CHAPTER 5
Logarithmic, Exponential, and Other Transcendental Functions

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Section 5.1 The Natural Logarithmic Function: Differentiation

1. For $x > 1$, $\ln x = \int_1^x \frac{1}{t}\,dt > 0$.

For $0 < x < 1$, $\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt$.

2. $\ln 4 + \ln(n^{-1}) = \ln 4 - \ln 7$

$-\ln(n) = -\ln 7$

$n = 7$

3. The number e is the base for the natural logarithm and defined by the equation

$\ln e = \int_1^e \frac{1}{t}\,dt = 1.$

4. The Chain Rule version of the derivative of the natural logarithmic function is

$\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}$

Here, u is a function of x and $u > 0$.

5. (a) $\ln 45 \approx 3.8067$

(b) $\int_1^{45} \frac{1}{t}\,dt \approx 3.8067$

6. (a) $\ln 8.3 \approx 2.1163$

(b) $\int_1^{8.3} \frac{1}{t}\,dt \approx 2.1163$

7. (a) $\ln 0.8 \approx -0.2231$

(b) $\int_1^{0.8} \frac{1}{t}\,dt \approx -0.2231$

8. (a) $\ln 0.6 \approx -0.5108$

(b) $\int_1^{0.6} \frac{1}{t}\,dt \approx -0.5108$

9. $f(x) = \ln x + 1$

Vertical shift 1 unit upward

Matches (b)

10. $f(x) = -\ln x$

Reflection in the x-axis

Matches (d)

11. $f(x) = \ln(x - 1)$

Horizontal shift 1 unit to the right

Matches (a)

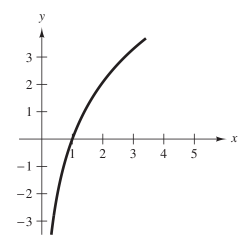
12. $f(x) = -\ln(-x)$

Reflection in the y-axis and the x-axis

Matches (c)

