrac{78}{20} = \dfrac{39}{10}$

$\bar{y} = \dfrac{3(1) + 2(2) + 6(-1) + 9(5)}{3 + 2 + 6 + 9} = \dfrac{46}{20} = \dfrac{23}{10}$

$(\bar{x}, \bar{y}) = \left(\dfrac{39}{10}, \dfrac{23}{10}\right)$

43. $A = \displaystyle\int_{-1}^{3}\Big[(2x + 3) - x^2\Big]\,dx = \left[x^2 + 3x - \frac{1}{3}x^3\right]_{-1}^{3} = \frac{32}{3}$

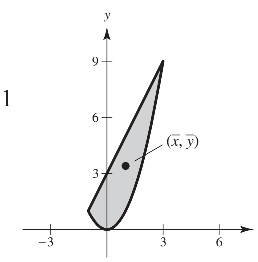
$\dfrac{1}{A} = \dfrac{3}{32}$

$$\bar{x} = \frac{3}{32}\int_{-1}^{3} x\left(2x + 3 - x^2\right)dx = \frac{3}{32}\int_{-1}^{3}\left(3x + 2x^2 - x^3\right)dx = \frac{3}{32}\left[\frac{3}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4\right]_{-1}^{3} = 1$$

$$\bar{y} = \left(\frac{3}{32}\right)\frac{1}{2}\int_{-1}^{3}\Big[(2x + 3)^2 - x^4\Big]\,dx = \frac{3}{64}\int_{-1}^{3}\left(9 + 12x + 4x^2 - x^4\right)dx$$

$$= \frac{3}{64}\left[9x + 6x^2 + \frac{4}{3}x^3 - \frac{1}{5}x^5\right]_{-1}^{3} = \frac{17}{5}$$

$(\bar{x}, \bar{y}) = \left(1, \dfrac{17}{5}\right)$



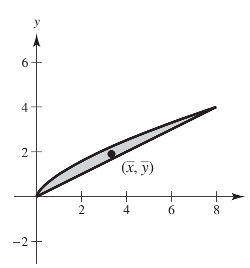
44. $A = \displaystyle\int_0^8\left(x^{2/3} - \frac{1}{2}x\right)dx = \left[\frac{3}{5}x^{5/3} - \frac{1}{4}x^2\right]_0^8 = \frac{16}{5}$

$\dfrac{1}{A} = \dfrac{5}{16}$

$$\bar{x} = \frac{5}{16}\int_0^8 x\left(x^{2/3} - \frac{1}{2}x\right)dx = \frac{5}{16}\left[\frac{3}{8}x^{8/3} - \frac{1}{6}x^3\right]_0^8 = \frac{10}{3}$$

$$\bar{y} = \left(\frac{5}{16}\right)\frac{1}{2}\int_0^8\left(x^{4/3} - \frac{1}{4}x^2\right)dx = \frac{1}{2}\left(\frac{5}{16}\right)\left[\frac{3}{7}x^{7/3} - \frac{1}{12}x^3\right]_0^8 = \frac{40}{21}$$

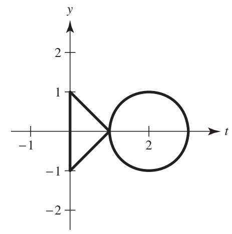
$(\bar{x}, \bar{y}) = \left(\dfrac{10}{3}, \dfrac{40}{21}\right)$



45. Answers will vary. *Sample answer*:

The centroid of the triangle is $\left(\frac{1}{3}, 0\right)$.

The centroid of the circle is $(2, 0)$.



$A = 1 + \pi$

$\bar{x} = \dfrac{1(1/3) + \pi(2)}{1 + \pi} = \dfrac{1 + 6\pi}{3(1 + \pi)} \approx 1.596$

$(\bar{x}, \bar{y}) \approx (1.596, 0)$

46. $r = 4$ is the distance between the center of the circle and the y-axis.

$A = \pi(2)^2 = 4\pi$ is the area of the circle. So,

$V = 2\pi rA = 2\pi(4)(4\pi) = 32\pi^2$.

