C H A P T E R 6
Differential Equations

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Section 6.1 Slope Fields and Euler's Method

1. A function $f(x)$ is a solution of a differential equation if the equation is satisfied when y and its derivatives are replaced by $f(x)$ and its derivatives.

2. The general solution of a first-order differential equation represents a family of curves known as solution curves, one for each value assigned to the arbitrary constant.

3. The line segments show the general shape of all the solutions of a differential equation and give a visual perspective of the directions of the solutions of the differential equation.

4. Euler's Method allows you to approximate the solution to a first-order initial value problem.

5. Differential equation: $y' = 5y$

Solution: $y = Ce^{5x}$

Check: $y' = 5Ce^{5x} = 5y$

6. Differential equation: $3y' + 5y = -e^{-2x}$

Solution: $y = e^{-2x}$

$$y' = -2e^{-2x}$$

Check: $3\left(-2e^{-2x}\right) + 5\left(e^{-2x}\right) = -e^{-2x}$

7. Differential equation: $y'' + y = 0$

Solution:
$$\begin{aligned} y &= C_1 \sin x - C_2 \cos x \\ y' &= C_1 \cos x + C_2 \sin x \\ y'' &= -C_1 \sin x + C_2 \cos x \end{aligned}$$

Check: $y'' + y = \left(-C_1 \sin x + C_2 \cos x\right) + \left(C_1 \sin x - C_2 \cos x\right) = 0$

8. Differential equation: $y'' + 2y' + 2y = 0$

Solution: $y = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x$

Check:
$$\begin{aligned} y' &= -(C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x \\ y'' &= 2C_1 e^{-x} \sin x - 2C_2 e^{-x} \cos x \\ y'' + 2y' + 2y &= 2C_1 e^{-x} \sin x - 2C_2 e^{-x} \cos x + \\ &\quad 2\left(-(C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x\right) + 2\left(C_1 e^{-x} \cos x + C_2 e^{-x} \sin x\right) \\ &= (2C_1 - 2C_1 - 2C_2 + 2C_2)e^{-x} \sin x + (-2C_2 - 2C_1 + 2C_2 + 2C_1)e^{-x} \cos x = 0 \end{aligned}$$

9. Differential equation: $y'' + y = \tan x$

Solution:
$$\begin{aligned} y &= -\cos x \ln\left|\sec x + \tan x\right| \\ y' &= (-\cos x)\frac{1}{\sec x + \tan x}\left(\sec x \cdot \tan x + \sec^2 x\right) + \sin x \ln\left|\sec x + \tan x\right| \\ &= \frac{(-\cos x)}{\sec x + \tan x}(\sec x)(\tan x + \sec x) + \sin x \ln\left|\sec x + \tan x\right| \\ &= -1 + \sin x \ln\left|\sec x + \tan x\right| \\ y'' &= (\sin x)\frac{1}{\sec x + \tan x}\left(\sec x \cdot \tan x + \sec^2 x\right) + \cos x \ln\left|\sec x + \tan x\right| \\ &= (\sin x)(\sec x) + \cos x \ln\left|\sec x + \tan x\right| \end{aligned}$$

Check: $y'' + y = (\sin x)(\sec x) + \cos x \ln\left|\sec x + \tan x\right| - \cos x \ln\left|\sec x + \tan x\right| = \tan x.$

10. Differential equation: $y'' + 4y' = 2e^x$

Solution: $y = \frac{2}{5}\left(e^{-4x} + e^x\right)$

$y' = \frac{2}{5}\left(-4e^{-4x} + e^x\right) = -\frac{8}{5}e^{-4x} + \frac{2}{5}e^x$

$y'' = \frac{32}{5}e^{-4x} + \frac{2}{5}e^x$

Check: $y'' + 4y' = \left(\frac{32}{5}e^{-4x} + \frac{2}{5}e^x\right) + 4\left(-\frac{8}{5}e^{-4x} + \frac{2}{5}e^x\right) = \left(\frac{2}{5} + \frac{8}{5}\right)e^x = 2e^x$

11. $y = \sin x \cos x - \cos^2 x$

$y' = -\sin^2 x + \cos^2 x + 2\cos x \sin x$

$= -1 + 2\cos^2 x + \sin 2x$

Differential equation:

$2y + y' = 2\left(\sin x \cos x - \cos^2 x\right) + \left(-1 + 2\cos^2 x + \sin 2x\right)$

$= 2\sin x \cos x - 1 + \sin 2x$

$= 2\sin 2x - 1$

Initial condition $\left(\frac{\pi}{4}, 0\right)$:

$\sin\frac{\pi}{4}\cos\frac{\pi}{4} - \cos^2\frac{\pi}{4} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} - \left(\frac{\sqrt{2}}{2}\right)^2 = 0$

12. $y = 6x - 4\sin x + 1$

$y' = 6 - 4\cos x$

Differential equation: $y' = 6 - 4\cos x$

Initial condition $(0, 1)$: $0 - 0 + 1 = 1$

13. $y = 4e^{-6x^2}$

$y' = 4e^{-6x^2}(-12x) = -48xe^{-6x^2}$

Differential equation:

$y' = -12xy = -12x\left(4e^{-6x^2}\right) = -48xe^{-6x^2}$

Initial condition $(0, 4)$: $4e^0 = 4$

14. $y = e^{-\cos x}$

$y' = e^{-\cos x}(\sin x) = \sin x \cdot e^{-\cos x}$

Differential equation:

$y' = \sin x \cdot e^{-\cos x} = \sin x(y) = y\sin x$

Initial condition $\left(\frac{\pi}{2}, 1\right)$: $e^{-\cos(\pi/2)} = e^0 = 1$

In Exercises 15–22, the differential equation is $y^{(4)} - 16y = 0$.

15. $y = 3\cos 2x$

$y^{(4)} = 48\cos 2x$

$y^{(4)} - 16y = 48\cos 2x - 48\cos 2x = 0,$

Yes

16. $y = 3\sin 2x$

$y^{(4)} = 48\sin 2x$

$y^{(4)} - 16y = 48\sin 2x - 16(3\sin 2x) = 0$

Yes

17. $y = 3\cos x$

$y^{(4)} = 3\cos x$

$y^{(4)} - 16y = -45\cos x \neq 0,$

No

18. $y = 2\sin x$

$y^{(4)} = 2\sin x$

$y^{(4)} - 16y = 2\sin x - 16(2\sin x) \neq 0$

No

19.
$$y = e^{-2x}$$
$$y^{(4)} = 16e^{-2x}$$
$$y^{(4)} - 16y = 16e^{-2x} - 16e^{-2x} = 0,$$
Yes

20.
$$y = 5 \ln x$$
$$y^{(4)} = -\frac{30}{x^4}$$
$$y^{(4)} - 16y = -\frac{30}{x^4} - 80 \ln x \neq 0,$$
No

21. $y = \ln x + e^{2x} + Cx^4$
$$y^{(4)} = 16e^{2x} - \frac{6}{x^4} + 24C$$
$$y^{(4)} - 16y = 16e^{2x} - \frac{6}{x^4} + 24C - \ln x - e^{2x} - Cx^4 \neq 0,$$
No

22.
$$y = 3e^{2x} - 4 \sin 2x$$
$$y^{(4)} = 48e^{2x} - 64 \sin 2x$$
$$y^{(4)} - 16y = \left(48e^{2x} - 64 \sin 2x\right) - 16\left(3e^{2x} - 4 \sin 2x\right) = 0,$$
Yes

In Exercises 23–30, the differential equation is $xy' - 2y = x^3e^x$.

23. $y = x^2 + e^x,\ y' = 2x + e^x$
$$\begin{aligned} xy' - 2y &= x\left(2x + e^x\right) - 2\left(x^2 + e^x\right) \\ &= xe^x - 2e^x \\ &\neq x^3e^x \end{aligned}$$
No

24. $y = x^3 - e^{-x},\ y' = 3x^2 + e^{-x}$
$$\begin{aligned} xy' - 2y &= x\left(3x^2 + e^{-x}\right) - 2\left(x^3 - e^{-x}\right) \\ &= x^3 + xe^{-x} + 2e^{-x} \\ &\neq x^3e^x \end{aligned}$$
No

25. $y = x^2e^x,\ y' = x^2e^x + 2xe^x = e^x\left(x^2 + 2x\right)$
$$xy' - 2y = x\left(e^x\left(x^2 + 2x\right)\right) - 2\left(x^2e^x\right) = x^3e^x,$$
Yes

26. $y = x^2\left(2 + e^x\right),\ y' = x^2\left(e^x\right) + 2x\left(2 + e^x\right)$
$$\begin{aligned} xy' - 2y &= x\left[x^2e^x + 2xe^x + 4x\right] - 2\left[x^2e^x + 2x^2\right] \\ &= x^3e^x, \end{aligned}$$
Yes

27. $y = e^x - \sin x,\ y' = e^x - \cos x$
$$\begin{aligned} xy' - 2y &= x\left(e^x - \cos x\right) - 2\left(e^x - \sin x\right) \\ &= xe^x - x\cos x - 2e^x + 2\sin x \\ &\neq x^3e^x \end{aligned}$$
No

28. $y = x^2e^x + \sin x + \cos x,\ y' = x^2e^x + 2xe^x + \cos x - \sin x$
$$\begin{aligned} xy' - 2y &= x\left(x^2e^x + 2xe^x + \cos x - \sin x\right) - 2\left(x^2e^x + \sin x + \cos x\right) \\ &\neq x^3e^x \end{aligned}$$
No

29. $y = 2e^x \ln x,\ y' = 2e^x \ln x + \dfrac{2}{x}e^x$
$$\begin{aligned} xy' - 2y &= x\left(2e^x \ln x + \frac{2}{x}e^x\right) - 2\left(2e^x \ln x\right) \\ &\neq x^3e^x \end{aligned}$$
No

30. $y = x^2e^x - 5x^2,\ y' = x^2e^x + 2xe^x - 10x$
$$\begin{aligned} xy' - 2y &= x\left[x^2e^x + 2xe^x - 10x\right] - 2\left[x^2e^x - 5x^2\right] \\ &= x^3e^x, \end{aligned}$$
Yes

31. $y = Ce^{-x/2}$ passes through $(0, 3)$.

$3 = Ce^0 = C \Rightarrow C = 3$

Particular solution: $y = 3e^{-x/2}$

32. $y(x^2 + y) = C$ passes through $(0, 2)$.

$2(0 + 2) = C \Rightarrow C = 4$

Particular solution: $y(x^2 + y) = 4$

33. $y^2 = Cx^3$ passes through $(4, 4)$.

$16 = C(64) \Rightarrow C = \frac{1}{4}$

Particular solution: $y^2 = \frac{1}{4}x^3$ or $4y^2 = x^3$

34. $2x^2 - y^2 = C$ passes through $(3, 4)$.

$2(9) - 16 = C \Rightarrow C = 2$

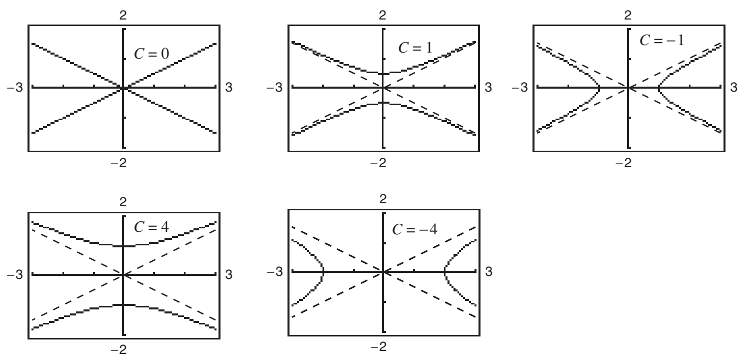
Particular solution: $2x^2 - y^2 = 2$

35. Differential equation: $4yy' - x = 0$

General solution: $4y^2 - x^2 = C$

Particular solutions: $C = 0$, Two intersecting lines

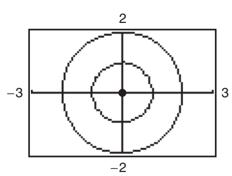
$C = \pm 1, C = \pm 4$, Hyperbolas



36. Differential equation: $yy' + x = 0$

General solution: $x^2 + y^2 = C$

Particular solutions: $C = 0$, Point $C = 1, C = 4$, Circles



37. Differential equation: $y' + 6y = 0$

General Solution: $y = Ce^{-6x}$

$y' + 6y = (-6Ce^{-6x}) + 6(Ce^{-6x}) = 0$

Initial condition $(0, 3)$: $3 = Ce^0 = C$

Particular solution: $y = 3e^{-6x}$

38. Differential equation: $3x + 2yy' = 0$

General solution: $3x^2 + 2y^2 = C$

$$6x + 4yy' = 0$$
$$2(3x + 2yy') = 0$$
$$3x + 2yy' = 0$$

Initial condition $(1, 3)$: $3(1)^2 + 2(3)^2 = 3 + 18 = 21 = C$

Particular solution: $3x^2 + 2y^2 = 21$

39. Differential equation: $y'' + 9y = 0$

General solution: $y = C_1 \sin 3x + C_2 \cos 3x$

$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x,$$
$$y'' = -9C_1 \sin 3x - 9C_2 \cos 3x$$
$$y'' + 9y = (-9C_1 \sin 3x - 9C_2 \cos 3x) + 9(C_1 \sin 3x + C_2 \cos 3x) = 0$$

Initial conditions $\left(\frac{\pi}{6}, 2\right)$ and $y' = 1$ when $x = \frac{\pi}{6}$:

$$2 = C_1 \sin\left(\frac{\pi}{2}\right) + C_2 \cos\left(\frac{\pi}{2}\right) \Rightarrow C_1 = 2$$
$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x$$
$$1 = 3C_1 \cos\left(\frac{\pi}{2}\right) - 3C_2 \sin\left(\frac{\pi}{2}\right) = -3C_2 \Rightarrow C_2 = -\frac{1}{3}$$

Particular solution: $y = 2 \sin 3x - \frac{1}{3} \cos 3x$

40. Differential equation: $xy'' + y' = 0$

General solution: $y = C_1 + C_2 \ln x$

$$y' = C_2\left(\frac{1}{x}\right), y'' = -C_2\left(\frac{1}{x^2}\right)$$
$$xy'' + y' = x\left(-C_2\frac{1}{x^2}\right) + C_2\frac{1}{x} = 0$$

Initial conditions $(2, 0)$ and $y' = \frac{1}{2}$ when $x = 2$:

$$0 = C_1 + C_2 \ln 2$$
$$y' = \frac{C_2}{x}$$
$$\frac{1}{2} = \frac{C_2}{2} \Rightarrow C_2 = 1, C_1 = -\ln 2$$

Particular solution: $y = -\ln 2 + \ln x = \ln \frac{x}{2}$

41. Differential equation: $x^2y'' - 3xy' + 3y = 0$

General solution: $y = C_1x + C_2x^3$

$$y' = C_1 + 3C_2x^2, y'' = 6C_2x$$
$$x^2y'' - 3xy' + 3y = x^2(6C_2x) - 3x(C_1 + 3C_2x^2) + 3(C_1x + C_2x^3) = 0$$

Initial conditions $(2, 0)$ and $y' = 4$ when $x = 2$:

$$0 = 2C_1 + 8C_2$$
$$y' = C_1 + 3C_2x^2$$
$$4 = C_1 + 12C_2$$
$$\left.\begin{aligned} C_1 + 4C_2 &= 0 \\ C_1 + 12C_2 &= 4 \end{aligned}\right\} C_2 = \tfrac{1}{2}, C_1 = -2$$

Particular solution: $y = -2x + \frac{1}{2}x^3$

42. Differential equation: $9y'' - 12y' + 4y = 0$

General solution: $y = e^{2x/3}(C_1 + C_2x)$

$$y' = \tfrac{2}{3}e^{2x/3}(C_1 + C_2x) + C_2e^{2x/3} = e^{2x/3}\left(\tfrac{2}{3}C_1 + C_2 + \tfrac{2}{3}C_2x\right)$$

$$y'' = \tfrac{2}{3}e^{2x/3}\left(\tfrac{2}{3}C_1 + C_2 + \tfrac{2}{3}C_2x\right) + e^{2x/3}\tfrac{2}{3}C_2 = \tfrac{2}{3}e^{2x/3}\left(\tfrac{2}{3}C_1 + 2C_2 + \tfrac{2}{3}C_2x\right)$$

$$9y'' - 12y' + 4y = 9\left(\tfrac{2}{3}e^{2x/3}\right)\left(\tfrac{2}{3}C_1 + 2C_2 + \tfrac{2}{3}C_2x\right) - 12\left(e^{2x/3}\right)\left(\tfrac{2}{3}C_1 + C_2 + \tfrac{2}{3}C_2x\right) + 4\left(e^{2x/3}\right)(C_1 + C_2x) = 0$$

Initial conditions $(0, 4)$ and $(3, 0)$:

$$0 = e^2(C_1 + 3C_2)$$

$$4 = (1)(C_1 + 0) \Rightarrow C_1 = 4$$

$$0 = e^2(4 + 3C_2) \Rightarrow C_2 = -\tfrac{4}{3}$$

Particular solution: $y = e^{2x/3}\left(4 - \tfrac{4}{3}x\right)$

43. $\dfrac{dy}{dx} = 12x^2$

$$y = \int 12x^2\,dx = 4x^3 + C$$

44. $\dfrac{dy}{dx} = 3x^8 - 2x$

$$y = \int (3x^8 - 2x)\,dx = \frac{x^9}{3} - x^2 + C$$

45. $\dfrac{dy}{dx} = \dfrac{x}{1 + x^2}$

$$y = \int \frac{x}{1 + x^2}\,dx = \frac{1}{2}\ln(1 + x^2) + C$$

$(u = 1 + x^2, du = 2x\,dx)$

46. $\dfrac{dy}{dx} = \dfrac{e^x}{4 + e^x}$

$$y = \int \frac{e^x}{4 + e^x}\,dx = \ln(4 + e^x) + C$$

47. $\dfrac{dy}{dx} = \sin 2x$

$$y = \int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C$$

$(u = 2x, du = 2\,dx)$

48. $\dfrac{dy}{dx} = \tan^2 x = \sec^2 x - 1$

$$y = \int (\sec^2 x - 1)\,dx = \tan x - x + C$$

49. $\dfrac{dy}{dx} = x\sqrt{x - 6}$

Let $u = \sqrt{x - 6}$, then $x = u^2 + 6$ and $dx = 2u\,du$.

$$\begin{aligned} y = \int x\sqrt{x - 6}\,dx &= \int (u^2 + 6)(u)(2u)\,du \\ &= 2\int (u^4 + 6u^2)\,du \\ &= 2\left(\frac{u^5}{5} + 2u^3\right) + C \\ &= \frac{2}{5}(x - 6)^{5/2} + 4(x - 6)^{3/2} + C \\ &= \frac{2}{5}(x - 6)^{3/2}(x - 6 + 10) + C \\ &= \frac{2}{5}(x - 6)^{3/2}(x + 4) + C \end{aligned}$$

50. $\dfrac{dy}{dx} = 2x\sqrt{4x^2 + 1}$

$$\begin{aligned} y &= \int 2x\sqrt{4x^2 + 1}\,dx \\ &= \frac{1}{4}\int \sqrt{4x^2 + 1}\,(8x)\,dx \\ &= \frac{1}{4}\frac{(4x^2 + 1)^{3/2}}{(3/2)} + C \\ &= \frac{1}{6}(4x^2 + 1)^{3/2} + C \end{aligned}$$

51. $\dfrac{dy}{dx} = xe^{x^2}$

$$y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$$

$(u = x^2, du = 2x\,dx)$

52. $\dfrac{dy}{dx} = 5(\sin x)e^{\cos x}, u = \cos x, du = -\sin x\, dx$

$y = -\int 5e^{\cos x}(-\sin x\, dx)$

$= -5e^{\cos x} + C$

53.

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	-4	Undef.	0	1	$\frac{4}{3}$	2

54.

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	6	2	4	2	2	0

55.

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	$-2\sqrt{2}$	-2	0	0	$-2\sqrt{2}$	-8

56.

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	$\sqrt{3}$	0	$-\sqrt{3}$	$-\sqrt{3}$	0	$\sqrt{3}$

57. $\dfrac{dy}{dx} = \sin 2x$

For $x = 0, \dfrac{dy}{dx} = 0$. Matches (b).

58. $\dfrac{dy}{dx} = \dfrac{1}{2}\cos x$

For $x = 0, \dfrac{dy}{dx} = \dfrac{1}{2}$. Matches (c).

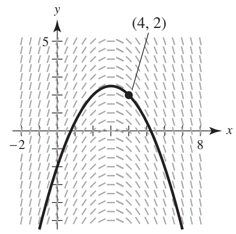
59. $\dfrac{dy}{dx} = e^{-2x}$

As $x \to \infty, \dfrac{dy}{dx} \to 0$. Matches (d).

60. $\dfrac{dy}{dx} = \dfrac{x}{x^2 + 1}$

For $x = 0, \dfrac{dy}{dx} = 0$ and for $x = 3, \dfrac{dy}{dx} = \dfrac{3}{10} > 0$.
Matches (a).

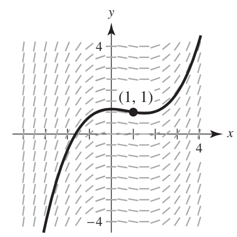
61. (a), (b)



(c) As $x \to \infty, y \to -\infty$

As $x \to -\infty, y \to -\infty$

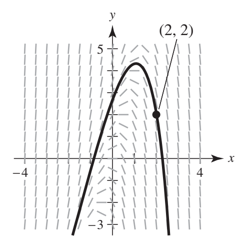
62. (a), (b)



(c) As $x \to \infty, y \to \infty$

As $x \to -\infty, y \to -\infty$

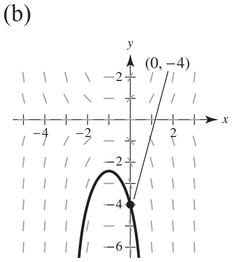
63. (a), (b)



(c) As $x \to \infty, y \to -\infty$

As $x \to -\infty, y \to -\infty$

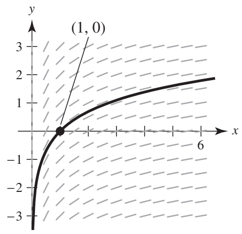
64. (a), (b)



(c) As $x \to \infty, y \to -\infty$

As $x \to -\infty, y \to -\infty$

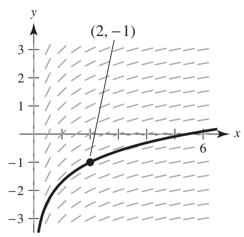
65. (a) $y' = \dfrac{1}{x}, (1, 0)$



As $x \to \infty, y \to \infty$

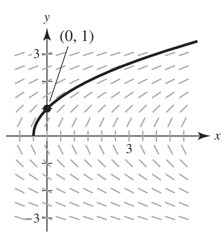
[Note: The solution is $y = \ln x$.]