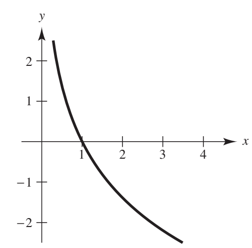
13. $f(x) = 3 \ln x$

Domain: $x > 0$



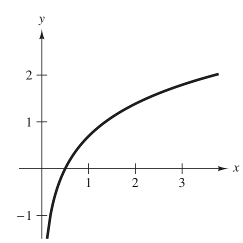
14. $f(x) = -2 \ln x$

Domain: $x > 0$



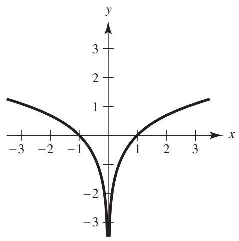
15. $f(x) = \ln 2x$

Domain: $x > 0$



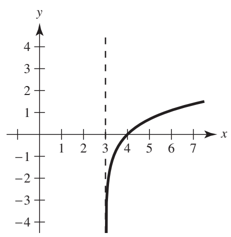
16. $f(x) = \ln|x|$

Domain: $x \neq 0$



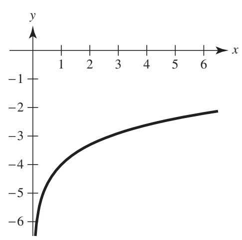
17. $f(x) = \ln(x - 3)$

Domain: $x > 3$



18. $f(x) = \ln x - 4$

Domain: $x > 0$



19. (a) $\ln 6 = \ln 2 + \ln 3 \approx 1.7917$

(b) $\ln \frac{2}{3} = \ln 2 - \ln 3 \approx -0.4055$

(c) $\ln 81 = \ln 3^4 = 4 \ln 3 \approx 4.3944$

(d) $\ln \sqrt{3} = \ln 3^{1/2} = \frac{1}{2} \ln 3 \approx 0.5493$

20. (a) $\ln 0.25 = \ln \frac{1}{4} = \ln 1 - 2 \ln 2 \approx -1.3862$

(b) $\ln 24 = 3 \ln 2 + \ln 3 \approx 3.1779$

(c) $\ln \sqrt[3]{12} = \frac{1}{3}(2 \ln 2 + \ln 3) \approx 0.8283$

(d) $\ln \frac{1}{72} = \ln 1 - (3 \ln 2 + 2 \ln 3) \approx -4.2765$

21. $\ln \dfrac{x}{4} = \ln x - \ln 4$

22. $\ln \sqrt{x^5} = \ln x^{5/2} = \frac{5}{2} \ln x$

23. $\ln \dfrac{xy}{z} = \ln x + \ln y - \ln z$

24. $\ln(xyz) = \ln x + \ln y + \ln z$

25. $\ln\left(x\sqrt{x^2 + 5}\right) = \ln x + \ln\left(x^2 + 5\right)^{1/2}$

$= \ln x + \frac{1}{2} \ln\left(x^2 + 5\right)$

26. $x \ln \sqrt{x - 4} = x \ln(x - 4)^{1/2} = \dfrac{1}{2} x \ln(x - 4)$

27. $\ln\sqrt{\dfrac{x - 1}{x}} = \ln\left(\dfrac{x - 1}{x}\right)^{1/2} = \dfrac{1}{2} \ln\left(\dfrac{x - 1}{x}\right)$

$= \dfrac{1}{2}\left[\ln(x - 1) - \ln x\right]$

$= \dfrac{1}{2} \ln(x - 1) - \dfrac{1}{2} \ln x$

28. $\ln 3e^2 = \ln 3 + 2 \ln e = 2 + \ln 3$

29. $\ln z(z - 1)^2 = \ln z + \ln(z - 1)^2$

$= \ln z + 2 \ln(z - 1)$

30. $\ln \dfrac{z}{e} = \ln z - \ln e = \ln z - 1$

31. $\ln(x - 2) - \ln(x + 2) = \ln \dfrac{x - 2}{x + 2}$

32. $3 \ln x + 2 \ln y - 4 \ln z = \ln x^3 + \ln y^2 - \ln z^4$

$= \ln \dfrac{x^3 y^2}{z^4}$

33. $\dfrac{1}{3}\left[2 \ln(x + 3) + \ln x - \ln\left(x^2 - 1\right)\right] = \dfrac{1}{3} \ln \dfrac{x(x + 3)^2}{x^2 - 1}$

$= \ln \sqrt[3]{\dfrac{x(x + 3)^2}{x^2 - 1}}$

34. $2\left[\ln x - \ln(x + 1) - \ln(x - 1)\right] = 2 \ln \dfrac{x}{(x + 1)(x - 1)}$

$= \ln\left(\dfrac{x}{x^2 - 1}\right)^2$

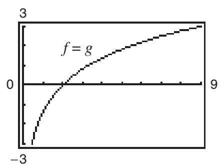
35. $4 \ln 2 - \dfrac{1}{2} \ln\left(x^3 + 6x\right) = \ln 2^4 - \ln \sqrt{x^3 + 6x}$

$= \ln \dfrac{16}{\sqrt{x^3 + 6x}}$

36. $\frac{3}{2}\left[\ln\left(x^2+1\right)-\ln(x+1)-\ln(x-1)\right]=\frac{3}{2}\ln\frac{x^2+1}{(x+1)(x-1)}$

$=\ln\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^3}$

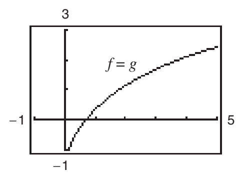
37. (a)



(b) $f(x)=\ln\frac{x^2}{4}=\ln x^2-\ln 4=2\ln x-\ln 4=g(x)$

because $x>0$.