47. The weight density of water is 62.4 lb/ft^3.

$P = wh = (62.4)(3) = 187.2 \text{ lb/ft}^2$

Fluid force $= F = PA = (187.2)(2) = 374.4$ lb

48. The weight density of water is 62.4 lb/ft^3.

$P = wh = (62.4)(3) = 187.2 \text{ lb/ft}^2$

Fluid force $= F = PA = (187.2)(15) = 2808$ lb

49. $h(y) = 9 - y$

$L(y) = 4 - \frac{4}{3}y$

$$\begin{aligned} F &= 64\int_0^3 (9 - y)\left(4 - \frac{4}{3}y\right) dy \\ &= 64\int_0^3 \left(36 - 16y + \frac{4}{3}y^2\right) dy \\ &= 64\left[36y - 8y^2 + \frac{4}{9}y^3\right]_0^3 \\ &= 64\left[36(3) - 8(9) + 4(3)\right] = 64(48) \\ &= 3072 \text{ lb} \end{aligned}$$

50. $h(y) = 5 - y$

$L(y) = 7$

$$\begin{aligned} F &= 140.7\int_0^5 (5 - y)(7)\, dy \\ &= 140.7(7)\left[5y - \frac{y^2}{2}\right]_0^5 \\ &= 984.9\left(25 - \frac{25}{2}\right) \\ &= 12{,}311.25 \text{ lb} \end{aligned}$$

51. From Exercise 29 in Section 7.7: $F = wk(\pi r^2) = (64.0)(1600)\left(\pi(1.5)^2\right) \approx 723{,}822.95$ lb

Problem Solving for Chapter 7

1. $T = \frac{1}{2}c(c^2) = \frac{1}{2}c^3$

$R = \int_0^c (cx - x^2)\, dx = \left[\frac{cx^2}{2} - \frac{x^3}{3}\right]_0^c = \frac{c^3}{2} - \frac{c^3}{3} = \frac{c^3}{6}$

$$\lim_{c\to 0^+} \frac{T}{R} = \lim_{c\to 0^+} \frac{\frac{1}{2}c^3}{\frac{1}{6}c^3} = 3$$

2. (a) By symmetry, $M_x = 0$ for L

(b) Because

$$(M_y \text{ for } L) + (M_y \text{ for } A) = (M_y \text{ for } B),$$

you have

$$(M_y \text{ for } L) = (M_y \text{ for } B) - (M_y \text{ for } A)$$

(c) M_y for $B = 0$, because B is a circle at the origin

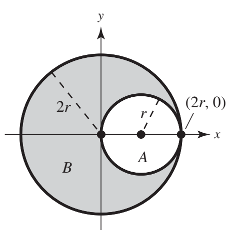
For A, $\bar{x} = \dfrac{M_y}{\text{Area}} \Rightarrow M_y = r(\pi r^2) = \pi r^3$

So, $(M_y \text{ for } L) = 0 - \pi r^3 = -\pi r^3$

(d) $\bar{y} = 0$ by symmetry.

$$\bar{x} = \frac{M_y \text{ of } L}{\text{Area of } L} = \frac{-\pi r^3}{4\pi r^2 - \pi r^2} = -\frac{r}{3}$$

$$(\bar{x}, \bar{y}) = \left(-\frac{r}{3}, 0\right)$$



3. $R = \int_0^1 x(1 - x)\,dx = \left[\dfrac{x^2}{2} - \dfrac{x^3}{3}\right]_0^1 = \dfrac{1}{2} - \dfrac{1}{3} = \dfrac{1}{6}$

Let (c, mc) be the intersection of the line and the parabola.

Then, $mc = c(1 - c) \Rightarrow m = 1 - c$ or $c = 1 - m$.

$$\frac{1}{2}\left(\frac{1}{6}\right) = \int_0^{1-m} (x - x^2 - mx)\,dx$$

$$\frac{1}{12} = \left[\frac{x^2}{2} - \frac{x^3}{3} - m\frac{x^2}{2}\right]_0^{1-m}$$

$$= \frac{(1-m)^2}{2} - \frac{(1-m)^3}{3} - m\frac{(1-m)^2}{2}$$

$$1 = 6(1-m)^2 - 4(1-m)^3 - 6m(1-m)^2$$

$$= (1-m)^2(6 - 4(1-m) - 6m)$$

$$= (1-m)^2(2 - 2m)$$

$$\frac{1}{2} = (1-m)^3$$

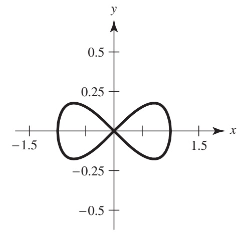
$$\left(\frac{1}{2}\right)^{1/3} = 1 - m$$

$$m = 1 - \left(\frac{1}{2}\right)^{1/3} \approx 0.2063$$

So, $y = 0.2063x$.