(b) $y' = \dfrac{1}{x}, (2, -1)$



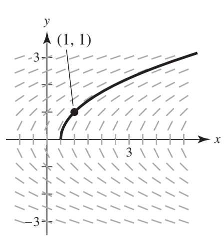
As $x \to \infty, y \to \infty$

66. (a) $y' = \dfrac{1}{y}, (0, 1)$



As $x \to \infty, y \to \infty$

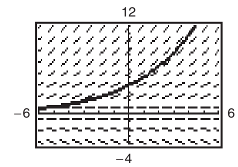
(b) $y' = \dfrac{1}{y}, (1, 1)$



As $x \to \infty, y \to \infty$

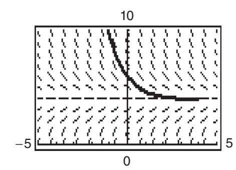
67. $\dfrac{dy}{dx} = 0.25y, y(0) = 4$

(a), (b)



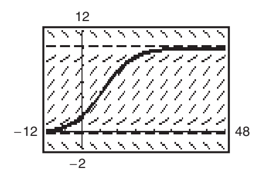
68. $\dfrac{dy}{dx} = 4 - y, y(0) = 6$

(a), (b)



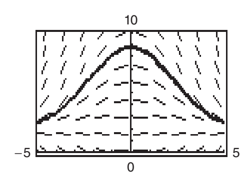
69. $\dfrac{dy}{dx} = 0.02y(10 - y), y(0) = 2$

(a), (b)



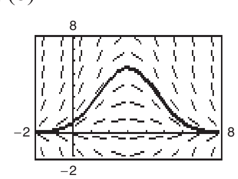
70. $\dfrac{dy}{dx} = 0.2x(2 - y), y(0) = 9$

(a), (b)



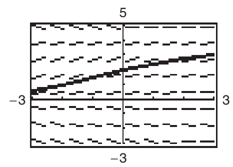
71. $\dfrac{dy}{dx} = 0.4y(3 - x), y(0) = 1$

(a), (b)



72. $\dfrac{dy}{dx} = \dfrac{1}{2}e^{-x/8}\sin\dfrac{\pi y}{4},\ y(0) = 2$

(a), (b)



73. $y' = x + y, \quad y(0) = 2, \quad n = 10, \quad h = 0.1$

$y_1 = y_0 + hF(x_0, y_0) = 2 + (0.1)(0 + 2) = 2.2$

$y_2 = y_1 + hF(x_1, y_1) = 2.2 + (0.1)(0.1 + 2.2) = 2.43$, etc.

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	2	2.2	2.43	2.693	2.992	3.332	3.715	4.146	4.631	5.174	5.781

74. $y' = x + y, \quad y(0) = 2, \quad n = 20, \quad h = 0.05$

$y_1 = y_0 + hF(x_0, y_0) = 2 + (0.05)(0 + 2) = 2.1$

$y_2 = y_1 + hF(x_1, y_1) = 2.1 + (0.05)(0.05 + 2.1) = 2.2075$, etc.

The table shows the values for $n = 0, 2, 4, \ldots, 20$.

n	0	2	4	6	8	10	12	14	16	18	20
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	2	2.208	2.447	2.720	3.032	3.387	3.788	4.240	4.749	5.320	5.960

75. $y' = 3x - 2y, \quad y(0) = 3, \quad n = 10, \quad h = 0.05$

$y_1 = y_0 + hF(x_0, y_0) = 3 + (0.05)(3(0) - 2(3)) = 2.7$

$y_2 = y_1 + hF(x_1, y_1) = 2.7 + (0.05)(3(0.05) + 2(2.7)) = 2.4375$, etc.

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
y_n	3	2.7	2.438	2.209	2.010	1.839	1.693	1.569	1.464	1.378	1.308

76. $y' = 0.5x(3 - y), \quad y(0) = 1, \quad n = 5, \quad h = 0.4$

$y_1 = y_0 + hF(x_0, y_0) = 1 + (0.4)(0.5(0)(3 - 1)) = 1$

$y_2 = y_1 + hF(x_1, y_1) = 1 + (0.4)(0.5(0.4)(3 - 1)) = 1.16$, etc.

n	0	1	2	3	4	5
x_n	0	0.4	0.8	1.2	1.6	2.0
y_n	1	1	1.16	1.454	1.825	2.201

77. $y' = e^{xy}, \quad y(0) = 1, \quad n = 10, \quad h = 0.1$

$y_1 = y_0 + hF(x_0, y_0) = 1 + (0.1)e^{0(1)} = 1.1$

$y_2 = y_1 + hF(x_1, y_1) = 1.1 + (0.1)e^{(0.1)(1.1)} \approx 1.2116$, etc.

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	1	1.1	1.212	1.339	1.488	1.670	1.900	2.213	2.684	3.540	5.958

78. $y' = \cos x + \sin y, y(0) = 5, n = 10, h = 0.1$

$y_1 = y_0 + hF(x_0, y_0) = 5 + (0.1)(\cos 0 + \sin 5) \approx 5.0041$

$y_2 = y_1 + hF(x_1, y_1) = 5.0041 + (0.1)(\cos(0.1) + \sin(5.0041)) \approx 5.0078$, etc.

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	5	5.004	5.008	5.010	5.010	5.007	4.999	4.985	4.965	4.938	4.903

79. $\dfrac{dy}{dx} = y, y = 3e^x, (0, 3)$

x	0	0.2	0.4	0.6	0.8	1
$y(x)$ (exact)	3	3.6642	4.4755	5.4664	6.6766	8.1548
$y(x)\ (h = 0.2)$	3	3.6000	4.3200	5.1840	6.2208	7.4650
$y(x)\ (h = 0.1)$	3	3.6300	4.3923	5.3147	6.4308	7.7812

80. $\dfrac{dy}{dx} = \dfrac{2x}{y}, y = \sqrt{2x^2 + 4}, \quad (0, 2)$

x	0	0.2	0.4	0.6	0.8	1
$y(x)$ (exact)	2	2.0199	2.0785	2.1726	2.2978	2.4495
$y(x)\ (h = 0.2)$	2	2.000	2.0400	2.1184	2.2317	2.3751
$y(x)\ (h = 0.1)$	2	2.0100	2.0595	2.1460	2.2655	2.4131

81. $\dfrac{dy}{dx} = y + \cos x, y = \dfrac{1}{2}(\sin x - \cos x + e^x), \quad (0, 0)$

x	0	0.2	0.4	0.6	0.8	1
$y(x)$ (exact)	0	0.2200	0.4801	0.7807	1.1231	1.5097
$y(x)\ (h = 0.2)$	0	0.2000	0.4360	0.7074	1.0140	1.3561
$y(x)\ (h = 0.1)$	0	0.2095	0.4568	0.7418	1.0649	1.4273

82. As h increases (from 0.1 to 0.2), the error increases.

83. $\frac{dy}{dt} = -\frac{1}{2}(y - 72), \quad (0, 140), h = 0.1$

(a)

t	0	1	2	3
Euler	140	112.7	96.4	86.6

(b) $y = 72 + 68e^{-t/2}$ exact

t	0	1	2	3
Exact	140	113.24	97.016	87.173

(c) $\frac{dy}{dt} = -\frac{1}{2}(y - 72), \quad (0, 140), h = 0.05$

t	0	1	2	3
Euler	140	112.98	96.7	86.9

The approximations are better using $h = 0.05$.

84. When $x = 0$, $y' = 0$, therefore (d) is not possible.

When $x, y > 0$, $y' < 0$ (decreasing function) therefore (c) is the equation.

85. Euler's Method produces an exact solution to an initial value problem when the exact solution is a line.

86. $y = Ce^{kx}$

$\frac{dy}{dx} = Cke^{kx}$

Because $dy/dx = 0.07y$, you have $Cke^{kx} = 0.07Ce^{kx}$.

So, $k = 0.07$.

C cannot be determined.

87. False. Consider Example 2. $y = x^3$ is a solution to $xy' - 3y = 0$, but $y = x^3 + 1$ is not a solution.

88. False. A slope field shows the slopes of all solutions.

89. $\frac{dy}{dx} = -2y, y(0) = 4, y = 4e^{-2x}$

(a)

x	0	0.2	0.4	0.6	0.8	1
y	4	2.6813	1.7973	1.2048	0.8076	0.5413
y_1	4	2.5600	1.6384	1.0486	0.6711	0.4295
y_2	4	2.4000	1.4400	0.8640	0.5184	0.3110
e_1	0	0.1213	0.1589	0.1562	0.1365	0.1118
e_2	0	0.2813	0.3573	0.3408	0.2892	0.2303
r		0.4312	0.4447	0.4583	0.4720	0.4855

(b) If h is halved, then the error is approximately halved $(r \approx 0.5)$.

(c) When $h = 0.05$, the errors will again be approximately halved.

90. $\frac{dy}{dx} = x - y, y(0) = 1, y = x - 1 + 2e^{-x}$

(a)

x	0	0.2	0.4	0.6	0.8	1
y	1	0.8375	0.7406	0.6976	0.6987	0.7358
y_1	1	0.8200	0.7122	0.6629	0.6609	0.6974
y_2	1	0.8000	0.6800	0.6240	0.6192	0.6554
e_1	0	0.0175	0.0284	0.0347	0.0378	0.0384
e_2	0	0.0375	0.0606	0.0736	0.0795	0.0804
r		0.47	0.47	0.47	0.48	0.48

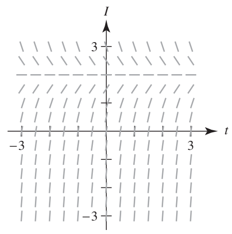
(b) If h is halved, then the error is halved $(r \approx 0.5)$.

(c) When $h = 0.05$, the error will again be approximately halved.

91. (a) $L\frac{dI}{dt} + RI = E(t)$

$$4\frac{dI}{dt} + 12I = 24$$

$$\frac{dI}{dt} = \frac{1}{4}(24 - 12I) = 6 - 3I$$



(b) As $t \to \infty$, $I \to 2$. That is, $\lim_{t\to\infty} I(t) = 2$. In fact, $I = 2$ is a solution to the differential equation.

92. False. The slope field could represent many different differential equations, such as $y' = 2x + 4y$.

93.
$$y = A\sin\omega t$$
$$y' = A\omega\cos\omega t$$
$$y'' = -A\omega^2\sin\omega t$$
$$y'' + 16y = 0$$
$$-A\omega^2\sin\omega t + 16A\sin\omega t = 0$$
$$A\sin\omega t\left[16 - \omega^2\right] = 0$$

If $A \neq 0$, then $\omega = \pm 4$

94.
$$y = e^{kt}$$
$$y' = ke^{kt}$$
$$y'' = k^2e^{kt}$$
$$y'' - 16y = 0$$
$$k^2e^{kt} - 16e^{kt} = 0$$
$$k^2 - 16 = 0 \qquad \left(\text{because } e^{kt} \neq 0\right)$$
$$k = \pm 4$$

95.
$$f(x) + f''(x) = -xg(x)f'(x), \qquad g(x) \geq 0$$
$$2f(x)f'(x) + 2f'(x)f''(x) = -2xg(x)\left[f'(x)\right]^2$$
$$\frac{d}{dx}\left[f(x)^2 + f'(x)^2\right] = -2x\,g(x)\left[f'(x)\right]^2$$

For $x < 0, -2x\,g(x)\left[f'(x)\right]^2 \geq 0$

For $x > 0, -2x\,g(x)\left[f'(x)\right]^2 \leq 0$

So, $f(x)^2 + f'(x)^2$ is increasing for $x < 0$ and decreasing for $x > 0$.

$f(x)^2 + f'(x)^2$ has a maximum at $x = 0$. So, it is bounded by its value at $x = 0$, $f(0)^2 + f'(0)^2$. So, f (and f') is bounded.

96. Let the vertical line $x = k$ cut the graph of the solution $y = f(x)$ at (k, t). The tangent line at (k, t) is

$$y - t = f'(k)(x - k)$$

Because $y' + p(x)y = q(x)$, you have $y - t = \left[q(k) - p(k)t\right](x - k)$

For any value of t, this line passes through the point $\left(k + \frac{1}{p(k)}, \frac{q(k)}{p(k)}\right)$.

To see this, note that

$$\frac{q(k)}{p(k)} - t \stackrel{?}{=} \left[q(k) - p(k)t\right]\left(k + \frac{1}{p(k)} - k\right)$$
$$\stackrel{?}{=} q(k)k - p(k)tk + \frac{q(k)}{p(k)} - t - kq(k) + p(k)kt = \frac{q(k)}{p(k)} - t.$$

Section 6.2 Growth and Decay

1. In the model $y = Ce^{kt}$, C represents the initial value of y (when $t = 0$). k is the proportionality constant.

2. When $k > 0$ (and $C > 0$), the function $y = Ce^{kt}$ is increasing for $t > 0$. On the other hand, if $k < 0$, then $y = Ce^{kt}$ is decreasing.

3. $\dfrac{dy}{dx} = x + 3$

$$y = \int (x + 3)\,dx = \frac{x^2}{2} + 3x + C$$

4. $\dfrac{dy}{dx} = 5 - 8x$

$$y = \int (5 - 8x)\,dx = 5x - 4x^2 + C$$

5.
$$\begin{aligned}\frac{dy}{dx} &= y + 3\\ \frac{dy}{y+3} &= dx\\ \int \frac{1}{y+3}\,dy &= \int dx\\ \ln|y+3| &= x + C_1\\ y + 3 &= e^{x+C_1} = Ce^x\\ y &= Ce^x - 3\end{aligned}$$

6.
$$\begin{aligned}\frac{dy}{dx} &= 6 - y\\ \frac{dy}{6-y} &= dx\\ \int \frac{-1}{6-y}\,dy &= \int -dx\\ \ln|6-y|dy &= -x + C_1\\ 6 - y &= e^{-x+C_1} = Ce^{-x}\\ y &= 6 - Ce^{-x}\end{aligned}$$

7.
$$\begin{aligned}y' &= \frac{5x}{y}\\ yy' &= 5x\\ \int yy'\,dx &= \int 5x\,dx\\ \int y\,dy &= \int 5x\,dx\\ \frac{1}{2}y^2 &= \frac{5}{2}x^2 + C_1\\ y^2 - 5x^2 &= C\end{aligned}$$

8.
$$\begin{aligned}y' &= -\frac{\sqrt{x}}{4y}\\ 4y\,y' &= -\sqrt{x}\\ \int 4y\,dy &= \int -\sqrt{x}\,dx\\ 2y^2 &= -\frac{2}{3}x^{3/2} + C_1\\ 6y^2 + 2x^{3/2} &= C\end{aligned}$$

9.
$$\begin{aligned}y' &= \sqrt{x}y\\ \frac{y'}{y} &= \sqrt{x}\\ \int \frac{y'}{y}\,dx &= \int \sqrt{x}\,dx\\ \int \frac{dy}{y} &= \int \sqrt{x}\,dx\\ \ln|y| &= \frac{2}{3}x^{3/2} + C_1\\ y &= e^{(2/3)x^{3/2}+C_1}\\ &= e^{C_1}e^{(2/3)x^{3/2}}\\ &= Ce^{(2x^{3/2})/3}\end{aligned}$$

10.
$$\begin{aligned}y' &= x(1 + y)\\ \frac{y'}{1+y} &= x\\ \int \frac{y'}{1+y}\,dx &= \int x\,dx\\ \int \frac{dy}{1+y} &= \int x\,dx\\ \ln(1+y) &= \frac{x^2}{2} + C_1\\ 1 + y &= e^{(x^2/2)+C_1}\\ y &= e^{C_1}e^{x^2/2} - 1\\ &= Ce^{x^2/2} - 1\end{aligned}$$

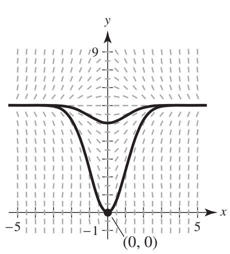
11.
$$\begin{aligned}(1 + x^2)y' - 2xy &= 0\\ y' &= \frac{2xy}{1+x^2}\\ \frac{y'}{y} &= \frac{2x}{1+x^2}\\ \int \frac{y'}{y}\,dx &= \int \frac{2x}{1+x^2}\,dx\\ \int \frac{dy}{y} &= \int \frac{2x}{1+x^2}\,dx\\ \ln|y| &= \ln(1+x^2) + C_1\\ \ln|y| &= \ln(1+x^2) + \ln C\\ \ln|y| &= \ln\left[C(1+x^2)\right]\\ y &= C(1+x^2)\end{aligned}$$

12.
$$xy + y' = 100x$$
$$y' = 100x + xy = x(100 - y)$$
$$\frac{y'}{100 - y} = x$$
$$\int \frac{y'}{100 - y}\,dx = \int x\,dx$$
$$\int \frac{1}{100 - y}\,dy = \int x\,dx$$
$$-\ln(100 - y) = \frac{x^2}{2} + C_1$$
$$\ln(100 - y) = -\frac{x^2}{2} - C_1$$
$$100 - y = e^{-(x^2/2) - C_1}$$
$$-y = e^{-C_1}e^{-x^2/2} - 100$$
$$y = 100 - Ce^{-x^2/2}$$

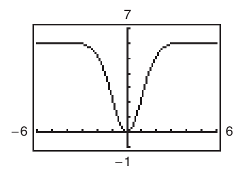
13.
$$\frac{dQ}{dt} = \frac{k}{t^2}$$
$$\int \frac{dQ}{dt}\,dt = \int \frac{k}{t^2}\,dt$$
$$\int dQ = -\frac{k}{t} + C$$
$$Q = -\frac{k}{t} + C$$

14.
$$\frac{dP}{dt} = k(25 - t)$$
$$\int \frac{dP}{dt}\,dt = \int k(25 - t)\,dt$$
$$\int dP = -\frac{k}{2}(25 - t)^2 + C$$
$$P = -\frac{k}{2}(25 - t)^2 + C$$

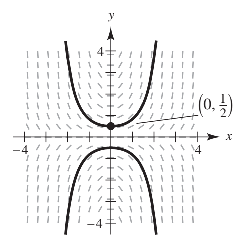
15. (a)



(b)
$$\frac{dy}{dx} = x(6 - y), \quad (0, 0)$$
$$\frac{dy}{y - 6} = -x\,dx$$
$$\ln|y - 6| = \frac{-x^2}{2} + C$$
$$y - 6 = e^{-x^2/2 + C} = C_1e^{-x^2/2}$$
$$y = 6 + C_1e^{-x^2/2}$$
$$(0, 0):\ 0 = 6 + C_1 \Rightarrow C_1 = -6$$
$$y = 6 - 6e^{-x^2/2}$$

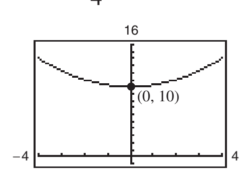


16. (a)



(b)
$$\frac{dy}{dx} = xy, \quad \left(0, \frac{1}{2}\right)$$
$$\frac{dy}{y} = x\,dx$$
$$\ln|y| = \frac{x^2}{2} + C$$
$$y = e^{x^2/2 + C} = C_1e^{x^2/2}$$
$$\left(0, \frac{1}{2}\right):\ \frac{1}{2} = C_1e^0 \Rightarrow C_1 = \frac{1}{2}$$
$$y = \frac{1}{2}e^{x^2/2}$$

17.
$$\frac{dy}{dt} = \frac{1}{2}t, \quad (0, 10)$$
$$\int dy = \int \frac{1}{2}t\,dt$$
$$y = \frac{1}{4}t^2 + C$$
$$10 = \frac{1}{4}(0)^2 + C \Rightarrow C = 10$$
$$y = \frac{1}{4}t^2 + 10$$



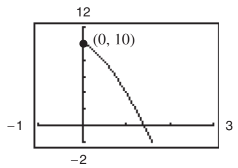
18. $\dfrac{dy}{dt} = -9\sqrt{t}, \quad (0, 10)$

$$\int dy = \int -9\sqrt{t}\, dt$$

$$y = -6t^{3/2} + C$$

$$10 = 0 + C \Rightarrow C = 10$$

$$y = -6t^{3/2} + 10$$



19. $\dfrac{dy}{dt} = -\dfrac{1}{2}y, \quad (0, 10)$

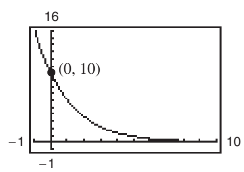
$$\int \frac{dy}{y} = \int -\frac{1}{2}\, dt$$

$$\ln|y| = -\frac{1}{2}t + C_1$$

$$y = e^{-(t/2)+C_1} = e^{C_1}e^{-t/2} = Ce^{-t/2}$$

$$10 = Ce^0 \Rightarrow C = 10$$

$$y = 10e^{-t/2}$$



20. $\dfrac{dy}{dt} = \dfrac{3}{4}y, \quad (0, 10)$

$$\int \frac{dy}{y} = \int \frac{3}{4}\, dt$$

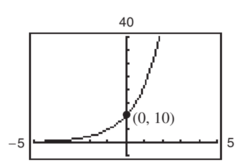
$$\ln y = \frac{3}{4}t + C_1$$

$$y = e^{(3/4)t + C_1}$$

$$= e^{C_1}e^{(3/4)t} = Ce^{3t/4}$$

$$10 = Ce^0 \Rightarrow C = 10$$

$$y = 10e^{3t/4}$$



21. $\dfrac{dN}{dt} = kN$

$$N = Ce^{kt} \qquad \text{(Theorem 6.1)}$$

$$(0, 250):\ C = 250$$

$$(1, 400):\ 400 = 250e^k \Rightarrow k = \ln\frac{400}{250} = \ln\frac{8}{5}$$

$$N = 250e^{\ln(8/5)t} \approx 250e^{0.4700t}$$

When $t = 4$, $N = 250e^{4\ln(8/5)} = 250e^{\ln(8/5)^4}$

$$= 250\left(\frac{8}{5}\right)^4 = \frac{8192}{5}.$$

22. $\dfrac{dP}{dt} = kP$

$$P = Ce^{kt} \qquad \text{(Theorem 6.1)}$$

$$(0, 5000):\ C = 5000$$

$$(1, 4750):\ 4750 = 5000e^k \Rightarrow k = \ln\left(\frac{19}{20}\right)$$

$$P = 5000e^{\ln(19/20)t} \approx 5000e^{-0.0513t}$$

When $t = 5$, $P = 5000e^{\ln(19/20)(5)}$

$$= 5000\left(\frac{19}{20}\right)^5 \approx 3868.905.$$

23. $y = Ce^{kt}, (0, 2), (4, 3)$

$$C = 2$$

$$y = 2e^{kt}$$

$$3 = 2e^{4k}$$

$$k = \frac{\ln(3/2)}{4}$$

$$y = 2e^{[(1/4)\ln(3/2)]t} \approx 2e^{0.1014t}$$

24. $y = Ce^{kt}, \quad (0, 4), \left(5, \dfrac{1}{2}\right)$

$$C = 4$$

$$y = 4e^{kt}$$

$$\frac{1}{2} = 4e^{5k}$$

$$k = \frac{\ln(1/8)}{5} \approx -0.4159$$

$$y = 4e^{-0.4159t}$$

25. $y = Ce^{kt}, \quad (1, 5), (5, 2)$

$$5 = Ce^{k} \Rightarrow 10 = 2Ce^{k}$$
$$2 = Ce^{5k} \Rightarrow 10 = 5Ce^{k}$$
$$2Ce^{k} = 5Ce^{5k}$$
$$2e^{k} = 5e^{5k}$$
$$\frac{2}{5} = e^{4k}$$
$$k = \frac{1}{4}\ln\left(\frac{2}{5}\right) = \ln\left(\frac{2}{5}\right)^{1/4}$$
$$C = 5e^{-k} = 5e^{-1/4\ln(2/5)} = 5\left(\frac{2}{5}\right)^{-1/4} = 5\left(\frac{5}{2}\right)^{1/4}$$
$$y = 5\left(\frac{5}{2}\right)^{1/4} e^{[1/4\ln(2/5)]t} \approx 6.2872\,e^{-0.2291t}$$

26. $y = Ce^{kt}, \quad \left(3, \frac{1}{2}\right), (4, 5)$

$$\frac{1}{2} = Ce^{3k} \Rightarrow 1 = 2Ce^{3k}$$
$$5 = Ce^{4k} \Rightarrow 1 = \frac{1}{5}Ce^{4k}$$
$$2Ce^{3k} = \frac{1}{5}Ce^{4k}$$
$$10e^{3k} = e^{4k}$$
$$10 = e^{k}$$
$$k = \ln 10 \approx 2.3026$$
$$y = Ce^{2.3026t}$$
$$5 = Ce^{2.3026(4)}$$
$$C \approx 0.0005$$
$$y = 0.0005e^{2.3026t}$$

27. $\dfrac{dy}{dx} = \dfrac{1}{2}xy$

$\dfrac{dy}{dx} > 0$ when $xy > 0$. Quadrants I and III.