38. (a)



(b) $f(x)=\ln\sqrt{x\left(x^2+1\right)}=\frac{1}{2}\ln\left[x\left(x^2+1\right)\right]$

$=\frac{1}{2}\left[\ln x+\ln\left(x^2+1\right)\right]=g(x)$

39. $\lim_{x\to 3^+}\ln(x-3)=-\infty$

40. $\lim_{x\to 6^-}\ln(6-x)=-\infty$

41. $\lim_{x\to 2^-}\ln\left[x^2(3-x)\right]=\ln 4\approx 1.3863$

42. $\lim_{x\to 5^+}\ln\frac{x}{\sqrt{x-4}}=\ln 5\approx 1.6094$

43. $f(x)=\ln(3x)$

$f'(x)=\frac{1}{3x}(3)=\frac{1}{x}$

44. $f(x)=\ln(x-1)$

$f'(x)=\frac{1}{x-1}$

45. $f(x)=\ln\left(x^2+3\right)$

$f'(x)=\frac{1}{x^2+3}(2x)=\frac{2x}{x^2+3}$

46. $h(x)=\ln\left(2x^2+1\right)$

$h'(x)=\frac{1}{2x^2+1}(4x)=\frac{4x}{2x^2+1}$

47. $y=(\ln x)^4$

$\frac{dy}{dx}=4(\ln x)^3\left(\frac{1}{x}\right)=\frac{4(\ln x)^3}{x}$

48. $y=x^2\ln x$

$y'=x^2\left(\frac{1}{x}\right)+2x\ln x=x+2x\ln x=x(1+2\ln x)$

49. $y=\ln(t+1)^2=2\ln(t+1)$

$y'=2\frac{1}{t+1}=\frac{2}{t+1}$

50. $y=\ln\sqrt{x^2-4}=\frac{1}{2}\ln\left(x^2-4\right)$

$\frac{dy}{dx}=\frac{1}{2}\left(\frac{2x}{x^2-4}\right)=\frac{x}{x^2-4}$

51. $y=\ln\left[x\sqrt{x^2-1}\right]=\ln x+\frac{1}{2}\ln\left(x^2-1\right)$

$\frac{dy}{dx}=\frac{1}{x}+\frac{1}{2}\left(\frac{2x}{x^2-1}\right)=\frac{2x^2-1}{x\left(x^2-1\right)}$

52. $y=\ln\left[t\left(t^2+3\right)^3\right]=\ln t+3\ln\left(t^2+3\right)$

$y'=\frac{1}{t}+\frac{3}{t^2+3}(2t)=\frac{1}{t}+\frac{6t}{t^2+3}$

53. $f(x)=\ln\frac{x}{x^2+1}=\ln x-\ln\left(x^2+1\right)$

$f'(x)=\frac{1}{x}-\frac{2x}{x^2+1}=\frac{1-x^2}{x\left(x^2+1\right)}$

54. $f(x)=\ln\left(\frac{2x}{x+3}\right)=\ln 2x-\ln(x+3)$

$f'(x)=\frac{1}{x}-\frac{1}{x+3}=\frac{3}{x(x+3)}$

55. $g(t)=\frac{\ln t}{t^2}$

$g'(t)=\frac{t^2(1/t)-2t\ln t}{t^4}=\frac{1-2\ln t}{t^3}$

56. $h(t)=\frac{\ln t}{t^3+5}$

$h'(t)=\frac{\left(t^3+5\right)(1/t)-\ln t\left(3t^2\right)}{\left(t^3+5\right)^2}$

$=\frac{t^3+5-3t^3\ln t}{t\left(t^3+5\right)^2}$

57. $y = \ln(\ln x^2)$

$$\frac{dy}{dx} = \frac{1}{\ln x^2}\frac{d}{dx}(\ln x^2) = \frac{(2x/x^2)}{\ln x^2} = \frac{2}{x \ln x^2} = \frac{1}{x \ln x}$$

58. $y = \ln(\ln x)$

$$\frac{dy}{dx} = \frac{1/x}{\ln x} = \frac{1}{x \ln x}$$

59. $y = \ln\sqrt{\dfrac{x+1}{x-1}} = \dfrac{1}{2}\left[\ln(x+1) - \ln(x-1)\right]$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x+1} - \frac{1}{x-1}\right] = \frac{1}{1-x^2}$$