4. $8y^2 = x^2(1 - x^2)$

$$y = \pm\frac{|x|\sqrt{1 - x^2}}{2\sqrt{2}}$$



For $x > 0$, $y' = \dfrac{1 - 2x^2}{2\sqrt{2}\sqrt{1 - x^2}}$.

$$S = 2(2\pi)\int_0^1 x\sqrt{1 + \left(\frac{1 - 2x^2}{2\sqrt{2}\sqrt{1 - x^2}}\right)^2}\,dx$$

$$= \frac{5\sqrt{2}\pi}{3}$$

5. $\bar{y} = 0$ by symmetry.

For the trapezoid:

$$m = \left[(4)(6) - (1)(6)\right]\rho = 18\rho$$

$$M_y = \rho\int_0^6 x\left[\left(\frac{1}{6}x + 1\right) - \left(-\frac{1}{6}x - 1\right)\right] dx = \rho\int_0^6 \left(\frac{1}{3}x^2 + 2x\right) dx = \rho\left[\frac{x^3}{9} + x^2\right]_0^6 = 60\rho$$

For the semicircle:

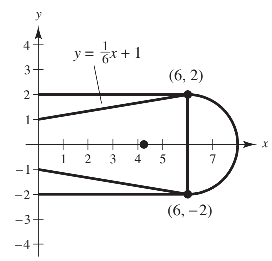
$$m = \left(\frac{1}{2}\right)(\pi)(2)^2 \rho = 2\pi\rho$$

$$M_y = \rho\int_6^8 x\left[\sqrt{4 - (x - 6)^2} - \left(-\sqrt{4 - (x - 6)^2}\right)\right] dx = 2\rho\int_6^8 x\sqrt{4 - (x - 6)^2}\, dx$$

Let $u = x - 6$, then $x = u + 6$ and $dx = du$. When $x = 6, u = 0$. When $x = 8, u = 2$.

$$M_y = 2\rho\int_0^2 (u + 6)\sqrt{4 - u^2}\, du = 2\rho\int_0^2 u\sqrt{4 - u^2}\, du + 12\rho\int_0^2 \sqrt{4 - u^2}\, du$$

$$= 2\rho\left[\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(4 - u^2\right)^{3/2}\right]_0^2 + 12\rho\left[\frac{\pi(2)^2}{4}\right] = \frac{16\rho}{3} + 12\pi\rho = \frac{4\rho(4 + 9\pi)}{3}$$



So, you have: $\bar{x}(18\rho + 2\pi\rho) = 60\rho + \dfrac{4\rho(4 + 9\pi)}{3}$

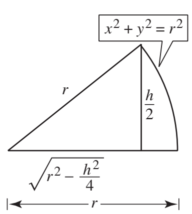
$$\bar{x} = \frac{180\rho + 4\rho(4 + 9\pi)}{3} \cdot \frac{1}{2\rho(9 + \pi)} = \frac{2(9\pi + 49)}{3(\pi + 9)}$$

The centroid of the blade is $\left(\dfrac{2(9\pi + 49)}{3(\pi + 9)}, 0\right)$.

6. $V = 2(2\pi)\int_{\sqrt{r^2 - (h^2/4)}}^{r} x\sqrt{r^2 - x^2}\, dx$

$$= -2\pi\left[\frac{2}{3}\left(r^2 - x^2\right)^{3/2}\right]_{\sqrt{r^2 - (h^2/4)}}^{r}$$

$$= \frac{-4\pi}{3}\left[-\frac{h^3}{8}\right] = \frac{\pi h^3}{6} \text{ which does not depend on } r$$