28. $\dfrac{dy}{dx} = \dfrac{1}{2}x^2y$

$\dfrac{dy}{dx} > 0$ when $y > 0$. Quadrants I and II.

29. Because the initial quantity is 20 grams,

$$y = 20e^{kt}.$$

Because the half-life is 1599 years,

$$10 = 20e^{k(1599)}$$
$$k = \tfrac{1}{1599}\ln\left(\tfrac{1}{2}\right).$$

So, $y = 20e^{[\ln(1/2)/1599]t}$.

When $t = 1000$, $y = 20e^{[\ln(1/2)/1599](1000)} \approx 12.96$ g.

When $t = 10{,}000$, $y \approx 0.26$ g.

30. Because the half-life is 1599 years,

$$\tfrac{1}{2} = 1e^{k(1599)}$$
$$k = \tfrac{1}{1599}\ln\left(\tfrac{1}{2}\right).$$

Because there are 1.5 g after 1000 years,

$$1.5 = Ce^{[\ln(1/2)/1599](1000)}$$
$$C \approx 2.314.$$

So, the initial quantity is approximately 2.314 g.

When $t = 10{,}000$, $y = 2.314e^{[\ln(1/2)/1599](10{,}000)} \approx 0.03$ g.

31. Because the half-life is 1599 years,

$$\tfrac{1}{2} = 1e^{k(1599)}$$
$$k = \tfrac{1}{1599}\ln\left(\tfrac{1}{2}\right).$$

Because there are 0.1 gram after 10,000 years,

$$0.1 = Ce^{[\ln(1/2)/1599](10{,}000)}$$
$$C \approx 7.63.$$

So, the initial quantity is approximately 7.63 g.

When $t = 1000$, $y = 7.63e^{[\ln(1/2)/1599](1000)} \approx 4.95$ g.

32. Because the half-life is 5715 years,

$$\tfrac{1}{2} = 1e^{k(5715)}$$
$$k = \tfrac{1}{5715}\ln\left(\tfrac{1}{2}\right).$$

Because there are 3 grams after 10,000 years,

$$3 = Ce^{[\ln(1/2)/5715](10{,}000)}$$
$$C \approx 10.089.$$

So, the initial quantity is approximately 10.09 g.

When $t = 1000$, $y = 10.09e^{[\ln(1/2)/5715](1000)} \approx 8.94$ g.

33. Because the initial quantity is 5 grams, $C = 5$.

Because the half-life is 5715 years,

$$2.5 = 5e^{k(5715)}$$
$$k = \frac{1}{5715}\ln\left(\frac{1}{2}\right).$$

When $t = 1000$ years, $y = 5e^{[\ln(1/2)/5715](1000)} \approx 4.43$ g.

When $t = 10{,}000$ years, $y = 5e^{[\ln(1/2)/5715](10{,}000)}$
≈ 1.49 g.

34. Because the half-life is 5715 years,

$$\frac{1}{2} = 1e^{k(5715)}$$
$$k = \frac{1}{5715}\ln\left(\frac{1}{2}\right).$$

Because there are 1.6 grams when $t = 1000$ years,

$$1.6 = Ce^{[\ln(1/2)/5715](1000)}$$
$$C \approx 1.806.$$

So, the initial quantity is approximately 1.806 g.

When $t = 10{,}000$, $y = 1.806e^{[\ln(1/2)/5715](10{,}000)}$
≈ 0.54 g.

35. Because the half-life is 24,100 years,

$$\frac{1}{2} = 1e^{k(24{,}100)}$$
$$k = \frac{1}{24{,}100}\ln\left(\frac{1}{2}\right).$$

Because there are 2.1 grams after 1000 years,

$$2.1 = Ce^{[\ln(1/2)/24{,}100](1000)}$$
$$C \approx 2.161.$$

So, the initial quantity is approximately 2.161 g.

When $t = 10{,}000$, $y = 2.161e^{[\ln(1/2)/24{,}100](10{,}000)}$
≈ 1.62 g.

36. Because the half-life is 24,100 years,

$$\frac{1}{2} = 1e^{k(24{,}100)}$$
$$k = \frac{1}{24{,}100}\ln\left(\frac{1}{2}\right).$$

Because there are 0.4 grams after 10,000 years,

$$0.4 = Ce^{[\ln(1/2)/24{,}100](10{,}000)}$$
$$C \approx 0.533.$$

So, the initial quantity is approximately 0.533 g.

When $t = 1000$, $y = 0.533e^{[\ln(1/2)/24{,}100](1000)}$
≈ 0.52 g.

37.

$$y = Ce^{kt}$$
$$\frac{1}{2}C = Ce^{k(1599)}$$
$$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$$

When $t = 100$, $y = Ce^{[\ln(1/2)/1599](100)}$
$\approx 0.9576C$

Therefore, 95.76% remains after 100 years.

38.

$$y = Ce^{kt}$$
$$\frac{1}{2}C = Ce^{k(5715)}$$
$$k = \frac{1}{5715}\ln\left(\frac{1}{2}\right)$$
$$0.15C = Ce^{[\ln(1/2)/5715]t}$$
$$\ln(0.15) = \frac{\ln\left(\frac{1}{2}\right)t}{5715}$$
$$t \approx 15{,}641.8 \text{ yr}$$

39. Because $A = 1000e^{0.12t}$, the time to double is given by

$$2000 = 1000e^{0.12t}$$
$$2 = e^{0.12t}$$
$$t = \frac{\ln 2}{0.12} \approx 5.78 \text{ years}$$

The amount after 10 years is

$$A = 1000e^{0.12(10)} \approx \$3320.12.$$

40. Because $A = 28{,}000e^{0.025t}$, the time to double is given by

$$56{,}000 = 28{,}000e^{0.025t}$$
$$2 = e^{0.025t}$$
$$t = \frac{\ln 2}{0.025} \approx 27.73 \text{ years.}$$

The amount after 10 years is

$$A = 28{,}000e^{0.025(10)} \approx \$35{,}952.71.$$

41. Because $A = 150e^{rt}$ and $A = 300$ when $t = 15$, you have

$$300 = 150e^{r(15)}$$
$$2 = e^{15r}$$
$$r = \frac{\ln 2}{15} \approx 0.0462 \text{ or } 4.62\%.$$

The amount after 10 years is

$$A = 150e^{0.0462(10)} \approx \$238.09.$$

42. Because $A = 31{,}000e^{rt}$ and $A = 62{,}000$ when $t = 8$, you have

$$62{,}000 = 31{,}000e^{r(8)}$$
$$2 = e^{8r}$$
$$r = \frac{\ln 2}{8} \approx 0.0866 = 8.66\%.$$

The amount after 10 years is

$$A = 31{,}000e^{0.0866(10)} \approx \$73{,}698.85.$$

43. Because $A = 900e^{rt}$ and $A = 1845.25$ when $t = 10$, you have

$$1845.25 = 900e^{r(10)}$$
$$2.0503 \approx e^{10r}$$
$$\ln(2.0503) = 10r$$
$$r \approx 0.0718 \text{ or } 7.18\%.$$

The time to double is given by

$$1800 = 900e^{0.0718t}$$
$$2 = e^{0.0718t}$$
$$t = \frac{\ln 2}{0.0718} \approx 9.65 \text{ years.}$$

44. Because $A = 6000e^{rt}$ and $A = 6840$ when $t = 10$, you have

$$6840 = 6000e^{r(10)}$$
$$1.14 = e^{10r}$$
$$\ln(1.14) = 10r$$
$$r = \frac{\ln(1.14)}{10} \approx 0.0131 = 1.31\%.$$

The time to double is given by

$$12{,}000 = 6000e^{0.0131t}$$
$$2 = e^{0.0131t}$$
$$t = \frac{\ln 2}{0.0131} \approx 52.91 \text{ years.}$$

45. $1{,}000{,}000 = P\left(1 + \frac{0.075}{12}\right)^{(12)(20)}$

$$P = 1{,}000{,}000\left(1 + \frac{0.075}{12}\right)^{-240}$$
$$\approx \$224{,}174.18$$

46. $1{,}000{,}000 = P\left(1 + \frac{0.06}{12}\right)^{(12)(40)}$

$$P = 1{,}000{,}000(1.005)^{-480} \approx \$91{,}262.08$$

47. $1{,}000{,}000 = P\left(1 + \frac{0.08}{12}\right)^{(12)(35)}$

$$P = 1{,}000{,}000\left(1 + \frac{0.08}{12}\right)^{-420}$$
$$= \$61{,}377.75$$

48. $1{,}000{,}000 = P\left(1 + \frac{0.09}{12}\right)^{(12)(25)}$

$$P = 1{,}000{,}000\left(1 + \frac{0.09}{12}\right)^{-300}$$
$$\approx \$106{,}287.83$$

49. (a) $2000 = 1000(1 + 0.07)^t$

$$2 = 1.07^t$$
$$\ln 2 = t \ln 1.07$$
$$t = \frac{\ln 2}{\ln 1.07} \approx 10.24 \text{ yr}$$

(b) $2000 = 1000\left(1 + \frac{0.07}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.007}{12}\right)^{12t}$$
$$\ln 2 = 12t \ln\left(1 + \frac{0.07}{12}\right)$$
$$t = \frac{\ln 2}{12 \ln(1 + (0.07/12))} \approx 9.93 \text{ yr}$$

(c) $2000 = 1000\left(1 + \frac{0.07}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.07}{365}\right)^{365t}$$
$$\ln 2 = 365t \ln\left(1 + \frac{0.07}{365}\right)$$
$$t = \frac{\ln 2}{365 \ln(1 + (0.07/365))} \approx 9.90 \text{ yr}$$

(d) $2000 = 1000e^{(0.07)t}$

$$2 = e^{0.07t}$$
$$\ln 2 = 0.07t$$
$$t = \frac{\ln 2}{0.07} \approx 9.90 \text{ yr}$$

50. (a) $2000 = 1000(1 + 0.055)^t$

$2 = 1.055^t$

$\ln 2 = t \ln 1.055$

$t = \dfrac{\ln 2}{\ln 1.055} \approx 12.95$ yr

(b) $2000 = 1000\left(1 + \dfrac{0.055}{12}\right)^{12t}$

$2 = \left(1 + \dfrac{0.055}{12}\right)^{12t}$

$\ln 2 = 12t \ln\left(1 + \dfrac{0.055}{12}\right)$

$t = \dfrac{1}{12}\dfrac{\ln 2}{\ln\left(1 + \dfrac{0.055}{12}\right)} \approx 12.63$ yr

(c) $2000 = 1000\left(1 + \dfrac{0.055}{365}\right)^{365t}$

$2 = \left(1 + \dfrac{0.055}{365}\right)^{365t}$

$\ln 2 = 365t \ln\left(1 + \dfrac{0.055}{365}\right)$

$t = \dfrac{1}{365}\dfrac{\ln 2}{\ln\left(1 + \dfrac{0.055}{365}\right)} \approx 12.60$ yr

(d) $2000 = 1000e^{0.055t}$

$2 = e^{0.055t}$

$\ln 2 = 0.055t$

$t = \dfrac{\ln 2}{0.055} \approx 12.60$ yr

51. (a) $P = Ce^{kt} = Ce^{-0.011t}$

$P(5) = 2.0 = Ce^{-0.011(5)} \Rightarrow C = 2.0e^{0.011(5)} \approx 2.113$

$P = 2.113e^{-0.011t}$

(b) For 2030, $t = 20$ and $P = 2.113e^{-0.011(20)} \approx 1.70$ million

(c) Because $k < 0$, the population is decreasing.

52. (a) $P = Ce^{kt} = Ce^{0.008t}$

$P(5) = 35.1 = Ce^{0.008(5)} \Rightarrow C = 35.1e^{-0.008(5)} \approx 33.724$

$P = 33.724e^{0.008t}$

(b) For 2030, $t = 20$ and $P = 33.724e^{0.008(20)} \approx 39.57$ million

(c) Because $k > 0$, the population is increasing.

53. (a) $P = Ce^{kt} = Ce^{0.012t}$

$P(5) = 6.8 = Ce^{0.012(5)} \Rightarrow C = 6.8e^{-0.012(5)} \approx 6.404$

$P = 6.404e^{0.012t}$

(b) For 2030, $t = 20$ and $P = 6.404e^{0.012(20)} \approx 8.14$ million

(c) Because $k > 0$, the population is increasing.

54. (a) $P = Ce^{kt} = Ce^{-0.006t}$

$P(5) = 44.4 = Ce^{-0.006(5)} \Rightarrow C = 44.4e^{0.006(5)} \approx 45.752$

$P = 45.752e^{-0.006t}$

(b) For 2030, $t = 20$ and $P = 45.752e^{-0.006(20)} \approx 40.58$ million

(c) Because $k < 0$, the population is decreasing.

55. (a) $N = 100.1596(1.2455)^t$

(b) $N = 400$ when $t = 6.3$ hr (graphing utility)

Analytically,

$$400 = 100.1596(1.2455)^t$$
$$1.2455^t = \frac{400}{100.1596} = 3.9936$$
$$t \ln 1.2455 = \ln 3.9936$$
$$t = \frac{\ln 3.9936}{\ln 1.2455} \approx 6.3 \text{ hr}$$

56. (a) Let $y = Ce^{kt}$.

At time 2: $125 = Ce^{k(2)} \Rightarrow C = 125e^{-2k}$

At time 4:

$$350 = Ce^{k(4)} \Rightarrow 350 = (125e^{-2k})(e^{4k})$$
$$\tfrac{14}{5} = e^{2k}$$
$$2k = \ln \tfrac{14}{5}$$
$$k = \tfrac{1}{2} \ln \tfrac{14}{5} \approx 0.5148$$

$$C = 125e^{-2k}$$
$$= 125e^{-2(1/2)\ln(14/5)}$$
$$= 125\left(\tfrac{5}{14}\right) = \tfrac{625}{14} \approx 44.64$$

Approximately 45 bacteria at time 0.

(b) $y = \frac{625}{14} e^{(1/2)\ln(14/5)t} \approx 44.64e^{0.5148t}$

(c) When $t = 8$,

$$y = \tfrac{625}{14}e^{(1/2)\ln(14/5)8} = \tfrac{625}{14}\left(\tfrac{14}{5}\right)^4 = 2744.$$

(d) $25{,}000 = \frac{625}{14}e^{(1/2)\ln(14/5)t} \Rightarrow t \approx 12.29$ hr

57. (a) $19 = 30(1 - e^{20k})$

$$30e^{20k} = 11$$
$$k = \frac{\ln(11/30)}{20} \approx -0.0502$$
$$N \approx 30(1 - e^{-0.0502t})$$

(b) $25 = 30(1 - e^{-0.0502t})$

$$e^{-0.0502t} = \frac{1}{6}$$
$$t = \frac{-\ln 6}{-0.0502} \approx 36 \text{ days}$$

58. (a) $20 = 30(1 - e^{30k})$

$$30e^{30k} = 10$$
$$k = \frac{\ln(1/3)}{30} = \frac{-\ln 3}{30} \approx -0.0366$$
$$N \approx 30(1 - e^{-0.0366t})$$

(b) $25 = 30(1 - e^{-0.0366t})$

$$e^{-0.0366t} = \frac{1}{6}$$
$$t = \frac{-\ln 6}{-0.0366} \approx 49 \text{ days}$$

59. (a) Because the population increases by a constant each month, the rate of change from month to month will always be the same. So, the slope is constant, and the model is linear.

(b) Although the percentage increase is constant each month, the rate of growth is not constant. The rate of change of y is given by

$$\frac{dy}{dt} = ry$$

which is an exponential model.

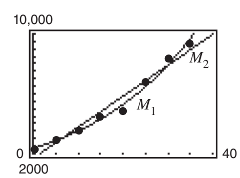
60. (a) Both functions represent exponential growth because the graphs are increasing.

(b) g has a greater k value because its graph is increasing at a greater rate than the graph of f.

61. (a) Using a graphing utility, $M_1 = 2335.3e^{0.0407t}$.

(b) Using a graphing utility, $M_2 = 206.9t + 1685$.

(c) One way to determine which model fits the data better is to use a graphing utility to graph the data with both models.



The exponential model fits the data better because the graph is closer to the data values than is the graph of the linear model.

(d) $15{,}000 = 2335.3e^{0.0407t}$

$$6.423 = e^{0.0407t}$$
$$t = \frac{\ln 6.423}{0.0407} \approx 46 \text{ years, or year 2026}$$

Yes. The exponential model indicates a reasonably slow growth rate.

62. $A(t) = V(t)e^{-0.10t}$

$= 100{,}000e^{0.8\sqrt{t}}e^{-0.10t} = 100{,}000e^{0.8\sqrt{t}-0.10t}$

$\dfrac{dA}{dt} = 100{,}000\left(\dfrac{0.4}{\sqrt{t}} - 0.10\right)e^{0.8\sqrt{t}-0.10t}$

$\dfrac{dA}{dt} = 0$ when $\dfrac{0.4}{\sqrt{t}} = 0.10 \Rightarrow t = 16.$

The timber should be harvested in the year 2026 $(2010 + 16)$.

Note: You could also use a graphing utility to graph $A(t)$ and find the maximum value. Use a viewing window of $0 \le x \le 30,\ 0 \le y \le 600{,}000$.

63. $\beta(I) = 10\log_{10}\dfrac{I}{I_0},\ I_0 = 10^{-16}$

(a) $\beta(10^{-14}) = 10\log_{10}\dfrac{10^{-14}}{10^{-16}} = 20$ decibels

(b) $\beta(10^{-9}) = 10\log_{10}\dfrac{10^{-9}}{10^{-16}} = 70$ decibels

(c) $\beta(10^{-4}) = 10\log_{10}\dfrac{10^{-4}}{10^{-16}} = 120$ decibels

64. $93 = 10\log_{10}\dfrac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-6.7 = \log_{10} I \Rightarrow I = 10^{-6.7}$

$80 = 10\log_{10}\dfrac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-8 = \log_{10} I \Rightarrow I = 10^{-8}$

Percentage decrease: $\left(\dfrac{10^{-6.7} - 10^{-8}}{10^{-6.7}}\right)(100) \approx 95\%$

65. (a) $\dfrac{dy}{dt} = k(y - 80)$

$\displaystyle\int \frac{1}{y - 80}\,dy = \int k\,dt$

$\ln(y - 80) = kt + C_1$

$y = 80 + e^{kt + C_1} = 80 + Ce^{kt}$

When $t = 0$, $y = 1500$, so $C = 1420$.

$y = 80 + 1420e^{kt}$

When $t = 1$, $y = 1120 = 80 + 1420e^{k(1)}$

$\Rightarrow e^k = \dfrac{1120 - 80}{1420} = \dfrac{52}{71}$

$\Rightarrow k = \ln\left(\dfrac{52}{71}\right) \approx -0.3114$

So, $y = 80 + 1420e^{-0.3114t}$

(b) At $t = 6$, $y = 80 + 1420e^{-0.3114(6)} \approx 299.2°$ F.

66. (a) $\dfrac{dy}{dt} = k(y - 20)$

$\displaystyle\int \frac{1}{y - 20}\,dy = \int k\,dt$

$\ln(y - 20) = kt + C_1$

$y = 20 + Ce^{kt}$

When $t = 0$, $y = 160 = 20 + Ce^{k(0)} \Rightarrow C = 140$

$y = 20 + 140e^{kt}$

When $t = 5$, $y = 60 = 20 + 140e^{k(5)}$

$\Rightarrow \dfrac{60 - 20}{140} = \dfrac{2}{7} = e^{5k} \Rightarrow k = \dfrac{1}{5}\ln\left(\dfrac{2}{7}\right) \approx -0.2506$

So, $y = 20 + 140e^{-0.2506t}$.

(b) $25 = 20 + 140e^{-0.2506t}$

$\dfrac{1}{28} = e^{-0.2506t} \Rightarrow t = \dfrac{\ln(1/28)}{-0.2506} \approx 13.3$

So, it takes about $13.3 - 5 = 8.3$ minutes longer.

67. False. The half-life of radium is 1599 years.

68. False. The prices are rising at a rate of 6.2% per year.

Section 6.3 Separation of Variables and the Logistic Equation

1. (a) $y = 2x^5y' - y' = (2x^5 - 1)\dfrac{dy}{dx}$

$$\frac{dx}{(2x^5 - 1)} = \frac{dy}{y}$$

Separable

(b) Not separable

2. Two families of curves are mutually orthogonal if each curve in one family is orthogonal to all curves in the other family.

3. The carrying capacity is the maximum population that can be sustained over-time

4. Answers will vary. *Sample answer:* Releasing deer into a forest that can only support a certain larger amount of deer.