60. $y = \ln\sqrt[3]{\dfrac{x-1}{x+1}} = \dfrac{1}{3}\left[\ln(x-1) - \ln(x+1)\right]$

$$y' = \frac{1}{3}\left[\frac{1}{x-1} - \frac{1}{x+1}\right] = \frac{1}{3}\frac{2}{x^2-1} = \frac{2}{3(x^2-1)}$$

61. $f(x) = \ln\dfrac{\sqrt{4+x^2}}{x} = \dfrac{1}{2}\ln(4+x^2) - \ln x$

$$f'(x) = \frac{x}{4+x^2} - \frac{1}{x} = \frac{-4}{x(x^2+4)}$$

62. $f(x) = \ln\left(x + \sqrt{4+x^2}\right)$

$$f'(x) = \frac{1}{x+\sqrt{4+x^2}}\left(1 + \frac{x}{\sqrt{4+x^2}}\right) = \frac{1}{\sqrt{4+x^2}}$$

63. $y = \ln|\sin x|$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

64. $y = \ln|\csc x|$

$$y' = \frac{-\csc x \cdot \cot x}{\csc x} = -\cot x$$

65. $y = \ln\left|\dfrac{\cos x}{\cos x - 1}\right|$

$$= \ln|\cos x| - \ln|\cos x - 1|$$

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} - \frac{-\sin x}{\cos x - 1}$$

$$= -\tan x + \frac{\sin x}{\cos x - 1}$$

66. $y = \ln|\sec x + \tan x|$

$$\frac{dy}{dx} = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x$$

67. (a) $y = \ln x^4 = 4 \ln x, \ (1, 0)$

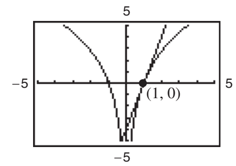
$$\frac{dy}{dx} = \frac{4}{x}$$

When $x = 1, \dfrac{dy}{dx} = 4.$

Tangent line: $y - 0 = 4(x - 1)$

$$y = 4x - 4$$

(b)



68. (a) $y = \ln x^{2/3} = \dfrac{2}{3}\ln|x|, \ (-1, 0)$

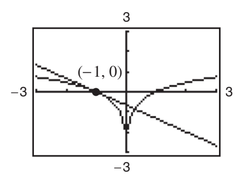
$$\frac{dy}{dx} = \frac{2}{3x}$$

When $x = -1, \dfrac{dy}{dx} = -\dfrac{2}{3}.$

Tangent line: $y - 0 = \dfrac{-2}{3}(x + 1)$

$$y = -\frac{2}{3}x - \frac{2}{3}$$

(b)



69. (a) $y = 3x^2 - \ln x, \quad (1, 3)$

$$\frac{dy}{dx} = 6x - \frac{1}{x}$$

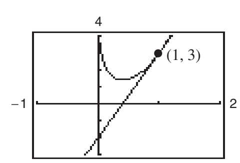
When $x = 1, \dfrac{dy}{dx} = 5.$

Tangent line: $y - 3 = 5(x - 1)$

$$y = 5x - 2$$

$$0 = 5x - y - 2$$

(b)



70. (a) $y = 4 - x^2 - \ln\left(\frac{1}{2}x + 1\right), \quad (0, 4)$

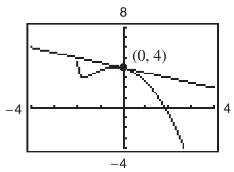
$$\frac{dy}{dx} = -2x - \frac{1}{(1/2)x + 1}\left(\frac{1}{2}\right) = -2x - \frac{1}{x + 2}$$

When $x = 0, \frac{dy}{dx} = -\frac{1}{2}.$

Tangent line: $y - 4 = -\frac{1}{2}(x - 0)$

$$y = -\frac{1}{2}x + 4$$

(b)



71. (a) $f(x) = \ln\sqrt{1 + \sin^2 x}$

$$= \frac{1}{2}\ln\left(1 + \sin^2 x\right), \quad \left(\frac{\pi}{4}, \ln\sqrt{\frac{3}{2}}\right)$$