7. By the Theorem of Pappus,

$$V = 2\pi rA$$

$$= 2\pi\left[d + \tfrac{1}{2}\sqrt{w^2 + l^2}\right] lw.$$

8. (a) Tangent at A: $y = x^3, y' = 3x^2$

$$y - 1 = 3(x - 1)$$

$$y = 3x - 2$$

To find point B:

$$x^3 = 3x - 2$$

$$x^3 - 3x + 2 = 0$$

$$(x - 1)^2(x + 2) = 0 \Rightarrow B = (-2, -8)$$

Tangent at B: $y = x^3, y' = 3x^2$

$$y + 8 = 12(x + 2)$$

$$y = 12x + 16$$

To find point C:

$$x^3 = 12x + 16$$

$$x^3 - 12x - 16 = 0$$

$$(x + 2)^2(x - 4) = 0 \Rightarrow C = (4, 64)$$

Area of $R = \int_{-2}^{1}\left(x^3 - 3x + 2\right) dx = \dfrac{27}{4}$

Area of $S = \int_{-2}^{4}\left(12x + 16 - x^3\right) dx = 108$

Area of $S = 16$(area of R) $\quad \left[\dfrac{\text{area } S}{\text{area } R} = 16\right]$

(b) Tangent at $A(a, a^3)$: $y - a^3 = 3a^2(x - a)$

$$y = 3a^2x - 2a^3$$

To find point B: $x^3 - 3a^2x + 2a^3 = 0$

$$(x - a)^2(x + 2a) = 0$$

$$\Rightarrow B = (-2a, -8a^3)$$

Tangent at B: $y + 8a^3 = 12a^2(x + 2a)$

$$y = 12a^2x + 16a^3$$

To find point C: $x^3 - 12a^2x - 16a^3 = 0$

$$(x + 2a)^2(x - 4a) = 0$$

$$\Rightarrow C = (4a, 64a^3)$$

Area of $R = \int_{-2a}^{a}\left[x^3 - 3a^2x + 2a^3\right] dx = \frac{27}{4}a^4$

Area of $S = \int_{-2a}^{4a}\left[12a^2x + 16a^3 - x^3\right] dx = 108a^4$

Area of $S = 16(\text{area of } R)$

9. $f'(x)^2 = e^x$

$$f'(x) = e^{x/2}$$

$$f(x) = 2e^{x/2} + C$$

$$f(0) = 0 \Rightarrow C = -2$$

$$f(x) = 2e^{x/2} - 2$$

10. $s(x) = \int_{\alpha}^{x}\sqrt{1 + f'(t)^2}\, dt$

(a) $s'(x) = \dfrac{ds}{dx} = \sqrt{1 + f'(x)^2}$

(b) $ds = \sqrt{1 + f'(x)^2}\, dx$

$$(ds)^2 = \left[1 + f'(x)^2\right](dx)^2$$

$$= \left[1 + \left(\frac{dy}{dx}\right)^2\right](dx)^2 = (dx)^2 + (dy)^2$$

(c) $s(x) = \int_1^x \sqrt{1 + \left(\frac{3}{2}t^{1/2}\right)^2}\, dt = \int_1^x \sqrt{1 + \frac{9}{4}t}\, dt$

(d) $s(2) = \int_1^2 \sqrt{1 + \frac{9}{4}t}\, dt$

$$= \left[\frac{8}{27}\left(1 + \frac{9}{4}t\right)^{3/2}\right]_1^2$$

$$= \frac{22}{27}\sqrt{22} - \frac{13}{27}\sqrt{13} \approx 2.0858$$

This is the length of the curve $y = x^{3/2}$ from $x = 1$ to $x = 2$.

11. Let ρ_f be the density of the fluid and ρ_0 the density of the iceberg. The buoyant force is

$$F = \rho_f g \int_{-h}^{0} A(y)\, dy$$

where $A(y)$ is a typical cross section and g is the acceleration due to gravity. The weight of the object is

$$W = \rho_0 g \int_{-h}^{L-h} A(y)\, dy.$$

$$F = W$$

$$\rho_f g \int_{-h}^{0} A(y)\, dy = \rho_0 g \int_{-h}^{L-h} A(y)\, dy$$

$$\frac{\rho_0}{\rho_f} = \frac{\text{submerged volume}}{\text{total volume}}$$

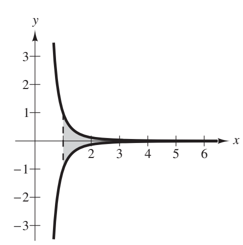
$$= \frac{0.92 \times 10^3}{1.03 \times 10^3} = 0.893 \text{ or } 89.3\%$$