5. $\dfrac{dy}{dx} = \dfrac{x}{y}$

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C_1$$

$$y^2 - x^2 = C$$

6. $\dfrac{dy}{dx} = \dfrac{3x^2}{y^2}$

$$\int y^2\,dy = \int 3x^2\,dx$$

$$\frac{y^3}{3} = x^3 + C_1$$

$$y^3 - 3x^3 = C$$

7. $\dfrac{dy}{dx} = \dfrac{x - 1}{y^3}$

$$\int y^3\,dy = \int (x - 1)\,dx$$

$$\frac{1}{4}y^4 = \frac{1}{2}x^2 - x + C_1$$

$$y^4 - 2x^2 + 4x = C$$

8. $\dfrac{dy}{dx} = \dfrac{6 - x^2}{2y^3}$

$$\int 2y^3\,dy = \int (6 - x^2)\,dx$$

$$\frac{y^4}{2} = 6x - \frac{x^3}{3} + C_1$$

$$3y^4 + 2x^3 - 36x = C$$

9. $\dfrac{dr}{ds} = \dfrac{4}{9}r$

$$\int \frac{dr}{r} = \int \frac{4}{9}\,ds$$

$$\ln|r| = \frac{4}{9}s + C_1$$

$$r = e^{4/9s + C_1}$$

$$r = Ce^{4/9s}$$

10. $\dfrac{dr}{ds} = \dfrac{9}{4}s$

$$\int dr = \int \frac{9}{4}s\,ds$$

$$r = \frac{9}{8}s^2 + C$$

11. $(2 + x)y' = 3y$

$$\int \frac{dy}{y} = \int \frac{3}{2 + x}\,dx$$

$$\ln|y| = 3\ln|2 + x| + \ln C = \ln\left|C(2 + x)^3\right|$$

$$y = C(x + 2)^3$$

12. $xy' = y$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + \ln C = \ln Cx$$

$$y = Cx$$

13. $y^2\dfrac{dy}{dx} = \sin 9x$

$$\int y^2\,dy = \int \sin 9x\,dx$$

$$\frac{y^3}{3} = -\frac{1}{9}\cos 9x + C_1$$

$$y^3 = C - \frac{1}{3}\cos 9x$$

14. $yy' = -8\cos(\pi x)$

$$y\frac{dy}{dx} = -8\cos(\pi x)$$

$$\int y\,dy = \int -8\cos(\pi x)\,dx$$

$$\frac{y^2}{2} = \frac{-8\sin(\pi x)}{\pi} + C$$

$$y^2 = \frac{-16}{\pi}\sin(\pi x) + C$$

15. $\sqrt{1-4x^2}\,y' = x$

$$dy = \frac{x}{\sqrt{1-4x^2}}\,dx$$

$$\int dy = \int \frac{x}{\sqrt{1-4x^2}}\,dx$$

$$= -\frac{1}{8}\int\left(1-4x^2\right)^{-1/2}(-8x\,dx)$$

$$y = -\frac{1}{4}\sqrt{1-4x^2} + C$$

16. $\sqrt{x^3-5}\,\dfrac{dy}{dx} = x^2$

$$\int dy = \int \frac{x^2}{\left(x^3-5\right)^{1/2}}\,dx$$

$$= \frac{1}{3}\int\left(x^3-5\right)^{-1/2}\left(3x^2\,dx\right)$$

$$y = \frac{2}{3}\sqrt{x^3-5} + C$$

17. $y \ln x - xy' = 0$

$$\int \frac{dy}{y} = \int \frac{\ln x}{x}\,dx \quad \left(u = \ln x,\ du = \frac{dx}{x}\right)$$

$$\ln|y| = \frac{1}{2}(\ln x)^2 + C_1$$

$$y = e^{(1/2)(\ln x)^2 + C_1} = Ce^{(\ln x)^2/2}$$

18. $12yy' - 7e^x = 0$

$$12y\,\frac{dy}{dx} = 7e^x$$

$$\int 12y\,dy = \int 7e^x\,dx$$

$$6y^2 = 7e^x + C$$

19. $yy' - 2e^x = 0$

$$y\,\frac{dy}{dx} = 2e^x$$

$$\int y\,dy = \int 2e^x\,dx$$

$$\frac{y^2}{2} = 2e^x + C$$

Initial condition $(0, 3)$: $\dfrac{9}{2} = 2 + C \Rightarrow C = \dfrac{5}{2}$

Particular solution: $\dfrac{y^2}{2} = 2e^x + \dfrac{5}{2}$

$$y^2 = 4e^x + 5$$

20. $\sqrt{x} + \sqrt{y}\,y' = 0$

$$\int y^{1/2}dy = -\int x^{1/2}\,dx$$

$$\frac{2}{3}y^{3/2} = -\frac{2}{3}x^{3/2} + C_1$$

$$y^{3/2} + x^{3/2} = C$$

Initial condition $(1, 9)$:

$(9)^{3/2} + (1)^{3/2} = 27 + 1 = 28 = C$

Particular solution: $y^{3/2} + x^{3/2} = 28$

21. $y(x+1) + y' = 0$

$$\int \frac{dy}{y} = -\int (x+1)\,dx$$

$$\ln|y| = -\frac{(x+1)^2}{2} + C_1$$

$$y = Ce^{-(x+1)^2/2}$$

Initial condition $(-2, 1)$: $1 = Ce^{-1/2}$, $C = e^{1/2}$

Particular solution: $y = e^{\left[1-(x+1)^2\right]/2} = e^{-\left(x^2+2x\right)/2}$

22. $2xy' - \ln x^2 = 0$

$$2x\,\frac{dy}{dx} = 2\ln x$$

$$\int dy = \int \frac{\ln x}{x}\,dx$$

$$y = \frac{(\ln x)^2}{2} + C$$

Initial condition $(1, 2)$: $2 = C$

Particular solution: $y = \dfrac{1}{2}(\ln x)^2 + 2$

23. $y\left(1+x^2\right)y' = x\left(1+y^2\right)$

$$\frac{y}{1+y^2}\,dy = \frac{x}{1+x^2}\,dx$$

$$\frac{1}{2}\ln\left(1+y^2\right) = \frac{1}{2}\ln\left(1+x^2\right) + C_1$$

$$\ln\left(1+y^2\right) = \ln\left(1+x^2\right) + \ln C = \ln\left[C\left(1+x^2\right)\right]$$

$$1 + y^2 = C\left(1+x^2\right)$$

Initial condition $\left(0, \sqrt{3}\right)$: $1 + 3 = C \Rightarrow C = 4$

Particular solution: $1 + y^2 = 4\left(1+x^2\right)$

$$y^2 = 3 + 4x^2$$

24. $$y\sqrt{1-x^2}\,\frac{dy}{dx} = x\sqrt{1-y^2}$$

$$\int (1-y^2)^{-1/2}y\,dy = \int (1-x^2)^{-1/2}x\,dx$$

$$-(1-y^2)^{1/2} = -(1-x^2)^{1/2} + C$$

Initial condition $(0, 1)$: $0 = -1 + C \Rightarrow C = 1$

Particular solution: $\sqrt{1-y^2} = \sqrt{1-x^2} - 1$

25. $$\frac{du}{dv} = uv\sin v^2$$

$$\int \frac{du}{u} = \int (\sin v^2)v\,dv$$

$$\ln|u| = -\frac{1}{2}\cos v^2 + C$$

Initial condition $(e^2, 0)$: $\ln e^2 = -\frac{1}{2}\cos 0 + C$

$$2 = -\frac{1}{2} + C$$

$$C = \frac{5}{2}$$

Particular solution: $\ln u = -\frac{1}{2}\cos v^2 + \frac{5}{2}$

$$u = e^{(5-\cos v^2)/2}$$

26. $$\frac{dr}{ds} = e^{r-2s}$$

$$\int e^{-r}dr = \int e^{-2s}ds$$

$$-e^{-r} = -\frac{1}{2}e^{-2s} + C$$

Initial condition:

$r(0) = 0$: $-1 = -\frac{1}{2} + C \Rightarrow C = -\frac{1}{2}$

Particular solution:

$$-e^{-r} = -\frac{1}{2}e^{-2s} - \frac{1}{2}$$

$$e^{-r} = \frac{1}{2}e^{-2s} + \frac{1}{2}$$

$$-r = \ln\left(\frac{1}{2}e^{-2s} + \frac{1}{2}\right) = \ln\left(\frac{1+e^{-2s}}{2}\right)$$

$$r = \ln\left(\frac{2}{1+e^{-2s}}\right)$$

27. $dP - kP\,dt = 0$

$$\int \frac{dP}{P} = k\int dt$$

$$\ln|P| = kt + C_1$$

$$P = Ce^{kt}$$

Initial condition: $P(0) = P_0,\ P_0 = Ce^0 = C$

Particular solution: $P = P_0e^{kt}$

28. $dT + k(T - 70)\,dt = 0$

$$\int \frac{dT}{T-70} = -k\int dt$$

$$\ln(T - 70) = -kt + C_1$$

$$T - 70 = Ce^{-kt}$$

Initial condition:

$T(0) = 140$: $140 - 70 = 70 = Ce^0 = C$

Particular solution:

$T - 70 = 70e^{-kt},\ T = 70(1 + e^{-kt})$

29. $$y' = \frac{dy}{dx} = \frac{x}{4y}$$

$$\int 4y\,dy = \int x\,dx$$

$$2y^2 = \frac{x^2}{2} + C$$

Initial condition $(0, 2)$: $2(2^2) = 0 + C \Rightarrow C = 8$

Particular solution: $2y^2 = \frac{x^2}{2} + 8$

$$4y^2 - x^2 = 16$$

30. $$\frac{dy}{dx} = \frac{-9x}{16y}$$

$$\int 16y\,dy = -\int 9x\,dx$$

$$8y^2 = \frac{-9}{2}x^2 + C$$

Initial condition $(1, 1)$: $8 = -\frac{9}{2} + C,\ C = \frac{25}{2}$

Particular solution: $8y^2 = \frac{-9}{2}x^2 + \frac{25}{2}$

$$16y^2 + 9x^2 = 25$$

31. $$y' = -\frac{y}{5x}$$
$$\int -\frac{dy}{y} = \int \frac{1}{5x}\,dx$$
$$\ln|y| = \frac{1}{5}\ln|x| + C_1$$
$$5\ln|y| = \ln|x| + \ln C = \ln|Cx|$$
$$y^5 = Cx$$

Initial condition $(3, 1)$: $1^5 = C(3) \Rightarrow C = \frac{1}{3}$

Particular solution: $y^5 = \frac{1}{3}x$
$$y = \left(\frac{x}{3}\right)^{1/5}$$

32. $$\frac{dy}{dx} = \frac{2y}{3x}$$
$$\int \frac{3}{y}\,dy = \int \frac{2}{x}\,dx$$
$$\ln y^3 = \ln x^2 + \ln C$$
$$y^3 = Cx^2$$

Initial condition $(8, 2)$: $2^3 = C(8^2), C = \frac{1}{8}$

Particular solution: $8y^3 = x^2, y = \frac{1}{2}x^{2/3}$

33. $$m = \frac{dy}{dx} = \frac{0 - y}{(x+2) - x} = -\frac{y}{2}$$
$$\int \frac{dy}{y} = \int -\frac{1}{2}\,dx$$
$$\ln|y| = -\frac{1}{2}x + C_1$$
$$y = Ce^{-x/2}$$

34. $$m = \frac{dy}{dx} = \frac{y - 0}{x - 0} = \frac{y}{x}$$
$$\int \frac{dy}{y} = \int \frac{dx}{x}$$
$$\ln y = \ln x + C_1 = \ln x + \ln C = \ln Cx$$
$$y = Cx$$

35. (a) $\frac{dy}{dx} = k(y - 4)$

(b) The direction field satisfies $(dy/dx) = 0$ along $y = 4$; but not along $y = 0$. Matches (a).

36. (a) $\frac{dy}{dx} = k(x - 4)$

(b) The direction field satisfies $(dy/dx) = 0$ along $x = 4$. Matches (b).

37. (a) $\frac{dy}{dx} = ky(y - 4)$

(b) The direction field satisfies $(dy/dx) = 0$ along $y = 0$ and $y = 4$. Matches (c).

38. (a) $\frac{dy}{dx} = ky^2$

(b) The direction field satisfies $(dy/dx) = 0$ along $y = 0$, and grows more positive as y increases. Matches (d).

39. $\frac{dy}{dt} = ky, \quad y = Ce^{kt}$

Initial amount: $y(0) = y_0 = C$

Half-life: $\frac{y_0}{2} = y_0 e^{k(1599)}$
$$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$$
$$y = Ce^{[\ln(1/2)/1599]t}$$

When $t = 50, y = 0.9786C$ or 97.86%.

40. $\frac{dy}{dt} = ky, \; y = Ce^{kt}$

Initial conditions: $y(0) = 40, y(1) = 35$
$$40 = Ce^0 = C$$
$$35 = 40e^k$$
$$k = \ln\frac{7}{8}$$

Particular solution: $y = 40e^{t\ln(7/8)}$

When 75% has been changed:
$$10 = 40e^{t\ln(7/8)}$$
$$\frac{1}{4} = e^{t\ln(7/8)}$$
$$t = \frac{\ln(1/4)}{\ln(7/8)} \approx 10.38 \text{ hr}$$

41. (a)

$$\frac{dw}{dt} = k(1200 - w)$$

$$\int \frac{dw}{1200 - w} = \int k\,dt$$

$$\ln|1200 - w| = -kt + C_1$$

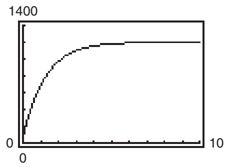
$$1200 - w = e^{-kt + C_1} = Ce^{-kt}$$

$$w = 1200 - Ce^{-kt}$$

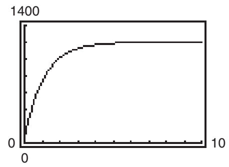
$$w(0) = 60 = 1200 - C \Rightarrow C = 1200 - 60 = 1140$$

$$w = 1200 - 1140e^{-kt}$$

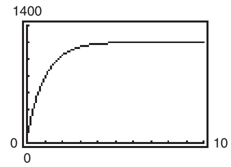
(b)



$k = 0.8$



$k = 0.9$



$k = 1$

(c)

$$800 = 1200 - 1140e^{-kt}$$

$$\frac{-400}{-1140} = e^{-kt} \Rightarrow kt = -\ln\frac{400}{1140} = \ln\frac{57}{20}$$

$k = 0.8$: $t = 1.31$ yr

$k = 0.9$: $t = 1.16$ yr

$k = 1.0$: $t = 1.05$ yr

(d) Maximum weight:

$$\lim_{t\to\infty} w = 1200 \text{ lb}$$

42. (a)

$$\frac{dw}{dt} = k(250 - w)$$

$$\int \left(\frac{dw}{250 - w}\right) = \int k\,dt$$

$$\ln|250 - w| = -kt + C_1$$

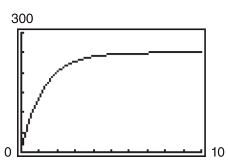
$$250 - w = e^{-kt + C_1} = Ce^{-kt}$$

$$w = 250 - Ce^{-kt}$$

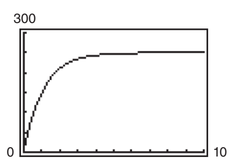
$$w(0) = 7 = 250 - C = 243$$

$$w = 250 - 243e^{-kt}$$

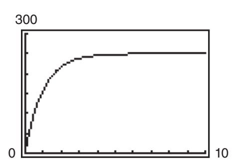
(b)



$k = 0.8$



$k = 0.9$



$k = 1$

(c)

$$175 = 250 - 243e^{-kt}$$

$$\frac{-75}{-243} = e^{-kt} \Rightarrow kt = -\ln\frac{75}{243} = \ln\frac{81}{25}$$

$k = 0.8$: $t \approx 1.47$ yr

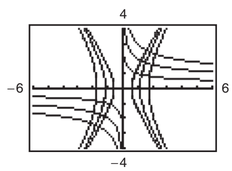
$k = 0.9$: $t \approx 1.31$ yr

$k = 1.0$: $t \approx 1.18$ yr

(d) Maximum weight: $\lim_{t\to\infty} w = 250$ lb

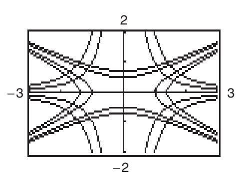
43. Given family (hyperbolas):
$$3x^2 - y^2 = C$$
$$6x - 2yy' = 0$$
$$y' = \frac{3x}{y}$$

Orthogonal trajectory:
$$y' = \frac{y}{3x}$$
$$\int \frac{1}{y}\,dy = \int \frac{1}{3x}\,dx$$
$$3\ln|y| = \ln|x| + \ln K = \ln Kx$$
$$y^3 = Kx \Rightarrow y = K\sqrt[3]{x}$$



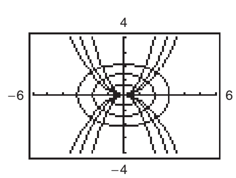
44. Given family (hyperbolas):
$$x^2 - 2y^2 = C$$
$$2x - 4yy' = 0$$
$$y' = \frac{x}{2y}$$

Orthogonal trajectory:
$$y' = \frac{-2y}{x}$$
$$\int \frac{dy}{y} = -\int \frac{2}{x}\,dx$$
$$\ln y = -2\ln x + \ln K$$
$$y = Kx^{-2} = \frac{K}{x^2}$$



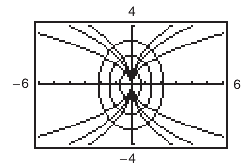
45. Given family (parabolas):
$$x^2 = Cy$$
$$2x = Cy'$$
$$y' = \frac{2x}{C} = \frac{2x}{x^2/y} = \frac{2y}{x}$$

Orthogonal trajectory (ellipses):
$$y' = -\frac{x}{2y}$$
$$2\int y\,dy = -\int x\,dx$$
$$y^2 = -\frac{x^2}{2} + K_1$$
$$x^2 + 2y^2 = K$$



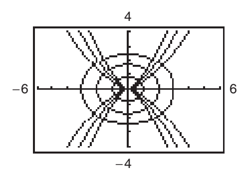
46. Given family (parabolas):
$$y^2 = 2Cx$$
$$2yy' = 2C$$
$$y' = \frac{C}{y} = \frac{y^2}{2x}\left(\frac{1}{y}\right) = \frac{y}{2x}$$

Orthogonal trajectory (ellipse):
$$y' = -\frac{2x}{y}$$
$$\int y\,dy = -\int 2x\,dx$$
$$\frac{y^2}{2} = -x^2 + K_1$$
$$2x^2 + y^2 = K$$



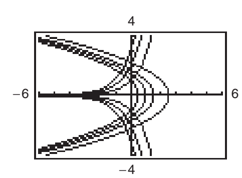
47. Given family:
$$y^2 = Cx^3$$
$$2yy' = 3Cx^2$$
$$y' = \frac{3Cx^2}{2y} = \frac{3x^2}{2y}\left(\frac{y^2}{x^3}\right) = \frac{3y}{2x}$$

Orthogonal trajectory (ellipses):
$$y' = -\frac{2x}{3y}$$
$$3\int y\,dy = -2\int x\,dx$$
$$\frac{3y^2}{2} = -x^2 + K_1$$
$$3y^2 + 2x^2 = K$$



48. Given family (exponential functions):
$$y = Ce^x$$
$$y' = Ce^x = y$$

Orthogonal trajectory (parabolas):
$$y' = -\frac{1}{y}$$
$$\int y\,dy = -\int dx$$
$$\frac{y^2}{2} = -x + K_1$$
$$y^2 = -2x + K$$



49. $y = \dfrac{12}{1 + e^{-x}}$

Because $y(0) = 6$, it matches (c) or (d).

Because (d) approaches its horizontal asymptote slower than (c), it matches (d).

50. $y = \dfrac{12}{1 + 3e^{-x}}$

Because $y(0) = \dfrac{12}{4} = 3$, it matches (a).

51. $y = \dfrac{12}{1 + \frac{1}{2}e^{-x}}$

Because $y(0) = \dfrac{12}{\left(\frac{3}{2}\right)} = 8$, it matches (b).

52. $y = \dfrac{12}{1 + e^{-2x}}$

Because $y(0) = 6$, it matches (c) or (d).

Because y approaches $L = 12$ faster for (c), it matches (c).

53. $P(t) = \dfrac{2100}{1 + 29e^{-0.75t}}$

(a) $k = 0.75$

(b) $L = 2100$

(c) $P(0) = \dfrac{2100}{1 + 29} = 70$

(d)
$$\begin{aligned} 1050 &= \frac{2100}{1 + 29e^{-0.75t}} \\ 1 + 29e^{-0.75t} &= 2 \\ e^{-0.75t} &= \frac{1}{29} \\ -0.75t &= \ln\left(\frac{1}{29}\right) = -\ln 29 \\ t &= \frac{\ln 29}{0.75} \approx 4.4897 \text{ yr} \end{aligned}$$

(e) $\dfrac{dP}{dt} = 0.75P\left(1 - \dfrac{P}{2100}\right), \quad P(0) = 70$

54. $P(t) = \dfrac{5000}{1 + 39e^{-0.2t}}$

(a) $k = 0.2$

(b) $L = 5000$

(c) $P(0) = \dfrac{5000}{1 + 39} = 125$

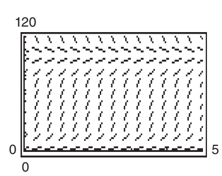
(d)
$$\begin{aligned} 2500 &= \frac{5000}{1 + 39e^{-0.2t}} \\ 1 + 39e^{-0.2t} &= 2 \\ e^{-0.2t} &= \frac{1}{39} \\ -0.2t &= \ln\left(\frac{1}{39}\right) = -\ln 39 \\ t &= \frac{\ln 39}{0.2} \approx 18.3178 \end{aligned}$$

(e) $\dfrac{dP}{dt} = 0.2P\left(1 - \dfrac{P}{5000}\right), \quad P(0) = 125$

55. $\dfrac{dP}{dt} = 3P\left(1 - \dfrac{P}{100}\right)$ (a) $k = 3$

(b) $L = 100$

(c)



(d) $\dfrac{d^2P}{dt^2} = 3P'\left(1 - \dfrac{P}{100}\right) + 3P\left(\dfrac{-P'}{100}\right)$

$$= 3\left[3P\left(1 - \frac{P}{100}\right)\right]\left(1 - \frac{P}{100}\right) - \frac{3P}{100}\left[3P\left(1 - \frac{P}{100}\right)\right] = 9P\left(1 - \frac{P}{100}\right)\left(1 - \frac{P}{100} - \frac{P}{100}\right) = 9P\left(1 - \frac{P}{100}\right)\left(1 - \frac{2P}{100}\right)$$

$\dfrac{d^2P}{dt^2} = 0$ for $P = 50$, and by the first Derivative Test, this is a maximum. $\left(\text{Note: } P = 50 = \dfrac{L}{2} = \dfrac{100}{2}\right)$

56. $\frac{dP}{dt} = 0.1P - 0.0004P^2$

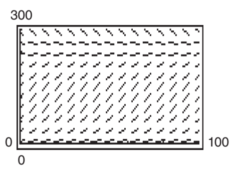
$= 0.1P(1 - 0.004P)$

$= 0.1P\left(1 - \frac{P}{250}\right)$

(a) $k = 0.1 = \frac{1}{10}$

(b) $L = 250$

(c)



(d) $P = \frac{250}{2} = 125$. (Same argument as in Exercise 77)

57. $\frac{dy}{dt} = y\left(1 - \frac{y}{36}\right), \quad (0, 4)$

$k = 1, L = 36$

$y = \frac{L}{1 + be^{-kt}} = \frac{36}{1 + be^{-t}}$

$(0, 4)$: $4 = \frac{36}{1 + b} \Rightarrow b = 8$

Solution: $y = \frac{36}{1 + 8e^{-t}}$

58. $\frac{dy}{dt} = 4.2y\left(1 - \frac{y}{21}\right), \quad (0, 9)$

$k = 4.2, L = 21$

$y = \frac{L}{1 + be^{-kt}} = \frac{21}{1 + be^{-4.2t}}$

$(0, 9)$: $9 = \frac{21}{1 + b} \Rightarrow b = \frac{4}{3}$

Solution: $y = \frac{21}{1 + \frac{4}{3}e^{-4.2t}}$

59. $\frac{dy}{dt} = \frac{4y}{5} - \frac{y^2}{150} = \frac{4}{5}y\left(1 - \frac{y}{120}\right), \quad (0, 8)$

$k = \frac{4}{5} = 0.8, L = 120$

$y = \frac{L}{1 + be^{-kt}} = \frac{120}{1 + be^{-0.8t}}$

$(0, 8)$: $8 = \frac{120}{1 + b} \Rightarrow b = 14$

Solution: $y = \frac{120}{1 + 14e^{-0.8t}}$

60. $\frac{dy}{dt} = \frac{3y}{20} - \frac{y^2}{1600} = \frac{3}{20}y\left(1 - \frac{y}{240}\right), \quad (0, 15)$

$k = \frac{3}{20}, L = 240$

$y = \frac{L}{1 + be^{-kt}} = \frac{240}{1 + be^{(-3/20)t}}$

$(0, 15)$: $15 = \frac{240}{1 + b} \Rightarrow b = 15$

Solution: $y = \frac{240}{1 + 15e^{(-3/20)t}}$

61. (a) $P = \frac{L}{1 + be^{-kt}}, L = 200, P(0) = 25$

$25 = \frac{200}{1 + b} \Rightarrow b = 7$

$39 = \frac{200}{1 + 7e^{-k(2)}}$

$1 + 7e^{-2k} = \frac{200}{39}$

$e^{-2k} = \frac{23}{39}$

$k = -\frac{1}{2}\ln\left(\frac{23}{39}\right) = \frac{1}{2}\ln\left(\frac{39}{23}\right) \approx 0.2640$

$P = \frac{200}{1 + 7e^{-0.2640t}}$

(b) For $t = 5, P \approx 70$ panthers.