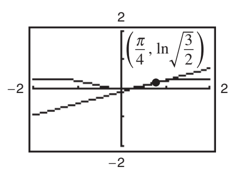
$$f'(x) = \frac{2\sin x \cos x}{2\left(1 + \sin^2 x\right)} = \frac{\sin x \cos x}{1 + \sin^2 x}$$

$$f'\left(\frac{\pi}{4}\right) = \frac{\left(\sqrt{2}/2\right)\left(\sqrt{2}/2\right)}{(3/2)} = \frac{1}{3}$$

Tangent line: $y - \ln\sqrt{\frac{3}{2}} = \frac{1}{3}\left(x - \frac{\pi}{4}\right)$

$$y = \frac{1}{3}x + \frac{1}{2}\ln\left(\frac{3}{2}\right) - \frac{\pi}{12}$$

(b)



72. (a) $f(x) = \sin(2x)\ln\left(x^2\right) = 2\sin(2x)\ln x, \quad (1, 0)$

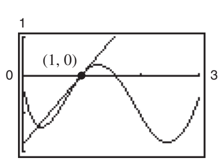
$$f'(x) = 4\cos(2x)\ln x + \frac{2\sin(2x)}{x}$$

$$f'(1) = 2\sin(2)$$

Tangent line: $y - 0 = 2\sin(2)(x - 1)$

$$y = 2\sin(2)x - 2\sin(2)$$

(b)



73. (a) $y = x^3 \ln x^4, (-1, 0)$

$$\frac{dy}{dx} = x^3 \cdot \frac{1}{x^4}\left(4x^3\right) + 3x^2 \ln x^4$$

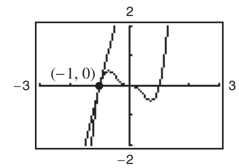
$$= 4x^2 + 3x^2 \ln x^4$$

When $x = -1, \frac{dy}{dx} = 4.$

Tangent line: $y - 0 = 4(x + 1)$

$$y = 4x + 4$$

(b)



74. (a) $f(x) = \frac{1}{2}x\ln\left(x^2\right), \quad (-1, 0)$

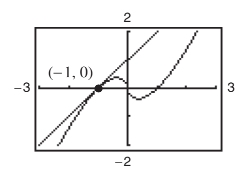
$$f'(x) = \frac{1}{2}\ln\left(x^2\right) + \frac{1}{2}x\left(\frac{2x}{x^2}\right) = \frac{1}{2}\ln\left(x^2\right) + 1$$

$$f'(-1) = 1$$

Tangent line: $y - 0 = 1(x + 1)$

$$y = x + 1$$

(b)



75. $y = x\sqrt{x^2 + 1}$

$$\ln y = \ln x + \frac{1}{2}\ln\left(x^2 + 1\right)$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{x}{x^2 + 1}$$

$$\frac{dy}{dx} = y\left[\frac{2x^2 + 1}{x\left(x^2 + 1\right)}\right] = \frac{2x^2 + 1}{\sqrt{x^2 + 1}}$$

76.
$$\begin{aligned}
y &= \sqrt{x^2(x+1)(x+2)},\ x > 0\\
y^2 &= x^2(x+1)(x+2)\\
2\ln y &= 2\ln x + \ln(x+1) + \ln(x+2)\\
\frac{2}{y}\frac{dy}{dx} &= \frac{2}{x} + \frac{1}{x+1} + \frac{1}{x+2}\\
\frac{dy}{dx} &= \frac{y}{2}\left[\frac{2}{x} + \frac{1}{x+1} + \frac{1}{x+2}\right]\\
\frac{dy}{dx} &= \frac{\sqrt{x^2(x+1)(x+2)}}{2}\left[\frac{2(x+1)(x+2) + x(x+2) + x(x+1)}{x(x+1)(x+2)}\right] = \frac{4x^2 + 9x + 4}{2\sqrt{(x+1)(x+2)}}
\end{aligned}$$