12. (a) $\bar{y} = 0$ by symmetry

$$M_y = \int_1^6 x\left(\frac{1}{x^3} - \left(-\frac{1}{x^3}\right)\right) dx = \int_1^6 \frac{2}{x^2}\, dx = \left[-2\frac{1}{x}\right]_1^6 = \frac{5}{3}$$

$$m = 2\int_1^6 \frac{1}{x^3}\, dx = \left[-\frac{1}{x^2}\right]_1^6 = \frac{35}{36}$$

$$\bar{x} = \frac{5/3}{35/36} = \frac{12}{7} \qquad (\bar{x}, \bar{y}) = \left(\frac{12}{7}, 0\right)$$



(b) $m = 2\int_1^b \frac{1}{x^3}\,dx = \frac{b^2 - 1}{b^2}$

$M_y = 2\int_1^6 \frac{1}{x^2}\,dx = \frac{2(b - 1)}{b}$

$\bar{x} = \frac{2(b - 1)/b}{(b^2 - 1)/b^2} = \frac{2b}{b + 1} \qquad (\bar{x}, \bar{y}) = \left(\frac{2b}{b + 1}, 0\right)$

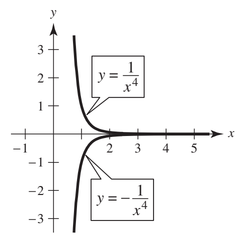
(c) $\lim\limits_{b\to\infty} \bar{x} = \lim\limits_{b\to\infty} \frac{2b}{b + 1} = 2 \qquad (\bar{x}, \bar{y}) = (2, 0)$

13. (a) $\bar{y} = 0$ by symmetry

$M_y = 2\int_1^6 x\frac{1}{x^4}\,dx = 2\int_1^6 \frac{1}{x^3}\,dx = \frac{35}{36}$

$m = 2\int_1^6 \frac{1}{x^4}\,dx = \frac{215}{324}$

$\bar{x} = \frac{35/36}{215/324} = \frac{63}{43} \qquad (\bar{x}, \bar{y}) = \left(\frac{63}{43}, 0\right)$



(b) $M_y = 2\int_1^b \frac{1}{x^3}\,dx = \frac{b^2 - 1}{b^2}$

$m = 2\int_1^b \frac{1}{x^4}\,dx = \frac{2(b^3 - 1)}{3b^3}$

$\bar{x} = \frac{(b^2 - 1)/b^2}{2(b^3 - 1)/3b^3} = \frac{3b(b + 1)}{2(b^2 + b + 1)} \qquad (\bar{x}, \bar{y}) = \left(\frac{3b(b + 1)}{2(b^2 + b + 1)}, 0\right)$

(c) $\lim\limits_{b\to\infty} \bar{x} = \frac{3b(b + 1)}{2(b^2 + b + 1)} = \frac{3}{2} \qquad (\bar{x}, \bar{y}) = \left(\frac{3}{2}, 0\right)$

14. (a) $W = \text{area} = 2 + 4 + 6 = 12$

(b) $W = \text{area} = 3 + (1 + 1) + 2 + \frac{1}{2} = 7\frac{1}{2}$

15. Point of equilibrium: $50 - 0.5x = 0.125x$

$x = 80,\ p = 10$

$(P_0, x_0) = (10, 80)$

Consumer surplus $= \int_0^{80} \left[(50 - 0.5x) - 10\right] dx = 1600$

Producer surplus $= \int_0^{80} (10 - 0.125x)\,dx = 400$

16. Point of equilibrium: $1000 - 0.4x^2 = 42x$

$x = 20,\ p = 840$

$(P_0, x_0) = (840, 20)$

Consumer surplus $= \int_0^{20} \left[(1000 - 0.4x^2) - 840\right] dx$

$= 2133.33$

Producer surplus $= \int_0^{20} (840 - 42x)\,dx = 8400$

17. Use Exercise 25, Section 7.7, which gives $F = wkhb$ for a rectangle plate.

Wall at shallow end

From Exercise 25: $F = 62.4(2)(4)(20) = 9984$ lb

Wall at deep end

From Exercise 25: $F = 62.4(4)(8)(20) = 39{,}936$ lb

Side wall

From Exercise 25: $F_1 = 62.4(2)(4)(40) = 19{,}968$ lb

$F_2 = 62.4\int_0^4 (8 - y)(10y)\,dy$

$= 624\int_0^4 (8y - y^2)\,dy = 624\left[4y^2 - \frac{y^3}{3}\right]_0^4$

$= 26{,}624$ lb

Total force: $F_1 + F_2 = 46{,}592$ lb