(c) $100 = \frac{200}{1 + 7e^{-0.264t}}$

$1 + 7e^{-0.264t} = 2$

$-0.264t = \ln\left(\frac{1}{7}\right)$

$t \approx 7.37$ years

(d) $\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$

$= 0.264P\left(1 - \frac{P}{200}\right), \; P(0) = 25$

Using Euler's Method, $P \approx 65.6$ when $t = 5$.

(e) P is increasing most rapidly where $P = 200/2 = 100$, corresponds to $t \approx 7.37$ years.

62. (a)

$$y = \frac{L}{1 + be^{-kt}},\ L = 20,\ y(0) = 1,\ y(2) = 4$$

$$1 = \frac{20}{1 + b} \Rightarrow b = 19$$

$$4 = \frac{20}{1 + 19e^{-2k}}$$

$$1 + 19e^{-2k} = 5$$

$$19e^{-2k} = 4$$

$$k = -\frac{1}{2}\ln\left(\frac{4}{19}\right) = \frac{1}{2}\ln\left(\frac{19}{4}\right) \approx 0.7791$$

$$y = \frac{20}{1 + 19e^{-0.7791t}}$$

(b) For $t = 5$, $y \approx 14.43$ grams

(c)

$$18 = \frac{20}{1 + 19e^{-0.7791t}}$$

$$1 + 19e^{-0.7791t} = \frac{20}{18} = \frac{10}{9}$$

$$19e^{-0.7791t} = \frac{1}{9}$$

$$e^{-0.7791t} = \frac{1}{171}$$

$$t = \frac{-1}{0.7791}\ln\left(\frac{1}{171}\right) \approx 6.60 \text{ hours}$$

(d) $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{L}\right) = \dfrac{1}{2}\ln\left(\dfrac{19}{4}\right)y\left(1 - \dfrac{y}{20}\right)$

t	0	1	2	3	4	5
Exact	1	2.06	4.00	7.05	10.86	14.43
Euler	1	1.74	2.98	4.95	7.86	11.57

(e) The weight is increasing most rapidly when $y = L/2 = 20/2 = 10$, corresponding to $t \approx 3.78$ hours.

63. Yes. Rewrite the equation as

$$\frac{dy}{dx} = f(x)\big(g(y) - h(y)\big)$$

$$\frac{dy}{g(y) - h(y)} = f(x)\,dx.$$

64. A logistic differential equation has the form

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$$

where k and L are positive constants. The slopes are horizontal for $y = 0$ and $y = L$ because $dy/dt = 0$. The slopes are positive for $0 < y < L$ because $dy/dt > 0$. The slopes are negative for $y < 0$ and $y > L$ because $dy/dt < 0$.

65.

$$y = \frac{1}{1 + be^{-kt}}$$

$$y' = \frac{-1}{\left(1 + be^{-kt}\right)^2}\left(-bke^{-kt}\right)$$

$$= \frac{k}{\left(1 + be^{-kt}\right)} \cdot \frac{be^{-kt}}{\left(1 + be^{-kt}\right)}$$

$$= \frac{k}{\left(1 + be^{-kt}\right)} \cdot \frac{1 + be^{-kt} - 1}{\left(1 + be^{-kt}\right)}$$

$$= \frac{k}{\left(1 + be^{-kt}\right)} \cdot \left(1 - \frac{1}{1 + be^{-kt}}\right) = ky(1 - y)$$

66. $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{L}\right),\ y(0) < L$

$$\frac{d^2y}{dt^2} = ky'\left(1 - \frac{y}{L}\right) + ky\left(-\frac{y'}{L}\right)$$

$$= k^2y\left(1 - \frac{y}{L}\right)^2 + ky\left[\frac{-ky\left(1 - \frac{y}{L}\right)}{L}\right]$$

$$= k^2\left(1 - \frac{y}{L}\right)y\left[\left(1 - \frac{y}{L}\right) - \frac{y}{L}\right]$$

$$= k^2\left(1 - \frac{y}{L}\right)y\left(1 - \frac{2y}{L}\right)$$

So, $\dfrac{d^2y}{dt^2} = 0$ when $1 - \dfrac{2y}{L} = 0 \Rightarrow y = \dfrac{L}{2}$.

By the First Derivative Test, this is a maximum.

67. (a) $$\frac{dv}{dt} = k(W - v)$$

$$\int \frac{dv}{W - v} = \int k\,dt$$

$$-\ln|W - v| = kt + C_1$$

$$v = W - Ce^{-kt}$$

Initial conditions:

$W = 20, v = 0$ when $t = 0$ and $v = 10$ when $t = 0.5$ so, $C = 20, k = \ln 4$.

Particular solution:

$$v = 20\left(1 - e^{-(\ln 4)t}\right) = 20\left(1 - \left(\frac{1}{4}\right)^t\right)$$

or

$$v = 20\left(1 - e^{-1.386t}\right)$$

(b) $s = \int 20\left(1 - e^{-1.386t}\right) dt \approx 20\left(t + 0.7215e^{-1.386t}\right) + C$

Because $s(0) = 0, C \approx -14.43$ and you have

$s \approx 20t + 14.43\left(e^{-1.386t} - 1\right)$.

68. The rate increases then decreases. *Sample answer*: There might be limits on available food or space.

69. $f(x, y) = x^3 - 4xy^2 + y^3$

$$f(tx, ty) = t^3x^3 - 4txt^2y^2 + t^3y^3$$

$$= t^3\left(x^3 - 4xy^2 + y^3\right)$$

Homogeneous of degree 3

70. $f(x, y) = x^4 + 2x^2y^2 + x + y$

$$f(tx, ty) = (tx)^4 + 2(tx)^2(ty)^2 + tx + ty$$

$$= t\left[t^3x^4 + 2t^3x^2y^2 + x + y\right]$$

$$\neq t^n f(x, y)$$

Not homogeneous

71. $f(x, y) = e^{x/y}$

$$f(tx, ty) = e^{tx/ty} = e^{x/y}$$

Homogenous of degree 0

72. $f(x, y) = x^2e^{4/x} + y^2$

$$f(tx, ty) = (tx)^2e^{ty/tx} + (ty)^2$$

$$= t^2\left[x^2e^{4/x} + y^2\right]$$

Homogenous of degree 2

73. $f(x, y) = 2\ln xy$

$$f(tx, ty) = 2\ln[txty]$$

$$= 2\ln\left[t^2xy\right] = 2\left(\ln t^2 + \ln xy\right) \neq t^n f(x, y)$$

Not homogeneous

74. $f(x, y) = \tan(x + y)$

$$f(tx, ty) = \tan(tx + ty) = \tan\left[t(x + y)\right] \neq t^n f(x, y)$$

Not homogeneous

75. $f(x, y) = 2\ln\dfrac{x}{y}$

$$f(tx, ty) = 2\ln\frac{tx}{ty} = 2\ln\frac{x}{y}$$

Homogeneous of degree 0

76. $f(x, y) = \tan\dfrac{y}{x}$

$$f(tx, ty) = \tan\frac{ty}{tx} = \tan\frac{y}{x}$$

Homogeneous of degree 0

77. $(x + y)dx - 2x\,dy = 0,\ y = ux,\ dy = x\,du + u\,dx$

$$(x + ux)dx - 2x(x\,du + u\,dx) = 0$$
$$(1 + u)dx - 2x\,du - 2u\,dx = 0$$
$$(1 - u)dx = 2x\,du$$
$$\frac{1}{x}dx = \frac{2}{1 - u}du$$
$$\int \frac{1}{x}dx = 2\int \frac{1}{1 - u}du$$
$$\ln|x| + \ln C = -2\ln|1 - u|$$
$$\ln|Cx| = \ln|1 - u|^{-2}$$
$$|Cx| = \frac{1}{(1 - u)^2} = \frac{1}{\left[1 - (y/x)\right]^2}$$
$$|Cx| = \frac{x^2}{(x - y)^2}$$
$$|x| = C(x - y)^2$$

78. $(x^3 + y^3)dx - xy^2\,dy = 0,\ y = ux,\ dy = x\,du + u\,dx$

$$\left[x^3 + (ux)^3\right]dx - x(ux)^2(x\,du + u\,dx) = 0$$
$$(1 + u^3)dx - u^2(x\,du + u\,dx) = 0$$
$$dx = xu^2\,du$$
$$\int \frac{dx}{x} = \int u^2\,du$$
$$\ln|x| + C_1 = \frac{u^3}{3} = \frac{1}{3}\left(\frac{y}{x}\right)^3$$
$$\left(\frac{y}{x}\right)^3 = 3\ln|x| + C$$
$$y^3 = 3x^3\ln|x| + Cx^3$$

79. $(x - y)dx - (x + y)dy = 0,\ y = ux,\ dy = x\,du + u\,dx$

$$(x - ux)dx - (x + ux)(x\,du + u\,dx) = 0$$
$$(1 - u)dx - (1 + u)(x\,du + u\,dx) = 0$$
$$(1 - 2u - u^2)dx = x(1 + u)du$$
$$-\frac{dx}{x} = \frac{1 + u}{u^2 + 2u - 1}du$$
$$-\int \frac{dx}{x} = \int \frac{u + 1}{u^2 + 2u - 1}du$$
$$-\ln|x| + \ln C = \frac{1}{2}\ln|u^2 + 2u - 1|$$
$$\ln\left|\frac{C}{x}\right| = \ln|u^2 + 2u - 1|^{1/2}$$
$$\frac{C^2}{x^2} = |u^2 + 2u - 1|$$
$$\frac{C}{x^2} = \left|\left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right) - 1\right|$$
$$C = |y^2 + 2yx - x^2|$$

80. $(x^2 + y^2)dx - 2x\,dy = 0,\ y = ux,\ dy = x\,du + u\,dx$

$$\left(x^2 + (ux)^2\right)dx - 2x(ux)(x\,du + u\,dx) = 0$$
$$(1 + u^2)dx - 2u(x\,du + u\,dx) = 0$$
$$(1 - u^2)dx = 2ux\,du$$
$$-\frac{dx}{x} = \frac{-2u}{1 - u^2}\,du$$
$$-\int \frac{dx}{x} = \int \frac{-2u\,du}{1 - u^2}$$
$$-\ln|x| + \ln C = \ln|1 - u^2| = \ln\left[u^2 - 1\right] = \ln|u^2 - 1|$$
$$\ln\left|\frac{C}{x}\right| = \ln|u^2 - 1|$$
$$\frac{C}{x} = u^2 - 1 = \left(\frac{y}{x}\right)^2 - 1$$
$$Cx = y^2 - x^2$$

81. $xydx + (y^2 - x^2)dy = 0,\ y = ux,\ dy = x\,du + u\,dx$

$$
\begin{aligned}
x(ux)\,dx + \left[(ux)^2 - x^2\right](x\,du + u\,dx) &= 0\\
u\,dx + (u^2 - 1)(x\,du + u\,dx) &= 0\\
u^3\,dx &= -(u^2 - 1)x\,du\\
\frac{dx}{x} &= \frac{1 - u^2}{u^3}\,du\\
\int \frac{dx}{x} &= \int \left(u^{-3} - \frac{1}{u}\right) du\\
\ln|x| + \ln|C_1| &= -\frac{1}{2u^2} - \ln|u|\\
\ln|C_1 xu| &= -\frac{1}{2u^2}\\
\ln|C_1 y| &= -\frac{1}{2(y/x)^2} = -\frac{x^2}{2y^2}\\
y &= Ce^{-x^2/(2y^2)}
\end{aligned}
$$

82. $(2x + 3y)dx - x\,dy = 0,\ y = ux,\ dy = x\,du + u\,dx$

$$
\begin{aligned}
(2x + 3ux)dx - x(x\,du + u\,dx) &= 0\\
(2 + 3u)dx - x\,du - u\,dx &= 0\\
(2 + 2u)dx &= x\,du\\
\frac{2dx}{x} &= \frac{du}{1 + u}\\
2\int \frac{1}{x}dx &= \int \frac{1}{u + 1}\,du\\
2\ln|x| + \ln C &= \ln|u + 1|\\
\ln x^2 C &= \ln|u + 1|\\
1 + u &= x^2 C\\
1 + \frac{y}{x} &= x^2 C\\
\frac{y}{x} &= Cx^2 - 1\\
y &= Cx^3 - x
\end{aligned}
$$

83. False. $\dfrac{dy}{dx} = \dfrac{x}{y}$ is separable, but $y = 0$ is not a solution.

84. True

$$\frac{dy}{dx} = (x - 2)(y + 1)$$

85. True

$$x^2 + y^2 = 2Cy \qquad x^2 + y^2 = 2Kx$$

$$\frac{dy}{dx} = \frac{x}{C - y} \qquad \frac{dy}{dx} = \frac{K - x}{y}$$

$$\frac{x}{C - y} \cdot \frac{K - x}{y} = \frac{Kx - x^2}{Cy - y^2} = \frac{2Kx - 2x^2}{2Cy - 2y^2} = \frac{x^2 + y^2 - 2x^2}{x^2 + y^2 - 2y^2} = \frac{y^2 - x^2}{x^2 - y^2} = -1$$

86.

$$fg' + gf' = f'g' \qquad \text{Product Rule}$$

$$(f - f')g' + gf' = 0$$

$$g' + \frac{f'}{f - f'}g = 0$$

Need $f - f' = e^{x^2} - 2xe^{x^2} = (1 - 2x)e^{x^2} \neq 0$, so avoid $x = \frac{1}{2}$.

$$\frac{g'}{g} = \frac{f'}{f' - f} = \frac{2xe^{x^2}}{(2x - 1)e^{x^2}} = 1 + \frac{1}{2x - 1}$$

$$\ln|g(x)| = x + \frac{1}{2}\ln|2x - 1| + C_1$$

$$g(x) = Ce^x|2x - 1|^{1/2}$$

So there exists g and interval (a, b), as long as $\frac{1}{2} \notin (a, b)$.

Section 6.4 First-Order Linear Differential Equations

1. The term "first-order" means that the derivative in the equation is first order.

2. To solve a first-order linear differential equation, write it in the form $y' + P(x)y = Q(x)$. Multiply the equation by the integrating factor $u(x) = e^{\int P(x)\,dx}$. The solution is

$$y = \frac{1}{u(x)} \int Q(x)u(x)\,dx.$$

3. $x^3y' + xy = e^x + 1$

$$y' + \frac{1}{x^2}y = \frac{1}{x^3}(e^x + 1)$$

Linear

4.

$$2xy - y'\ln x = y$$

$$(\ln x)y' + (1 - 2x)y = 0$$

$$y' + \frac{(1 - 2x)}{\ln x}y = 0$$

Linear

5. $y' - y\sin x = xy^2$

Not linear, because of the xy^2-term.

6.

$$\frac{2 - y'}{y} = 5x$$

$$2 - y' = 5xy$$

$$y' + 5xy = 2$$

Linear

7. $\frac{dy}{dx} + \left(\frac{1}{x}\right)y = 6x + 2$

Integrating factor: $e^{\int (1/x)\,dx} = e^{\ln x} = x$

$$xy = \int x(6x + 2)\,dx = 2x^3 + x^2 + C$$

$$y = 2x^2 + x + \frac{C}{x}$$

8. $\frac{dy}{dx} + \frac{2}{x}y = 3x - 5$

Integrating factor: $e^{\int 2/x\,dx} = e^{\ln x^2} = x^2$

$$x^2y = \int x^2(3x - 5)\,dx = \frac{3}{4}x^4 + \frac{5x^3}{3} + C$$

$$y = \frac{3}{4}x^2 + \frac{5}{3}x + \frac{C}{x^2}$$

9. $y' + 2xy = 10x$

Integrating factor: $e^{\int 2x\,dx} = e^{x^2}$

$$ye^{x^2} = \int 10xe^{x^2}\,dx = 5e^{x^2} + C$$

$$y = 5 + Ce^{-x^2}$$

10. $y' + 3x^2y = 6x^2$

Integrating factor: $e^{\int 3x^2\,dx} = e^{x^3}$

$$ye^{x^3} = \int 6x^2e^{x^3}\,dx = 2e^{x^3} + C$$

$$y = 2 + Ce^{-x^3}$$

11. $(y + 1)\cos x\,dx = dy$

$y' = (y + 1)\cos x = y\cos x + \cos x$

$y' - (\cos x)y = \cos x$

Integrating factor: $e^{\int -\cos x\,dx} = e^{-\sin x}$

$y'e^{-\sin x} - (\cos x)e^{-\sin x}y = (\cos x)e^{-\sin x}$

$ye^{-\sin x} = \int (\cos x)e^{-\sin x}\,dx$

$= -e^{-\sin x} + C$

$y = -1 + Ce^{\sin x}$

12. $[(y - 1)\sin x]\,dx - dy = 0$

$y' - (\sin x)y = -\sin x$

Integrating factor: $e^{\int -\sin x\,dx} = e^{\cos x}$

$ye^{\cos x} = \int -\sin x e^{\cos x}\,dx = e^{\cos x} + C$

$y = 1 + Ce^{-\cos x}$

13. $y' + 3y = e^{3x}$

Integrating factor: $e^{\int 3\,dx} = e^{3x}$

$ye^{3x} = \int e^{3x}e^{3x}dx = \int e^{6x}dx = \frac{1}{6}e^{6x} + C$

$y = \frac{1}{6}e^{3x} + Ce^{-3x}$

14. $xy' + y = x^2 \ln x$

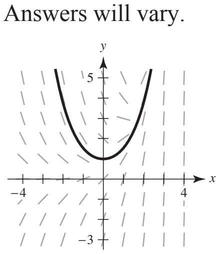
$y' + \frac{y}{x} = x\ln x$

Integrating factor: $e^{\int 1/x\,dx} = e^{\ln x} = x$

$xy = \int x^2 \ln x\,dx = \frac{x^3(3\ln x - 1)}{9} + C$

$y = \frac{x^2(3\ln x - 1)}{9} + \frac{C}{x}$

15. (a) Answers will vary.