77.
$$\begin{aligned}
y &= \frac{x^2\sqrt{3x-2}}{(x+1)^2}\\
\ln y &= 2\ln x + \frac{1}{2}\ln(3x-2) - 2\ln(x+1)\\
\frac{1}{y}\left(\frac{dy}{dx}\right) &= \frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x+1}\\
\frac{dy}{dx} &= y\left[\frac{3x^2 + 15x - 8}{2x(3x-2)(x+1)}\right]\\
&= \frac{3x^3 + 15x^2 - 8x}{2(x+1)^3\sqrt{3x-2}}
\end{aligned}$$

78.
$$\begin{aligned}
y &= \sqrt{\frac{x^2-1}{x^2+1}}\\
\ln y &= \frac{1}{2}\left[\ln(x^2-1) - \ln(x^2+1)\right]\\
\frac{1}{y}\left(\frac{dy}{dx}\right) &= \frac{1}{2}\left[\frac{2x}{x^2-1} - \frac{2x}{x^2+1}\right]\\
\frac{dy}{dx} &= \sqrt{\frac{x^2-1}{x^2+1}}\left[\frac{2x}{x^4-1}\right]\\
&= \frac{(x^2-1)^{1/2}2x}{(x^2+1)^{1/2}(x^2-1)(x^2+1)}\\
&= \frac{2x}{(x^2+1)^{3/2}(x^2-1)^{1/2}}
\end{aligned}$$

79.
$$\begin{aligned}
y &= \frac{x(x-1)^{3/2}}{\sqrt{x+1}}\\
\ln y &= \ln x + \frac{3}{2}\ln(x-1) - \frac{1}{2}\ln(x+1)\\
\frac{1}{y}\left(\frac{dy}{dx}\right) &= \frac{1}{x} + \frac{3}{2}\left(\frac{1}{x-1}\right) - \frac{1}{2}\left(\frac{1}{x+1}\right)\\
\frac{dy}{dx} &= \frac{y}{2}\left[\frac{2}{x} + \frac{3}{x-1} - \frac{1}{x+1}\right]\\
&= \frac{y}{2}\left[\frac{4x^2 + 4x - 2}{x(x^2-1)}\right] = \frac{(2x^2 + 2x - 1)\sqrt{x-1}}{(x+1)^{3/2}}
\end{aligned}$$

80.
$$\begin{aligned}
y &= \frac{(x+1)(x-2)}{(x-1)(x+2)}\\
\ln y &= \ln(x+1) + \ln(x-2) - \ln(x-1) - \ln(x+2)\\
\frac{1}{y}\left(\frac{dy}{dx}\right) &= \frac{1}{x+1} + \frac{1}{x-2} - \frac{1}{x-1} - \frac{1}{x+2}\\
\frac{dy}{dx} &= y\left[\frac{-2}{x^2-1} + \frac{4}{x^2-4}\right] = y\left[\frac{2x^2+4}{(x^2-1)(x^2-4)}\right]\\
&= \frac{(x+1)(x+2)}{(x-1)(x-2)}\cdot\frac{2x^2+4}{(x+1)(x-1)(x+2)(x-2)}\\
&= \frac{2(x^2+2)}{(x-1)^2(x-2)^2}
\end{aligned}$$

81.
$$\begin{aligned}
x^2 - 3\ln y + y^2 &= 10\\
2x - \frac{3}{y}\frac{dy}{dx} + 2y\frac{dy}{dx} &= 0\\
2x &= \frac{dy}{dx}\left(\frac{3}{y} - 2y\right)\\
\frac{dy}{dx} &= \frac{2x}{(3/y) - 2y} = \frac{2xy}{3 - 2y^2}
\end{aligned}$$

82.
$$\begin{aligned}
\ln(xy) + 5x &= 30\\
\ln x + \ln y + 5x &= 30\\
\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} + 5 &= 0\\
\frac{1}{y}\frac{dy}{dx} &= -\frac{1}{x} - 5\\
\frac{dy}{dx} &= -\frac{y}{x} - 5y = -\left(\frac{y + 5xy}{x}\right)
\end{aligned}$$