(b) $\frac{dy}{dx} = e^x - y$

$\frac{dy}{dx} + y = e^x$ Integrating factor: $e^{\int dx} = e^x$

$e^x y' + e^x y = e^{2x}$

$(ye^x) = \int e^{2x}dx$

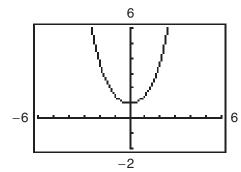
$ye^x = \frac{1}{2}e^{2x} + C$

$y(0) = 1 \Rightarrow 1 = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$

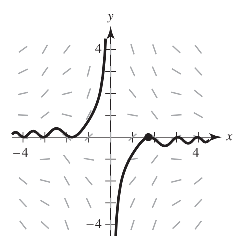
$ye^x = \frac{1}{2}e^{2x} + \frac{1}{2}$

$y = \frac{1}{2}e^x + \frac{1}{2}e^{-x} = \frac{1}{2}(e^x + e^{-x})$

(c)



16. (a)



(b) $y' + \frac{1}{x}y = \sin x^2$, $P(x) = \frac{1}{x}$, $Q(x) = \sin x^2$

$u(x) = e^{\int (1/x)\,dx} = e^{\ln x} = x$

$y'x + y = x\sin x^2$

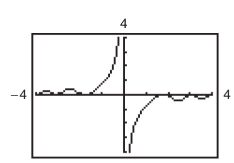
$yx = \int x\sin x^2\,dx = -\frac{1}{2}\cos x^2 + C$

$y = \frac{1}{x}\left[-\frac{1}{2}\cos x^2 + C\right]$

$0 = \frac{1}{\sqrt{\pi}}\left[-\frac{1}{2}\cos\pi + C\right] \Rightarrow C = -\frac{1}{2}$

$y = \frac{1}{x}\left[-\frac{1}{2}\cos x^2 - \frac{1}{2}\right]$

(c)



17. $y' + y = 6e^x$

Integrating factor: $e^{\int dx} = e^x$

$$ye^x = \int 6e^{2x}\,dx$$

$$y = \frac{1}{e^x}\left(3e^{2x} + C\right) = 3e^x + Ce^{-x}$$

Initial condition: $y(0) = 3,\ 3 = 3e^0 + Ce^0,\ C = 0$

Particular solution: $y = 3e^x$

18. $x^3y' + 2y = e^{1/x^2}$

$$y' + \left(\frac{2}{x^3}\right)y = \frac{1}{x^3}e^{1/x^2}$$

Integrating factor: $e^{\int(2/x^3)\,dx} = e^{-(1/x^2)}$

$$ye^{-1/x^2} = \int \frac{1}{x^3}\,dx = -\frac{1}{2x^2} + C_1$$

$$y = e^{1/x^2}\left(\frac{Cx^2 - 1}{2x^2}\right)$$

Initial condition: $y(1) = e,\ C = 3$

Particular solution: $y = e^{1/x^2}\left(\dfrac{3x^2 - 1}{2x^2}\right)$

19. $y' + y\tan x = \sec x + \cos x$

Integrating factor: $e^{\int \tan x\,dx} = e^{\ln|\sec x|} = \sec x$

$$y\sec x = \int \sec x(\sec x + \cos x)\,dx = \tan x + x + C$$

$$y = \sin x + x\cos x + C\cos x$$

Initial condition: $y(0) = 1,\ 1 = C$

Particular solution: $y = \sin x + (x + 1)\cos x$

20. $y' + y\sec x = \sec x$

Integrating factor:

$e^{\int \sec x\,dx} = e^{\ln|\sec x + \tan x|} = \sec x + \tan x$

$$y(\sec x + \tan x) = \int(\sec x + \tan x)\sec x\,dx$$

$$= \sec x + \tan x + C$$

$$y = 1 + \frac{C}{\sec x + \tan x}$$

Initial condition: $y(0) = 4,\ 4 = 1 + \dfrac{C}{1 + 0},\ C = 3$

Particular solution:

$$y = 1 + \frac{3}{\sec x + \tan x} = 1 + \frac{3\cos x}{1 + \sin x}$$

21. $y' + \left(\dfrac{1}{x}\right)y = 0$

Integrating factor: $e^{\int(1/x)\,dx} = e^{\ln|x|} = x$

Separation of variables:

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\int \frac{1}{y}\,dy = \int -\frac{1}{x}\,dx$$

$$\ln y = -\ln x + \ln C$$

$$\ln xy = \ln C$$

$$xy = C$$

Initial condition: $y(2) = 2,\ C = 4$

Particular solution: $xy = 4$

22. $y' + (2x - 1)y = 0$

Integrating factor: $e^{\int(2x-1)\,dx} = e^{x^2-x}$

$$ye^{x^2-x} = C$$

$$y = Ce^{x-x^2}$$

Separation of variables:

$$\int \frac{1}{y}\,dy = \int(1 - 2x)\,dx$$

$$\ln y + \ln C_1 = x - x^2$$

$$yC_1 = e^{x-x^2}$$

$$y = Ce^{x-x^2}$$

Initial condition: $y(1) = 2,\ 2 = C$

Particular solution: $y = 2e^{x-x^2}$

23. $x\,dy = (x + y + 2)\,dx$

$$\frac{dy}{dx} = \frac{x + y + 2}{x} = \frac{y}{x} + 1 + \frac{2}{x}$$

$$\frac{dy}{dx} - \frac{1}{x}y = 1 + \frac{2}{x} \qquad \text{Linear}$$

$$u(x) = e^{\int -(1/x)\,dx} = \frac{1}{x}$$

$$y = x\int\left(1 + \frac{2}{x}\right)\frac{1}{x}\,dx = x\int\left(\frac{1}{x} + \frac{2}{x^2}\right)dx$$

$$= x\left[\ln|x| + \frac{-2}{x} + C\right]$$

$$= -2 + x\ln|x| + Cx$$

$$y(1) = 10 = -2 + C \Rightarrow C = 12$$

$$y = -2 + x\ln|x| + 12x$$

24. $2xy' - y = x^3 - x$

$$\frac{dy}{dx} - \frac{1}{2x}y = \frac{x^2}{2} - \frac{1}{2} \qquad \text{Linear}$$

$$u(x) = e^{\int -(1/2x)\,dx} = \frac{1}{x^{1/2}}$$

$$y = x^{1/2}\int\left(\frac{x^2}{2} - \frac{1}{2}\right)\frac{1}{x^{1/2}}\,dx = x^{1/2}\int\left(\frac{x^{3/2}}{2} - \frac{x^{-1/2}}{2}\right)dx$$

$$= x^{1/2}\left[\frac{x^{5/2}}{5} - x^{1/2} + C\right]$$

$$= \frac{x^3}{5} - x + C\sqrt{x}$$

$$y(4) = 2 = \frac{64}{5} - 4 + 2C \Rightarrow C = -\frac{17}{5}$$

$$y = \frac{x^3}{5} - x - \frac{17}{5}\sqrt{x}$$

25. $\dfrac{dP}{dt} = kP + N$, N constant

$$\frac{dP}{kP + N} = dt$$

$$\int \frac{1}{kP + N}\,dP = \int dt$$

$$\frac{1}{k}\ln(kP + N) = t + C_1$$

$$\ln(kP + N) = kt + C_2$$

$$kP + N = e^{kt + C_2}$$

$$P = \frac{C_3e^{kt} - N}{k}$$

$$P = Ce^{kt} - \frac{N}{k}$$

When $t = 0$: $P = P_0$

$$P_0 = C - \frac{N}{k} \Rightarrow C = P_0 + \frac{N}{k}$$

$$P = \left(P_0 + \frac{N}{k}\right)e^{kt} - \frac{N}{k}$$

26. $\dfrac{dA}{dt} = rA + P$

$$\frac{dA}{rA + P} = dt$$

$$\int \frac{dA}{rA + P} = \int dt$$

$$\frac{1}{r}\ln(rA + P) = t + C_1$$

$$\ln(rA + P) = rt + C_2$$

$$rA + P = e^{rt + C_2}$$

$$A = \frac{C_3e^{rt} - P}{r}$$

$$A = Ce^{rt} - \frac{P}{r}$$

When $t = 0$: $A = 0$

$$0 = C - \frac{P}{r} \Rightarrow C = \frac{P}{r}$$

$$A = \frac{P}{r}\left(e^{rt} - 1\right)$$

27. (a) $A = \dfrac{P}{r}\left(e^{rt} - 1\right)$

$$A = \frac{275{,}000}{0.06}\left(e^{0.08(10)} - 1\right) \approx \$4{,}212{,}796.94$$

(b) $A = \dfrac{550{,}000}{0.05}\left(e^{0.059(25)} - 1\right) \approx \$31{,}424{,}909.75$

28. $1{,}000{,}000 = \dfrac{125{,}000}{0.08}\left(e^{0.08t} - 1\right)$

$$1.64 = e^{0.08t}$$

$$t = \frac{\ln(1.64)}{0.08} \approx 6.18 \text{ years}$$

29. (a) $\dfrac{dN}{dt} = k\,(75 - N)$

(b) $N' + kN = 75k$

Integrating factor: $e^{\int k\,dt} = e^{kt}$

$$N'e^{kt} + kNe^{kt} = 75\,ke^{kt}$$

$$\left(Ne^{kt}\right)' = 75\,ke^{kt}$$

$$Ne^{kt} = \int 75\,ke^{kt} = 75\,e^{kt} + C$$

$$N = 75 + Ce^{-kt}$$

(c) For $t = 1, N = 20$:

$$20 = 75 + Ce^{-k} \Rightarrow -55 = Ce^{-k}$$

For $t = 20, N = 35$:

$$35 = 75 + Ce^{-20k} \Rightarrow -40 = Ce^{-20k}$$

$$\frac{55}{40} = \frac{Ce^{-k}}{Ce^{-20k}} \Rightarrow e^{19k} = \frac{11}{8} \Rightarrow k = \frac{1}{19}\ln\left(\frac{11}{8}\right) \approx 0.0168$$

$$Ce^{-k} = -55$$

$$C = -55e^{k} \approx -55.9296$$

$$N = 75 - 55.9296\,e^{-0.0168t}$$

30. (a) $\dfrac{dQ}{dt} = q - kQ$, q constant

(b) $Q' + kQ = q$

Let $P(t) = k$, $Q(t) = q$, then the integrating factor is $u(t) = e^{kt}$.

$$Q = e^{-kt}\int qe^{kt}\,dt = e^{-kt}\left(\frac{q}{k}e^{kt} + C\right) = \frac{q}{k} + Ce^{-kt}$$

When $t = 0$: $Q = Q_0$

$$Q_0 = \frac{q}{k} + C \Rightarrow C = Q_0 - \frac{q}{k}$$

$$Q = \frac{q}{k} + \left(Q_0 - \frac{q}{k}\right)e^{-kt}$$

(c) $\displaystyle\lim_{t\to\infty} Q = \frac{q}{k}$

31. From Example 3, $v = \dfrac{-mg}{k}\left(1 - e^{-kt/m}\right)$, solution

$g = 9.8$ m/sec^2, mg $= (9.8)(4) = 39.2$,

$$v(5) = -31 = \frac{-39.2}{k}\left(1 - e^{-k(5)/4}\right)$$

Using a graphing utility, $k \approx 0.7984$, and

$$v = \frac{-39.2}{0.7984}\left(1 - e^{-0.7984t/4}\right)$$
$$= -49.0982\left(1 - e^{-0.1996t}\right)$$

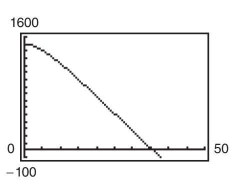
The limiting value is
$\displaystyle\lim_{t\to\infty} v = -49.0982$ meters per second.

32. $s(t) = \int v(t)\,dt$

$$= \int -49.0982\left(1 - e^{-0.1996t}\right)dt$$
$$= -49.0982t - 245.9830e^{-0.1996t} + C$$

$s(0) = 1500 = -245.9830 + C \Rightarrow C = 1745.98$

$s(t) = -49.0982t - 245.983e^{-0.1996t} + 1745.98$



$s(t) = 0$ when $t \approx 35.6$ seconds.

So, the velocity of the object when it reaches ground level is about $v(35.6) \approx -49.05$ meters per second.

33. $L\dfrac{dI}{dt} + RI = E_0$, $I' + \dfrac{R}{L}I = \dfrac{E_0}{L}$

Integrating factor: $e^{\int(R/L)\,dt} = e^{Rt/L}$

$$I\,e^{Rt/L} = \int \frac{E_0}{L}e^{Rt/L}\,dt = \frac{E_0}{R}e^{Rt/L} + C$$

$$I = \frac{E_0}{R} + Ce^{-Rt/L}$$

34. $I(0) = 0$, $E_0 = 120$ volts, $R = 600$ ohms, $L = 4$ henrys

$$I = \frac{E_0}{R} + Ce^{-Rt/L}$$

$$(0) = \frac{120}{600} + C \Rightarrow C = -\frac{1}{5}$$

$$I = \frac{1}{5} - \frac{1}{5}e^{-150t}$$

$$\lim_{t\to\infty} I = \frac{1}{5} \text{ amp}$$

$$(0.90)\frac{1}{5} = 0.18 = \frac{1}{5}\left(1 - e^{-150t}\right)$$
$$0.9 = 1 - e^{-150t}$$
$$e^{-150t} = 0.1$$
$$-150t = \ln(0.1)$$
$$t = \frac{\ln(0.1)}{-150} \approx 0.0154 \text{ sec}$$

35. Let Q be the number of pounds of concentrate in the solution at any time t. Because the number of gallons of solution in the tank at any time t is $v_0 + (r_1 - r_2)t$ and because the tank loses r_2 gallons of solution per minute, it must lose concentrate at the rate

$$\left[\frac{Q}{v_0 + (r_1 - r_2)t}\right]r_2.$$

The solution gains concentrate at the rate r_1q_1. Therefore, the net rate of change is

$$\frac{dQ}{dt} = q_1r_1 - \left[\frac{Q}{v_0 + (r_1 - r_2)t}\right]r_2$$

or

$$\frac{dQ}{dt} + \frac{r_2Q}{v_0 + (r_1 - r_2)t} = q_1r_1.$$

36. From Exercise 35, and using $r_1 = r_2 = r$,

$$\frac{dQ}{dt} + \frac{rQ}{v_0} = q_1r.$$

37. (a) From Exercise 35,

$$\frac{dQ}{dt} + \frac{r_2 Q}{u_0 + (r_1 - r_2)t} = q_1 r_1$$

You have $Q(0) = q_0 = 25$, $q_1 = 0$, $v_0 = 200$, and $r_1 = r_2 = 10$. Hence, the linear differential equation is

$$\frac{dQ}{dt} + \frac{1}{20}Q = 0.$$

By separating variables,

$$\int \frac{dQ}{Q} = -\int \frac{1}{20}\,dt$$

$$\ln Q = -\frac{1}{20}t + \ln C_1$$

$$Q = Ce^{-\frac{1}{20}t}.$$

The initial condition $Q(0) = 25$ implies that $C = 25$. Hence, $Q = 25e^{-\frac{1}{20}t}$.

(b) $15 = 25e^{-\frac{1}{20}t} \Rightarrow \frac{3}{5} = e^{-\frac{1}{20}t} \Rightarrow \ln\left(\frac{3}{5}\right) = -\frac{1}{20}t$

$\Rightarrow t = -20\ln\left(\frac{3}{5}\right) \approx 10.2$ min

(c) $\lim_{t\to\infty} Q = \lim_{t\to\infty} 25e^{-\frac{1}{20}t} = 0$

38. (a) The volume of the solution in the tank is given by $v_0 + (r_1 - r_2)t$. Therefore, $100 + (5 - 3)t = 200$ or $t = 50$ minutes.

(b) $Q' + \dfrac{r_2 Q}{v_0 + (r_1 - r_2)t} = q_1 r_1$

$Q(0) = q_0 = 0$, $q_1 = 0.5$, $v_0 = 100$, $r_1 = 5$, $r_2 = 3$, $Q' + \dfrac{3}{100 + 2t}Q = 2.5$

Integrating factor: $e^{\int[3/(100+2t)]\,dt} = (50 + t)^{3/2}$

$$Q(50 + t)^{3/2} = \int 2.5(50 + t)^{3/2}\,dt = (50 + t)^{5/2} + C$$

$$Q = (50 + t) + C(50 + t)^{-3/2}$$

Initial condition:

$Q(0) = 0$, $0 = 50 + C(50^{-3/2})$, $C = -50^{5/2}$

Particular solution:

$$Q = (50 + t) - 50^{-5/2}(50 + t)^{-3/2}$$

$$Q(50) = 100 - 50^{5/2}(100)^{-3/2}$$

$$= 100 - \frac{25}{\sqrt{2}} \approx 82.32 \text{ lb}$$

(c) The volume of the solution is given by $v_0 + (r_1 - r_2)t = 100 + (5 - 3)t = 200 \Rightarrow t = 50$ minutes.

$$Q' + \frac{r_2 Q}{v_0 + (r_1 - r_2)t} = q_1 r_1$$

$Q(0) = q_0 = 0$, $q_1 = 1$, $v_0 = 100$, $r_1 = 5$, $r_2 = 3$

$$Q' + \frac{3Q}{100 + 2t} = 5$$

Integrating factor is $(50 + t)^{3/2}$.

$$Q(50 + t)^{3/2} = \int 5(50 + t)^{3/2}\,dt = 2(50 + t)^{5/2} + C$$

$$Q = 2(50 + t) + C(50 + t)^{-3/2}$$

$Q(0) = 0$:

$0 = 100 + C(50)^{-3/2} \Rightarrow C = -100(50)^{3/2} = -2(50)^{5/2}$

$Q = 2(50 + t) - 2(50)^{5/2}(50 + t)^{-3/2}$

When $t = 50$:

$$Q = 200 - 2(50)^{5/2}(100)^{-3/2} = 200 - \frac{50}{\sqrt{2}}$$

≈ 164.64 lb (double the answer to part (b))

39. $y' + P(x)y = Q(x)$

Integrating factor: $u = e^{\int P(x)\,dx}$

$y'u + P(x)yu = Q(x)u$

$(uy)' = Q(x)u$

so $u'(x) = P(x)u$

Answer (a)

40. (a) At $t = 0$, $Q = 20$ pounds.

(b) The rate of solution withdrawn is greater.

(c) At $t = 25$, $Q = 0$. It takes 25 minutes to empty the tank.

41. Separation of variables:

$$\frac{dy}{dx} = x - 3xy = x(1 - 3y)$$

$$\frac{dy}{1 - 3y} = x\,dx$$

Integrating factor. $P(x) = 3x$, $Q(x) = x$, $u(x) = e^{\int 3x\,dx}$

42. You can omit the constant of integration because multiplying by the integrating factor $u(x) = e^{\int P(x)\,dx}$ always produces a derivative on the left:

$$y'e^{\int P(x)\,dx} + P(x)ye^{\int P(x)\,dx} = \left[ye^{\int P(x)\,dx}\right]'$$

43. $y' - 2x = 0$

$$\int dy = \int 2x\,dx$$

$$y = x^2 + C$$

Matches (c).

44. $y' - 2y = 0$

$$\int \frac{dy}{y} = \int 2\,dx$$

$$\ln y = 2x + C_1$$

$$y = Ce^{2x}$$

Matches (d).

45. $y' - 2xy = 0$

$$\int \frac{dy}{y} = \int 2x\,dx$$

$$\ln y = x^2 + C_1$$

$$y = Ce^{x^2}$$

Matches (a).

46. $y' - 2xy = x$

$$\int \frac{dy}{2y+1} = \int x\,dx$$

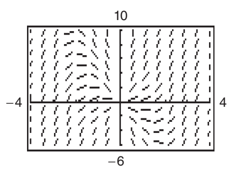
$$\frac{1}{2}\ln(2y+1) = \frac{1}{2}x^2 + C_1$$

$$2y + 1 = C_2e^{x^2}$$

$$y = -\frac{1}{2} + Ce^{x^2}$$

Matches (b).

47. (a)



(b) $\dfrac{dy}{dx} - \dfrac{1}{x}y = x^2$

Integrating factor: $e^{-1/x\,dx} = e^{-\ln x} = \dfrac{1}{x}$

$$\frac{1}{x}y' - \frac{1}{x^2}y = x$$

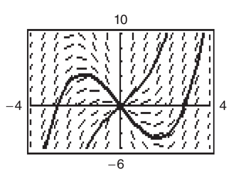
$$\left(\frac{1}{x}y\right) = \int x\,dx = \frac{x^2}{2} + C$$

$$y = \frac{x^3}{2} + Cx$$

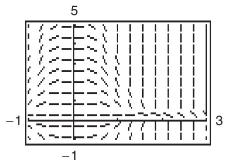
$(-2, 4)$: $4 = \dfrac{-8}{2} - 2C \Rightarrow C = -4 \Rightarrow y = \dfrac{x^3}{2} - 4x = \dfrac{1}{2}x(x^2 - 8)$

$(2, 8)$: $8 = \dfrac{8}{2} + 2C \Rightarrow C = 2 \Rightarrow y = \dfrac{x^3}{2} + 2x = \dfrac{1}{2}x(x^2 + 4)$

(c)



48. (a)



(b) $y' + 4x^3y = x^3$

Integrating factor: $e^{\int 4x^3\,dx} = e^{x^4}$

$$y'e^{x^4} + 4x^3ye^{x^4} = x^3e^{x^4}$$

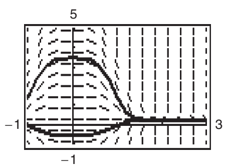
$$ye^{x^4} = \int x^3e^{x^4}\,dx = \tfrac{1}{4}e^{x^4} + C$$

$$y = \tfrac{1}{4} + Ce^{-x^4}$$

$\left(0, \tfrac{7}{2}\right)$: $\tfrac{7}{2} = \tfrac{1}{4} + C \Rightarrow C = \tfrac{13}{4} \Rightarrow y = \tfrac{1}{4} + \tfrac{13}{4}e^{-x^4}$

$\left(0, -\tfrac{1}{2}\right)$: $-\tfrac{1}{2} = \tfrac{1}{4} + C \Rightarrow C = -\tfrac{3}{4} \Rightarrow y = \tfrac{1}{4} - \tfrac{3}{4}e^{-x^4}$

(c)



49. $e^{2x+y}\,dx - e^{x-y}\,dy = 0$

Separation of variables:

$$e^{2x}e^{y}\,dx = e^{x}e^{-y}\,dy$$

$$\int e^{x}\,dx = \int e^{-2y}\,dy$$

$$e^{x} = -\tfrac{1}{2}e^{-2y} + C_1$$

$$2e^{x} + e^{-2y} = C$$

50. $y'\cos x^2 + \dfrac{y\cos x^2}{x} = \sec x^2$

Linear: $y' + \dfrac{y}{x} = \sec^2 x^2$

Integrating factor: $e^{\int 1/x\,dx} = e^{\ln x} = x$

$$xy' + y = x\sec^2 x^2$$

$$(xy)' = x\sec^2 x^2$$

$$xy = \int x\sec^2 x^2\,dx = \frac{1}{2}\tan^2 x^2 + C$$

$$y = \frac{1}{2x}\tan^2 x^2 + \frac{C}{x}$$

51. $(y\cos x - \cos x)\,dx + dy = 0$

Separation of variables:

$$\int \cos x\,dx = \int \frac{-1}{y-1}\,dy$$

$$\sin x = -\ln(y-1) + \ln C$$

$$\ln(y-1) = -\sin x + \ln C$$

$$y = Ce^{-\sin x} + 1$$

52. $y' = 2x\sqrt{1-y^2}$

Separation of variables:

$$\int \frac{1}{\sqrt{1-y^2}}\,dy = \int 2x\,dx$$

$$\arcsin y = x^2 + C$$

$$y = \sin\left(x^2 + C\right)$$

53. $\left(2y - e^x\right)dx + x\,dy = 0$

Linear: $y' + \left(\dfrac{2}{x}\right)y = \dfrac{1}{x}e^x$

Integrating factor: $e^{\int (2/x)\,dx} = e^{\ln x^2} = x^2$

$$yx^2 = \int x^2\frac{1}{x}e^x\,dx = e^x(x-1) + C$$

$$y = \frac{e^x}{x^2}(x-1) + \frac{C}{x^2}$$

54. $(x + y)\,dx - x\,dy = 0$

Linear: $y' - \dfrac{1}{x}y = 1$

Integrating factor: $e^{\int -(1/x)\,dx} = e^{\ln\left|x^{-1}\right|} = \dfrac{1}{x}$

$$y\frac{1}{x} = \int \frac{1}{x}\,dx = \ln|x| + C$$

$$y = x\left(\ln|x| + C\right)$$

55. $3(y - 4x^2)\,dx = -x\,dy$

$$x\frac{dy}{dx} = -3y + 12x^2$$

$$y' + \frac{3}{x}y = 12x$$

Integrating factor: $e^{\int (3/x)\,dx} = e^{3\ln x} = x^3$

$$y'x^3 + \frac{3}{x}x^3 y = 12x(x^3) = 12x^4$$

$$yx^3 = \int 12x^4\,dx = \frac{12}{5}x^5 + C$$

$$y = \frac{12}{5}x^2 + \frac{C}{x^3}$$

56. $x\,dx + (y + e^y)(x^2 + 1)\,dy = 0$

Separation of variables:

$$\int \frac{x}{x^2 + 1}\,dx = \int -(y + e^y)\,dy$$

$$\frac{1}{2}\ln(x^2 + 1) = -\frac{1}{2}y^2 - e^y + C_1$$

$$\ln(x^2 + 1) + y^2 + 2e^y = C$$

57. $y' + 3x^2 y = x^2 y^3$

$n = 3, Q = x^2, P = 3x^2$

$$y^{-2}e^{\int(-2)3x^2\,dx} = \int (-2)x^2 e^{\int(-2)3x^2\,dx}\,dx$$

$$y^{-2}e^{-2x^3} = -\int 2x^2 e^{-2x^3}\,dx$$

$$y^{-2}e^{-2x^3} = \frac{1}{3}e^{-2x^3} + C$$

$$y^{-2} = \frac{1}{3} + Ce^{2x^3}$$

$$\frac{1}{y^2} + Ce^{2x^3} + \frac{1}{3}$$

58. $y' + xy = xy^{-1}$

$n = -1, Q = x, P = x, e^{\int 2x\,dx} = e^{x^2}$

$$y^2 e^{x^2} = \int 2xe^{x^2}\,dx = e^{x^2} + C$$

$$y^2 = 1 + Ce^{-x^2}$$

59. $y' + \left(\frac{1}{x}\right)y = xy^2$

$n = 2, Q = x, P = x^{-1}$

$e^{\int -(1/x)\,dx} = e^{-\ln|x|} = x^{-1}$

$$y^{-1}x^{-1} = \int -x(x^{-1})\,dx = -x + C$$

$$\frac{1}{y} = -x^2 + Cx$$

$$y = \frac{1}{Cx - x^2}$$

60. $y' + \left(\frac{1}{x}\right)y = x\sqrt{y}$

$n = \frac{1}{2}, Q = x, P = x^{-1}$

$e^{(1/2)(1/x)\,dx} = e^{(1/2)\ln x} = \sqrt{x}$

$$y^{1/2}x^{1/2} = \int \frac{1}{2}x^{1/2}(x)\,dx$$

$$= \frac{1}{5}x^{5/2} + C_1 = \frac{x^{5/2} + C}{5}$$

$$y = \frac{(x^{5/2} + C)^2}{25x}$$

61. $xy' + y = xy^3$

$$y' + \frac{1}{x}y = y^3$$

$n = 3, Q = 1, P = \frac{1}{x},\quad e^{\int \frac{-2}{x}\,dx} = e^{-2\ln x} = x^{-2}$

$$y^{-2}x^{-2} = \int -2x^{-2}\,dx + C = 2x^{-1} + C$$

$$y^{-2} = 2x + Cx^2$$

$$y^2 = \frac{1}{2x + Cx^2} \quad \text{or} \quad \frac{1}{y^2} = 2x + Cx^2$$

62. $y' - y = y^3$

$n = 3, P = -1, Q = 1,\quad e^{\int -2(-1)\,dx} = e^{2x}$

$$y^{-2}e^{2x} = \int (-2)e^{2x}\,dx = -e^{2x} + C$$

$$y^{-2} = -1 + Ce^{-2x}$$

$$y^2 = \frac{1}{-1 + Ce^{-2x}}$$

63. $y' - y = e^x\sqrt[3]{y}, n = \frac{1}{3}, Q = e^x, P = -1$

$e^{\int -(2/3)\,dx} = e^{-(2/3)x}$

$$y^{2/3}e^{-(2/3)x} = \int \tfrac{2}{3}e^x e^{-(2/3)x}\,dx = \int \tfrac{2}{3}e^{(1/3)x}\,dx$$

$$y^{2/3}e^{-(2/3)x} = 2e^{(1/3)x} + C$$

$$y^{2/3} = 2e^x + Ce^{2x/3}$$

64. $yy' - 2y^2 = e^x$

$y' - 2y = e^x y^{-1}$

$n = -1, Q = e^x, P = -2$

$e^{\int 2(-2)\,dx} = e^{-4x}$

$y^2 e^{-4x} = \int 2e^{-4x}e^x\,dx = -\frac{2}{3}e^{-3x} + C$

$y^2 = -\frac{2}{3}e^x + Ce^{4x}$

65. False. The equation contains $\sqrt{y}$.

66. True. $y' + (x - e^x)y = 0$ is linear.

Review Exercises for Chapter 6

1. $y = x^3, y' = 3x^2$

$2xy' + 4y = 2x(3x^2) + 4(x^3) = 10x^3.$

Yes, it is a solution.

2. $y = 2\sin 2x$

$y' = 4\cos 2x$

$y'' = -8\sin 2x$

$y''' = -16\cos 2x$

$y''' - 8y = -16\cos 2x - 8(2\sin 2x) \neq 0$

Not a solution

3. $\dfrac{dy}{dx} = 4x^2 + 7$

$y = \int(4x^2 + 7)\,dx = \dfrac{4x^3}{3} + 7x + C$

4. $\dfrac{dy}{dx} = \dfrac{6 - x}{3x} = \dfrac{2}{x} - \dfrac{1}{3}$

$y = \int\left(\dfrac{2}{x} - \dfrac{1}{3}\right)dx = 2\ln x - \dfrac{1}{3}x + C$

5. $\dfrac{dy}{dx} = \cos 2x$

$y = \int\cos 2x\,dx = \dfrac{1}{2}\sin 2x + C$

6. $\dfrac{dy}{dx} = 8\csc x\cot x$

$y = 8\int\csc x\cot x\,dx$

$= -8\csc x + C$

7. $\dfrac{dy}{dx} = e^{2-x}$

$y = \int e^{2-x}\,dx = -e^{2-x} + C$

8. $\dfrac{dy}{dx} = 2e^{3x}$

$y = \int 2e^{3x}\,dx = \dfrac{2}{3}e^{3x} + C$

9. $\dfrac{dy}{dx} = 2x - y$

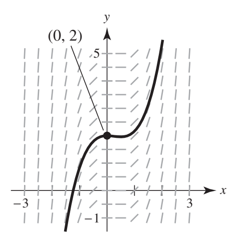
x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	-10	-4	-4	0	2	8

10. $\dfrac{dy}{dx} = x\sin\left(\dfrac{\pi y}{4}\right)$

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	-4	0	0	0	-4	0

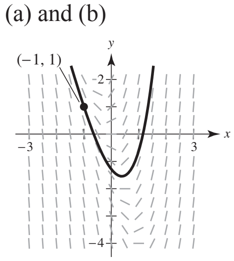
11. $y' = 2x^2 - x, \quad (0, 2)$

(a) and (b)



12. $y' = y + 4x, \quad (-1, 1)$

(a) and (b)



13. $y' = x - y, y(0) = 4, n = 10, h = 0.05$

$y_1 = y_0 + hF(x_0, y_0) = 4 + (0.05)(0 - 4) = 3.8$

$y_2 = y_1 + hF(x_1, y_1) = 3.8 + (0.05)(0.05 - 3.8) = 3.6125$, etc.

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
y_n	4	3.8	3.6125	3.437	3.273	3.119	2.975	2.842	2.717	2.601	2.494

14. $y' = 5x - 2y, y(0) = 2, n = 10, h = 0.1$

$y_1 = y_0 + hF(x_0, y_0) = 2 + (0.1)(5(0) - 2(2)) = 1.6$

$y_2 = y_1 + hF(x_1, y_1) = 1.6 + (0.1)(5(0.1) - 2(1.6)) = 1.33$, etc.

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	2	1.6	1.33	1.164	1.081	1.065	1.102	1.182	1.295	1.436	1.599

15. $\frac{dy}{dx} = 6x - x^3$

$$y = \int (6x - x^3)\,dx = 3x^2 - \frac{x^4}{4} + C$$

16.
$$\frac{dy}{dx} - 3y = 5$$
$$\frac{dy}{dx} = 3y + 5$$
$$\int \frac{dy}{3y + 5} = \int dx$$
$$\frac{1}{3}\ln|3y + 5| = x + C_1$$
$$\ln|3y + 5| = 3x + 3C_1$$
$$3y + 5 = e^{3(x + C_1)}$$
$$3y = -5 + Ce^{3x}$$
$$y = \frac{1}{3}(-5 + Ce^{3x})$$

17.
$$\frac{dy}{dx} = (y - 1)^2$$
$$\int (y - 1)^{-2}\,dy = \int dx$$
$$-(y - 1)^{-1} = x + C$$
$$-(y - 1) = \frac{1}{x + C}$$
$$y = \frac{-1}{x + C} + 1$$

18.
$$\frac{dy}{dx} = \frac{x}{x^2 + 2}$$
$$\int dy = \int \frac{x}{x^2 + 2}\,dx$$
$$y = \frac{1}{2}\ln(x^2 + 2) + C$$

19. $(2 + x)y' - xy = 0$
$$(2 + x)\frac{dy}{dx} = xy$$
$$\frac{1}{y}\,dy = \frac{x}{2 + x}dx$$
$$\frac{1}{y}\,dy = \left(1 - \frac{2}{2 + x}\right)dx$$
$$\ln|y| = x - 2\ln|2 + x| + C_1$$
$$y = Ce^x(2 + x)^{-2} = \frac{Ce^x}{(2 + x)^2}$$

20. $xy' - (x + 1)y = 0$
$$x\frac{dy}{dx} = (x + 1)y$$
$$\int \frac{dy}{y} = \int \frac{x + 1}{x}\,dx$$
$$\ln|y| = x + \ln|x| + C_1$$
$$y = Cxe^x$$

21. $\sqrt{x+1}\,y' - y = 0$

$$\sqrt{x+1}\,\frac{dy}{dx} = y$$

$$\frac{1}{y}\,dy = (x+1)^{-1/2}\,dx$$

$$\ln y = 2\sqrt{x+1} + C_1$$

$$y = e^{2\sqrt{x+1}+C_1} = Ce^{2\sqrt{x+1}}$$

22. $y' + \sqrt{x}\,y = 9\sqrt{x}$

$$y' = 9\sqrt{x} - \sqrt{x}\,y$$

$$y' = \sqrt{x}\,(9 - y)$$

$$\frac{y'}{9-y} = \sqrt{x}$$

$$\int \frac{dy}{9-y} = \int \sqrt{x}\,dx$$

$$-\ln|y-9| = \frac{2}{3}x^{3/2} + C_1$$

$$y - 9 = e^{-2/3\,x^{3/2}+C_1} = Ce^{-2/3\,x^{3/2}}$$

$$y = 9 + Ce^{-2/3\,x^{3/2}}$$

23. $\dfrac{dy}{dt} = \dfrac{k}{t^3}$

$$\int dy = \int kt^{-3}\,dt$$

$$y = -\frac{k}{2t^2} + C$$

24. $\dfrac{dy}{dt} = k(50 - t)$

$$\int dy = \int k(50-t)dt = \int (50k - kt)dt$$

$$y = 50kt - \frac{k}{2}t^2 + C$$

(Alternate form: $y = -\dfrac{k}{2}(50-t)^2 + C_1$ **)**

25. $y = Ce^{kt}$

$\left(0, \frac{3}{4}\right)$: $\frac{3}{4} = C$

$(5, 5)$: $5 = \frac{3}{4}e^{k(5)}$

$$\frac{20}{3} = e^{5k}$$

$$k = \frac{1}{5}\ln\left(\frac{20}{3}\right)$$

$$y = \frac{3}{4}e^{[\ln(20/3)/5]t} \approx \frac{3}{4}e^{0.379t}$$

26. $y = Ce^{kt}$

$(0, 5)$: $5 = C$

$\left(5, \frac{1}{6}\right)$: $\frac{1}{6} = 5e^{k(5)}$

$$\frac{1}{30} = e^{5k}$$

$$k = \frac{1}{5}\ln\frac{1}{30} = -\frac{1}{5}\ln 30$$

$$y = 5e^{[-\ln(30)/5]t} \approx 5e^{-0.6802t}$$

27. $y = Ce^{kt}$

$\left(2, \frac{3}{2}\right)$: $\frac{3}{2} = Ce^{2k} \Rightarrow C = \frac{3}{2}e^{-2k}$

$(4, 5)$: $5 = Ce^{4k} = \left(\frac{3}{2}e^{-2k}\right)e^{4k} = \frac{3}{2}e^{2k}$

$$\frac{10}{3} = e^{2k} \Rightarrow k = \frac{1}{2}\ln\left(\frac{10}{3}\right)$$

So, $C = \frac{3}{2}e^{-2(1/2)\ln(10/3)} = \frac{3}{2}\left(\frac{3}{10}\right) = \frac{9}{20}$.

$$y = \frac{9}{20}e^{1/2\ln(10/3)t} \approx \frac{9}{20}e^{0.602t}$$

28. $y = Ce^{kt}$

$(1, 4)$: $4 = Ce^{k(1)} = Ce^{k} \Rightarrow C = 4e^{-k}$

$(4, 1)$: $1 = Ce^{k(4)} = Ce^{4k}$

$$1 = \left(4e^{-k}\right)\left(e^{4k}\right) = 4e^{3k}$$

$$\frac{1}{4} = e^{3k} \Rightarrow k = \frac{1}{3}\ln\frac{1}{4} = -\frac{1}{3}\ln 4 \approx -0.4621$$

So, $C = 4e^{-k} = 4e^{0.4621} \approx 6.3496$.

$$y = 6.3496e^{-0.4621t}$$

29. $\dfrac{dP}{dh} = kP, \quad P(0) = 30$

$$P(h) = 30e^{kh}$$

$$P(18{,}000) = 30e^{18{,}000k} = 15$$

$$k = \frac{\ln(1/2)}{18{,}000} = \frac{-\ln 2}{18{,}000}$$

$$P(h) = 30e^{-(h\ln 2)/18{,}000}$$

$$P(35{,}000) = 30e^{-(35{,}000\ln 2)/18{,}000} \approx 7.79 \text{ inches}$$

30. $y = Ce^{kt} = 15e^{kt}$

$$7.5 = 15e^{k(1599)}$$

$$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right) \approx -0.000433$$

When $t = 750$, $y = 15e^{-0.000433(750)} \approx 10.84$ grams.

31. $$\begin{aligned} P &= Ce^{0.0185t} \\ 2C &= Ce^{0.0185t} \\ 2 &= e^{0.0185t} \\ \ln 2 &= 0.0185t \\ t &= \frac{\ln 2}{0.0185} \approx 37.5 \text{ years} \end{aligned}$$

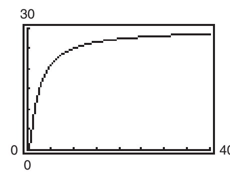
32. $A = Pe^{rt} = 400e^{0.02(11)} \approx \$498.43$

33. $S = Ce^{k/t}$

(a) $$\begin{aligned} S &= 5 \text{ when } t = 1. \\ 5 &= Ce^{k} \\ \lim_{t\to\infty} Ce^{k/t} &= C = 30 \\ 5 &= 30e^{k} \\ k &= \ln \tfrac{1}{6} \approx -1.7918 \\ S &= 30e^{-1.7918/t} \end{aligned}$$

(b) When $t = 5$, $S \approx 20.9646$ which is 20,965 units.

(c)



34. $S = 25\left(1 - e^{kt}\right)$

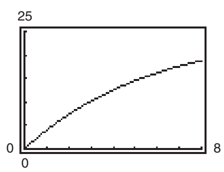
(a) $4 = 25\left(1 - e^{k(1)}\right) \Rightarrow 1 - e^{k} = \frac{4}{25} \Rightarrow e^{k} = \frac{21}{25} \Rightarrow k = \ln\left(\frac{21}{25}\right) \approx -0.1744$

$S = 25\left(1 - e^{-0.1744t}\right)$

(b) 25,000 units $\left(\lim_{t\to\infty} S = 25\right)$

(c) When $t = 5$, $S \approx 14.545$ which is 14,545 units.

(d)



35. $$\begin{aligned} \frac{dy}{dx} &= \frac{5x}{y} \\ \int y\,dy &= \int 5x\,dx \\ \frac{y^2}{2} &= \frac{5x^2}{2} + C_1 \\ y^2 &= 5x^2 + C \end{aligned}$$

36. $$\begin{aligned} \frac{dy}{dx} &= \frac{x^3}{2y^2} \\ \int 2y^2\,dy &= \int x^3\,dx \\ \frac{2y^3}{3} &= \frac{x^4}{4} + C_1 \\ 8y^3 &= 3x^2 + C \end{aligned}$$

37. $$\begin{aligned} y'e^{y-3x} &= e^{x+2y} \\ y'e^{y}e^{-3x} &= e^{x}e^{2y} \\ \int e^{-y}\,dy &= \int e^{4x}\,dx \\ -e^{-y} &= \frac{1}{4}e^{4x} + C \\ e^{-y} &= C - \frac{1}{4}e^{4x} \\ -y &= \ln\left(C - \frac{1}{4}e^{4x}\right) \\ y &= -\ln\left(C - \frac{1}{4}e^{4x}\right) \end{aligned}$$

38. $y' - e^y \sin x = 0$

$$\frac{dy}{dx} = e^y \sin x$$

$$\int e^{-y}\,dy = \int \sin x\,dx$$

$$-e^{-y} = -\cos x + C_1$$

$$e^y = \frac{1}{\cos x + C} \qquad (C = -C_1)$$

$$y = \ln\left|\frac{1}{\cos x + C}\right| = -\ln\left|\cos x + C\right|$$

39. $y^3y' - 3x = 0,\ y(2) = 2$

$$y^3\frac{dy}{dx} = 3x$$

$$\int y^3\,dy = \int 3x\,dx$$

$$\frac{y^4}{4} = \frac{3x^2}{2} + C_1$$

$$y^4 = 6x^2 + C$$

Initial condition: $y(2) = 2: 16 = 24 + C$

$$C = -8$$

Particular solution: $y^4 = 6x^2 - 8$

40. $yy' - 5e^{2x} = 0,\ y(0) = -3$

$$y\frac{dy}{dx} = 5e^{2x}$$

$$\int y\,dy = \int 5e^{2x}\,dx$$

$$\frac{y^2}{2} = \frac{5}{2}e^{2x} + C_1$$

$$y^2 = 5e^{2x} + C$$

Initial condition: $y(0) = -3: (-3)^2 = 5 + C$

$$C = 4$$

Particular solution: $y^2 = 5e^{2x} + 4$

41. $y^3(x^4 + 1)y' - x^3(y^4 + 1) = 0,\ y(0) = 1$

$$y^3(x^4 + 1)\frac{dy}{dx} = x^3(y^4 + 1)$$

$$\int \frac{y^3}{y^4 + 1}\,dy = \int \frac{x^3}{x^4 + 1}\,dx$$

$$\frac{1}{4}\ln(y^4 + 1) = \frac{1}{4}\ln(x^4 + 1) + \frac{1}{4}\ln C_1$$

$$\ln(y^4 + 1) = \ln\left[C(x^4 + 1)\right]$$

$$y^4 + 1 = C(x^4 + 1)$$

Initial condition: $y(0) = 1: 1 + 1 = C(0 + 1)$

$$C = 2$$

Particular solution: $y^4 + 1 = 2(x^4 + 1)$

$$y^4 = 2x^4 + 1$$

42. $y' + \sin x \cos x = 0,\ y(\pi) = -2$

$$\frac{dy}{dx} = -\sin x \cos x$$

$$\int dy = \int -\sin x \cos x\,dx$$

$$y = \frac{\cos^2 x}{2} + C$$

Initial condition:

$y(\pi) = -2: -2 = \frac{1}{2} + C \Rightarrow C = -\frac{5}{2}$

Particular solution: $y = \frac{1}{2}\cos^2 x - \frac{5}{2}$

43. $y' = \frac{2x}{y}$

$$y\,dy = 2x\,dx$$

$$\int y\,dy = \int 2x\,dx$$

$$\frac{y^2}{2} = x^2 + C$$

$(1, 3)$: $\frac{9}{2} = 1 + C \Rightarrow C = \frac{7}{2}$

$$\frac{y^2}{2} = x^2 + \frac{7}{2}$$

$$y^2 = 2x^2 + 7$$

44. $y' = \frac{y}{8x}$

$$\int \frac{dy}{y} = \int \frac{1}{8x}\,dx$$

$$\ln|y| = \frac{1}{8}\ln|x| + C_1 = \ln|x|^{1/8} + \ln C$$

$$y = Cx^{1/8}$$

$(1, -2)$: $-2 = C$

$$y = -2x^{1/8}$$

45. Given family (hyperbolas): $5x^2 - 4y^2 = C$

$$10x - 8yy' = 0$$

$$y' = \frac{5x}{4y}$$

Orthogonal trajectory:

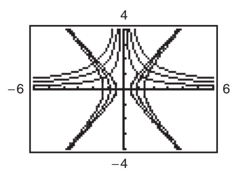
$$y' = \frac{-4y}{5x}$$

$$\int \frac{5}{y}\,dy = \int \frac{-4}{x}\,dx$$

$$5\ln|y| = -4\ln|x| + K_1 = -4\ln|x| + \ln K$$

$$y^5 = Kx^{-4}$$

$$y = Kx^{-4/5}$$



46. Given family: $x^3 = Cy$, $\quad C = \frac{x^3}{y}$

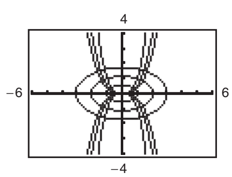
$$3x^2 = Cy' = \left(\frac{x^3}{y}\right)y' \Rightarrow y' = \frac{3y}{x}$$

Orthogonal trajectory: $\quad y' = \frac{-x}{3y}$

$$\int 3y\,dy = \int -x\,dx$$

$$3\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$3y^2 + x^2 = K$$



47. $P(t) = \frac{5250}{1 + 34e^{-0.55t}}$

(a) $k = 0.55$

(b) $L = 5250$

(c) $P(0) = \frac{5250}{1 + 34} = 150$

(d) $2625 = \frac{5250}{1 + 34e^{-0.55t}}$

$$1 + 34e^{-0.55t} = 2$$

$$e^{-0.55t} = \frac{1}{34}$$

$$t = \frac{-1}{0.55}\ln\left(\frac{1}{34}\right) \approx 6.41 \text{ yr}$$

(e) $\frac{dP}{dt} = 0.55P\left(1 - \frac{P}{5250}\right)$

48. $P(t) = \frac{4800}{1 + 14e^{-0.15t}}$

(a) $k = 0.15$

(b) $L = 4800$

(c) $P(0) = \frac{4800}{1 + 14} = 320$

(d) $2400 = \frac{4800}{1 + 14e^{-0.15t}}$

$$14e^{-0.15t} = 1$$

$$t = -\frac{1}{0.15}\ln\left(\frac{1}{14}\right) \approx 17.59 \text{ yr}$$

(e) $\frac{dP}{dt} = 0.15P\left(1 - \frac{P}{4800}\right)$

49. $\frac{dy}{dt} = y\left(1 - \frac{y}{80}\right)$, $\quad (0, 8)$

$k = 1$, $L = 80$

$$y = \frac{L}{1 + be^{-kt}} = \frac{80}{1 + be^{-t}}$$

$$y(0) = 8: \quad 8 = \frac{80}{1 + b} \Rightarrow b = 9$$

Solution: $y = \frac{80}{1 + 9e^{-t}}$

50. $\frac{dy}{dt} = 1.76y\left(1 - \frac{y}{8}\right)$, $\quad (0, 3)$

$k = 1.76$, $L = 8$

$$y = \frac{L}{1 + be^{-kt}} = \frac{8}{1 + be^{-1.76t}}$$

$$y(0) = 3: \quad 3 = \frac{8}{1 + b} \Rightarrow b = \frac{5}{3}$$

Solution: $y = \frac{8}{1 + \left(\frac{5}{3}\right)e^{-1.76t}}$

51. $\dfrac{dN}{dt} = k(380 - N(t))$

$$\int \frac{dN}{380 - N} = \int k\,dt$$

$$\ln|380 - N| = kt + C_1$$

$$380 - N = Ce^{kt}$$

$$N = 380 - Ce^{kt}$$

$$N(0) = 110 = 380 - C \Rightarrow C = 270$$

$$N = 380 - 270e^{kt}$$

$$N(4) = 150 = 380 - 270e^{k(4)}$$

$$e^{k(4)} = \frac{150 - 380}{-270} = \frac{23}{27}$$

$$k = \frac{1}{4}, \ln\left(\frac{23}{27}\right) \approx -0.04$$

$$N(t) = 380 - 270e^{-0.04t}$$

$$N(8) = 380 - 270e^{-0.04(8)} \approx 184 \text{ racoons}$$

52. (a) $L = 20{,}400,\ y(0) = 1200,\ y(1) = 2000$

$$y = \frac{20{,}400}{1 + be^{-kt}}$$

$$y(0) = 1200 = \frac{20{,}400}{1 + b} \Rightarrow b = 16$$

$$y(1) = 2000 = \frac{20{,}400}{1 + 16e^{-k}}$$

$$16e^{-k} = \frac{46}{5}$$

$$k = -\ln\frac{23}{40} = \ln\frac{40}{23} \approx 0.553$$

$$y = \frac{20{,}400}{1 + 16e^{-0.553t}}$$

(b) $y(8) \approx 17{,}118$ trout

(c) $10{,}000 = \dfrac{20{,}400}{1 + 16e^{-0.553t}} \Rightarrow t \approx 4.94$ yr

(d) $\dfrac{dy}{dt} = 0.553y\left(1 - \dfrac{y}{20{,}400}\right),\quad y(0) = 1200$

Use Euler's method with $h = 1$.

t	0	2	4	6	8
Exact	1200	3241	7414	12,915	17,117
Euler	1200	2743	5853	10,869	16,170

Euler's method gives $y(8) \approx 16{,}170$ trout.

(e) From Exercise 66 in Section 6.3, the point of inflection of a logistic equation occurs at $y = L/2$.

$$\frac{20{,}400}{2} = \frac{20{,}400}{1 + 16e^{-0.553t}}$$

$$1 + 16e^{-0.553t} = 2$$

$$16e^{-0.553t} = 1$$

$$e^{-0.553t} = \frac{1}{16}$$

$$-0.553t = \ln\frac{1}{16}$$

$$t \approx 5.014 \text{ yr}$$

53. $y' - y = 10$

$$P(x) = -1,\ Q(x) = 10$$

$$u(x) = e^{\int -dx} = e^{-x}$$

$$y = \frac{1}{e^{-x}}\int 10e^{-x}\,dx$$

$$= e^{x}\left(-10e^{-x} + C\right)$$

$$= -10 + Ce^{x}$$

54. $e^{x}y' + 4e^{x}y = 1$

$$y' + 4y = e^{-x}$$

$$P(x) = 4,\ Q(x) = e^{-x}$$

$$u(x) = e^{\int 4\,dx} = e^{4x}$$

$$y = \frac{1}{e^{4x}}\int e^{-x}e^{4x}\,dx = e^{-4x}\left(\frac{1}{3}e^{3x} + C\right) = \frac{1}{3}e^{-x} + Ce^{-4x}$$

55. $4y' = e^{x/y} + y$

$$y' - \frac{1}{4}y = \frac{1}{4}e^{x/4}$$

$$P(x) = -\frac{1}{4},\ Q(x) = \frac{1}{4}e^{x/4}$$

$$u(x) = e^{\int -(1/4)\,dx} = e^{-(1/4)x}$$

$$y = \frac{1}{e^{-(1/4)x}}\int \frac{1}{4}e^{x/4}e^{-(1/4)x}\,dx$$

$$= e^{(1/4)x}\left(\frac{1}{4}x + C\right)$$

$$= \frac{1}{4}xe^{x/4} + Ce^{x/4}$$

56. $\dfrac{dy}{dx} - \dfrac{5y}{x^2} = \dfrac{1}{x^2}$

$$P(x) = -\frac{5}{x^2},\ Q(x) = \frac{1}{x^2}$$

$$u(x) = e^{\int -\left(5/x^2\right)dx} = e^{5/x}$$

$$y = \frac{1}{e^{5/x}}\int \frac{1}{x^2}e^{5/x}\,dx = \frac{1}{e^{5/x}}\left(-\frac{1}{5}e^{5/x} + C\right) = -\frac{1}{5} + Ce^{-5/x}$$

57. $(x-2)y' + y = 1$

$$\frac{dy}{dx} + \frac{1}{x-2}y = \frac{1}{x-2}$$

$$P(x) = \frac{1}{x-2},\ Q(x) = \frac{1}{x-2}$$

$$u(x) = e^{\int (1/x-2)\,dx} = e^{\ln|x-2|} = x-2$$

$$y = \frac{1}{x-2}\int\left(\frac{1}{x-2}\right)(x-2)\,dx = \frac{1}{x-2}(x+C)$$

58. $(x+3)y' + 2y = 2(x+3)^2$

$$\frac{dy}{dx} + \frac{2}{x+3}y = 2(x+3)$$

$$P(x) = \frac{2}{x+3},\ Q(x) = 2(x+3)$$

$$u(x) = e^{\int (2/x+3)\,dx} = e^{2\ln(x+3)} = (x+3)^2$$

$$y = \frac{1}{(x+3)^2}\int 2(x+3)(x+3)^2\,dx$$

$$= \frac{1}{(x+3)^2}\left[\frac{(x+3)^4}{2} + C\right]$$

$$= \frac{(x+3)^2}{2} + \frac{C}{(x+3)^2}$$

59. $y' + 5y = e^{5x},\ y(0) = 3$

$$P(x) = 5,\ Q(x) = e^{5x}$$

$$u(x) = e^{\int 5\,dx} = e^{5x}$$

$$y = \frac{1}{e^{5x}}\int\left(e^{5x}\right)\left(e^{5x}\right)dx$$

$$= \frac{1}{e^{5x}}\int e^{10x}\,dx$$

$$= \frac{1}{e^{5x}}\int\left(\frac{1}{10}e^{10x} + C\right)$$

$$= \frac{1}{10}e^{5x} + Ce^{-5x}$$

Initial condition:

$y(0) = 3: 3 = \frac{1}{10}e^0 + Ce^0 \Rightarrow C = \frac{29}{10}$

Particular solution: $y = \frac{1}{10}e^{5x} + \frac{29}{10}e^{-5x}$

60. $y' - \left(\frac{3}{x}\right)y = 2x^3,\ y(1) = 1$

$$P(x) = -\frac{3}{x},\ Q(x) = 2x^3$$

$$u(x) = e^{\int -\frac{3}{x}\,dx} = e^{-3\ln|x|} = x^{-3}$$

$$y = \frac{1}{x^{-3}}\int 2x^3\left(x^{-3}\right)dx$$

$$= x^3\int 2\,dx$$

$$= x^3(2x + C)$$

$$= 2x^4 + Cx^3$$

Initial condition: $y(1) = 1: 1 = 2 + C \Rightarrow C = -1$

Particular solution: $y = 2x^4 - x^3$

61. $(3y+5)\cos x\,dx = dy,\quad y(\pi) = 0$

$$\frac{dy}{dx} - 3\cos x\,y = 5\cos x$$

$$P(x) = -3\cos x,\ Q(x) = 5\cos x$$

$$u(x) = e^{\int -3\cos x\,dx} = e^{-3\sin x}$$

$$\frac{dy}{dx}e^{-3\sin x} - 3\cos x\left(e^{-3\sin x}\right)y = 5\cos x\left(e^{-3\sin x}\right)$$

$$ye^{-3\sin x} = \int 5e^{-3\sin x}\cos x\,dx$$

$$= \frac{-5}{3}e^{-3\sin x} + C$$

$$y = -\frac{5}{3} + Ce^{3\sin x}$$

Initial condition: $y(\pi) = 0: 0 = -\frac{5}{3} + C \Rightarrow C = \frac{5}{3}$

Particular solution: $y = -\frac{5}{3} + \frac{5}{3}e^{3\sin x}$

62. $y' - 8x^3y = e^{2x^4},\quad y(0) = 2$

$$P(x) = -8x^3,\ Q(x) = e^{2x^4}$$

$$u(x) = e^{\int -8x^3\,dx} = e^{-2x^4}$$

$$y'e^{-2x^4} - 8x^3\left(e^{-2x^4}\right)y = e^{2x^4}\cdot e^{-2x^4} = 1$$

$$ye^{-2x^4} = \int 1\,dx = x + C$$

$$y = (x+C)e^{2x^4}$$

Initial condition: $y(0) = 2: 2 = (0+C)e^0 \Rightarrow C = 2$

Particular solution: $y = (x+2)e^{2x^4}$

Problem Solving for Chapter 6

1. (a)
$$\frac{dy}{dt} = y^{1.01}$$
$$\int y^{-1.01}\,dy = \int dt$$
$$\frac{y^{-0.01}}{-0.01} = t + C_1$$
$$\frac{1}{y^{0.01}} = -0.01t + C$$
$$y^{0.01} = \frac{1}{C - 0.01t}$$
$$y = \frac{1}{(C - 0.01t)^{100}}$$

$y(0) = 1$: $1 = \dfrac{1}{C^{100}} \Rightarrow C = 1$

So, $y = \dfrac{1}{(1 - 0.01t)^{100}}$.

For $T = 100$, $\lim_{t \to T^-} y = \infty$.

(b) $\int y^{-(1+\varepsilon)}\,dy = \int k\,dt$
$$\frac{y^{-\varepsilon}}{-\varepsilon} = kt + C_1$$
$$y^{-\varepsilon} = -\varepsilon kt + C$$
$$y = \frac{1}{(C - \varepsilon kt)^{1/\varepsilon}}$$

$$y(0) = y_0 = \frac{1}{C^{1/\varepsilon}} \Rightarrow C^{1/\varepsilon} = \frac{1}{y_0} \Rightarrow C = \left(\frac{1}{y_0}\right)^{\varepsilon}$$

So, $y = \dfrac{1}{\left(\dfrac{1}{y_0^{\varepsilon}} - \varepsilon kt\right)^{1/\varepsilon}}$.

For $t \to \dfrac{1}{y_0^{\varepsilon}\varepsilon k}$, $y \to \infty$.

2. (a) $\dfrac{dS}{dt} = k_1 S(L - S)$

$S = \dfrac{L}{1 + Ce^{-kt}}$ is a solution because
$$\frac{dS}{dt} = -L\left(1 + Ce^{-kt}\right)^{-2}\left(-Cke^{-kt}\right)$$
$$= \frac{LC\,ke^{-kt}}{\left(1 + Ce^{-kt}\right)^2}$$
$$= \left(\frac{k}{L}\right)\frac{L}{1 + Ce^{-kt}} \cdot \frac{C\,Le^{-kt}}{1 + Ce^{-kt}}$$
$$= \left(\frac{k}{L}\right)\frac{L}{1 + Ce^{-kt}} \cdot \left(L - \frac{L}{1 + Ce^{-kt}}\right)$$
$$= k_1 S(L - S), \text{ where } k_1 = \frac{k}{L}.$$

$L = 100$. Also, $S = 10$ when $t = 0 \Rightarrow C = 9$.

And, $S = 20$ when $t = 1 \Rightarrow k = -\ln\dfrac{4}{9}$.

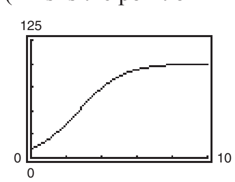
Particular Solution: $S = \dfrac{100}{1 + 9e^{\ln(4/9)t}} = \dfrac{100}{1 + 9e^{-0.8109t}}$

(b) $\dfrac{dS}{dt} = k_1 S(100 - S)$
$$\frac{d^2S}{dt^2} = k_1\left[S\left(-\frac{dS}{dt}\right) + (100 - S)\frac{dS}{dt}\right]$$
$$= k_1(100 - 2S)\frac{dS}{dt}$$
$$= 0 \text{ when } S = 50 \text{ or } \frac{dS}{dt} = 0.$$

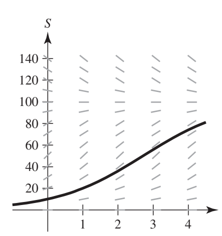
Choosing $S = 50$, you have:
$$50 = \frac{100}{1 + 9e^{\ln(4/9)t}}$$
$$2 = 1 + 9e^{\ln(4/9)t}$$
$$\frac{\ln(1/9)}{\ln(4/9)} = t$$
$$t \approx 2.7 \text{ months}$$

(This is the point of inflection.)

(c)

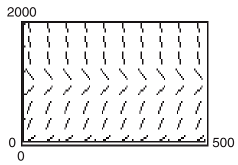


(d)



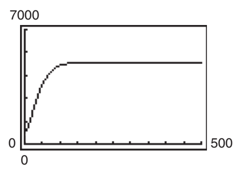
(e) Sales will decrease toward the line $S = L$.

3. (a)
$$\frac{dy}{dt} = k \ln\left(\frac{L}{y}\right)y$$
$$\frac{dy}{y[\ln L - \ln y]} = k\,dt$$
$$\ln[\ln L - \ln y] = -kt + C_1$$
$$\ln\frac{L}{y} = Ce^{-kt}$$
$$\frac{L}{y} = e^{Ce^{-kt}}$$
$$y = Le^{-Ce^{-kt}}$$

(b) 

(c) As $t \to \infty$, $y \to L$, the carrying capacity.

(d) $y_0 = 500 = 5000e^{-C} \Rightarrow e^C = 10 \Rightarrow C = \ln 10$



$$\frac{dy}{dt} = k \ln\left(\frac{L}{y}\right)y$$
$$\frac{d^2y}{dt^2} = k \ln\left(\frac{L}{y}\right)\frac{dy}{dt} + ky\frac{1}{(L/y)}\left(\frac{-L}{y^2}\right)\frac{dy}{dt}$$
$$= k\frac{dy}{dt}\left[\ln\left(\frac{L}{y}\right) - 1\right] = k^2 \ln\left(\frac{L}{y}\right)y\left[\ln\left(\frac{L}{y}\right) - 1\right]$$

So, $\frac{d^2y}{dt^2} = 0$ when

$$\ln\left(\frac{L}{y}\right) = 1 \Rightarrow \frac{L}{y} = e \Rightarrow y = \frac{L}{e}.$$

$y = \frac{L}{e} = \frac{5000}{e} \approx 1839.4$ and $t \approx 41.7$.

The graph is concave upward on $(0, 41.7)$ and downward on $(41.7, \infty)$.

4. $[f(x)g(x)]' \stackrel{?}{=} f'(x)g'(x)$

(a) Let $g(x) = x$, $g'(x) = 1$, then
$$[f(x)x]' = f'(x)$$
$$f'(x)x + f(x) = f'(x)$$
$$\frac{df}{dx}(x - 1) = -f(x)$$
$$\int\frac{df}{f} = \int\frac{dx}{1 - x}$$
$$\ln|f(x)| = -\ln|1 - x|$$
$$f(x) = \frac{1}{1 - x}$$

(b)
$$(fg)' = f'g'$$
$$f'g + fg' = f'g'$$
$$f'(g - g') = -fg'$$
$$\frac{f'}{f} = \frac{g'}{g' - g}$$
$$\ln|f| = \int\frac{g'}{g' - g}\,dx$$
$$f = e^{\int\frac{g'}{g'-g}\,dx}$$

(c) If $g(x) = e^x$, then $g'(x) - g(x) = e^x - e^x = 0$

Therefore, no f can exist.

5. $k = \left(\frac{1}{12}\right)^2 \pi$

$g = 32$

$x^2 + (y - 6)^2 = 36$ Equation of tank

$x^2 = 36 - (y - 6)^2 = 12y - y^2$

Area of cross section: $A(h) = (12h - h^2)\pi$

$$A(h)\frac{dh}{dt} = -k\sqrt{2gh}$$

$$(12h - h^2)\pi\frac{dh}{dt} = -\frac{1}{144}\pi\sqrt{64h}$$

$$(12h - h^2)\frac{dh}{dt} = -\frac{1}{18}h^{1/2}$$

$$\int\left(18h^{3/2} - 216h^{1/2}\right)dh = \int dt$$

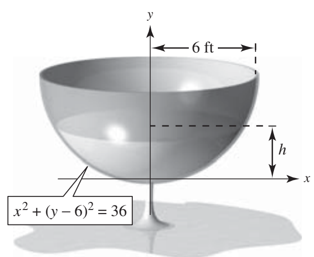
$$\frac{36}{5}h^{5/2} - 144h^{3/2} = t + C$$

$$\frac{h^{3/2}}{5}(36h - 720) = t + C$$

When $h = 6$, $t = 0$ and $C = \frac{6^{3/2}}{5}(-504) \approx -1481.45$.

The tank is completely drained when

$h = 0 \Rightarrow t = 1481.45$ sec $\approx$ 24 min, 41 sec



6. (a) $A(h)\frac{dh}{dt} = -k\sqrt{2gh}$

$$\pi r^2\frac{dh}{dt} = -k\sqrt{64h}$$

$$h^{-1/2}\,dh = \frac{-8k}{\pi r^2}\,dt = -C\,dt, \quad C = \frac{8k}{\pi r^2}$$

$$2\sqrt{h} = -Ct + C_1$$

$$2\sqrt{18} = C_1 \quad (\text{at } t = 0, h = 18)$$

So, $2\sqrt{h} = -Ct + 6\sqrt{2}$.

At $t = 30(60) = 1800$, $h = 12$:

$$2\sqrt{12} = -1800\,C + 6\sqrt{2}$$

$$\frac{6\sqrt{2} - 4\sqrt{3}}{1800} = C \approx 0.000865$$

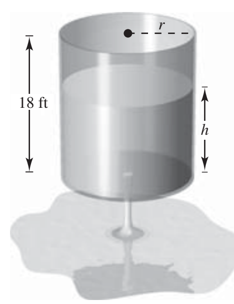
So, $2\sqrt{h} = -0.000865t + 6\sqrt{2}$.

$$h = 0 \Rightarrow t = \frac{6\sqrt{2}}{0.000865}$$

$$\approx 9809.1 \text{ seconds } (2 \text{ h}, 43 \text{ min}, 29 \text{ sec})$$

(b) $t = 3600$ sec $\Rightarrow 2\sqrt{h} = -0.000865(3600) + 6\sqrt{2}$

$\Rightarrow h \approx 7.21$ ft



7. $A(h)\dfrac{dh}{dt} = -k\sqrt{2gh}$

$$\pi 64 \frac{dh}{dt} = \frac{-\pi}{36}8\sqrt{h}$$

$$\int h^{-1/2}\,dh = \int \frac{-1}{288}\,dt$$

$$2\sqrt{h} = \frac{-t}{288} + C$$

$$h = 20: 2\sqrt{20} = C = 4\sqrt{5}$$

$$2\sqrt{h} = \frac{-t}{288} + 4\sqrt{5}$$

$$h = 0 \Rightarrow t = 4\sqrt{5}(288)$$

$$\approx 2575.95 \text{ sec} \approx 42 \text{ min, } 56 \text{ sec}$$

8. Let $u = \dfrac{1}{2}k\left(t - \dfrac{\ln b}{k}\right)$.

$$1 + \tanh u = 1 + \frac{e^u - e^{-u}}{e^u + e^{-u}} = \frac{2}{1 + e^{-2u}}$$

$$e^{-2u} = e^{-k(t-(\ln b/k))} = e^{\ln b}e^{-kt} = be^{-kt}$$

Finally,

$$\frac{1}{2}L\left[1 + \tanh\left(\frac{1}{2}k\left(t - \frac{\ln b}{k}\right)\right)\right] = \frac{L}{2}[1 + \tanh u]$$

$$= \frac{L}{2}\left(\frac{2}{1 + be^{-kt}}\right)$$

$$= \frac{L}{1 + be^{-kt}}.$$

Notice the graph of the logistics function is just a shift of the graph of the hyperbolic tangent. (See Section 5.8.)

9. $\dfrac{ds}{dt} = 3.5 - 0.019s$

(a) $\displaystyle\int \frac{-ds}{3.5 - 0.019s} = -\int dt$

$$\frac{1}{0.019}\ln|3.5 - 0.019s| = -t + C_1$$

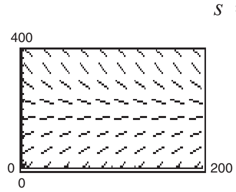
$$\ln|3.5 - 0.019s| = -0.019t + C_2$$

$$3.5 - 0.019s = C_3e^{-0.019t}$$

$$0.019s = 3.5 - C_3e^{-0.019t}$$

$$s = 184.21 - Ce^{-0.019t}$$

(b)



(c) As $t \to \infty$, $Ce^{-0.019t} \to 0$, and $s \to 184.21$.

10. $\dfrac{dy}{dt} = -ry$

$$\frac{dy}{y} = -r\,dt$$

$$\ln y = -rt + C_1$$

$$y = Ce^{-rt}$$

Initial condition:

$y(0) = 0.4$: $0.4 = Ce^{-r(0)} \Rightarrow C = 0.4$

General solution: $y = 0.4e^{-rt}$

11. (a) $\displaystyle\int \frac{dC}{C} = \int -\frac{R}{V}\,dt$

$$\ln|C| = -\frac{R}{V}t + K_1$$

$$C = Ke^{-Rt/V}$$

Since $C = C_0$ when $t = 0$, it follows that $K = C_0$ and the function is $C = C_0e^{-Rt/V}$.

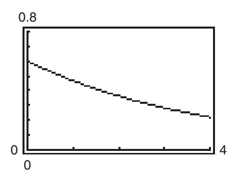
(b) Finally, as $t \to \infty$, we have

$$\lim_{t\to\infty} C = \lim_{t\to\infty} C_0e^{-Rt/V} = 0.$$

12. From Exercise 11, you have $C = C_0e^{-Rt/V}$.

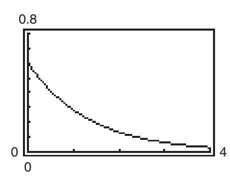
(a) For $V = 2$, $R = 0.5$, and $C_0 = 0.6$, you have

$C = 0.6e^{-0.25t}$



(b) For $V = 2$, $R = 1.5$, and $C_0 = 0.6$, you have

$C = 0.6e^{-0.75t}$.



13. (a) $\displaystyle\int \frac{1}{Q - RC}\,dC = \int \frac{1}{V}\,dt$

$$-\frac{1}{R}\ln|Q - RC| = \frac{t}{V} + K_1$$

$$Q - RC = e^{-R[(t/V)+K_1]}$$

$$C = \frac{1}{R}\left(Q - e^{-R[(t/V)+K_1]}\right)$$

$$= \frac{1}{R}\left(Q - Ke^{-Rt/V}\right)$$

Because $C = 0$ when $t = 0$, it follows that

$K = Q$ and you have $C = \dfrac{Q}{R}\left(1 - e^{-Rt/V}\right)$.

(b) As $t \to \infty$, the limit of C is Q/R.