83. $4x^3 + \ln y^2 + 2y = 2x$

$$12x^2 + \frac{2}{y}y' + 2y' = 2$$

$$\left(\frac{2}{y} + 2\right)y' = 2 - 12x^2$$

$$y' = \frac{2 - 12x^2}{2/y + 2}$$

$$y' = \frac{y - 6yx^2}{1 + y} = \frac{y(1 - 6x^2)}{1 + y}$$

84. $4xy + \ln x^2y = 7$

$$4xy + 2\ln x + \ln y = 7$$

$$4xy' + 4y + \frac{2}{x} + \frac{1}{y}y' = 0$$

$$\left(4x + \frac{1}{y}\right)y' = -4y - \frac{2}{x}$$

$$y' = \frac{-4y - \frac{2}{x}}{4x + \frac{1}{y}}$$

$$y' = \frac{-4xy^2 - 2y}{4x^2y + x}$$

85. $y = 2(\ln x) + 3$

$$y' = \frac{2}{x}$$

$$y'' = -\frac{2}{x^2}$$

$$xy'' + y' = x\left(-\frac{2}{x^2}\right) + \frac{2}{x} = 0$$

86. $y = x(\ln x) - 4x$

$$y' = x\left(\frac{1}{x}\right) + \ln x - 4 = -3 + \ln x$$

$$(x + y) - xy' = x + x\ln x - 4x - x(-3 + \ln x) = 0$$

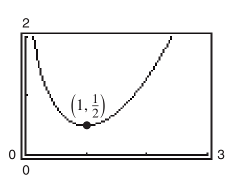
87. $y = \dfrac{x^2}{2} - \ln x$

Domain: $x > 0$

$$y' = x - \frac{1}{x} = \frac{(x+1)(x-1)}{x} = 0 \text{ when } x = 1.$$

$$y'' = 1 + \frac{1}{x^2} > 0$$

Relative minimum: $\left(1, \dfrac{1}{2}\right)$



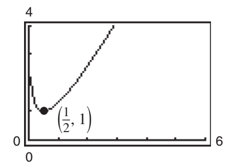
88. $y = 2x - \ln 2x = 2x - \ln 2 - \ln x$

Domain: $x > 0$

$$y' = 2 - \frac{1}{x} = \frac{2x - 1}{x} = 0 \text{ when } x = \frac{1}{2}.$$

$$y'' = \frac{1}{x^2} > 0$$

Relative minimum: $\left(\dfrac{1}{2}, 1\right)$



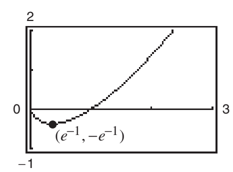
89. $y = x\ln x$

Domain: $x > 0$

$$y' = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x = 0 \text{ when } x = e^{-1}.$$

$$y'' = \frac{1}{x} > 0$$

Relative minimum: $\left(e^{-1}, -e^{-1}\right)$



90. $y = \dfrac{\ln x}{x}$

Domain: $x > 0$

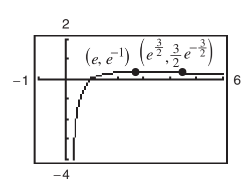
$$y' = \frac{x(1/x) - \ln x}{x^2} = \frac{1 - \ln x}{x^2} = 0 \text{ when } x = e.$$

$$y'' = \frac{x^2(-1/x) - (1 - \ln x)(2x)}{x^4}$$

$$= \frac{2(\ln x) - 3}{x^3} = 0 \text{ when } x = e^{3/2}.$$

Relative maximum: $\left(e, e^{-1}\right)$

Point of inflection: $\left(e^{3/2}, \dfrac{3}{2}e^{-3/2}\right)$



91. $y = \dfrac{x}{\ln x}$

Domain: $0 < x < 1, x > 1$

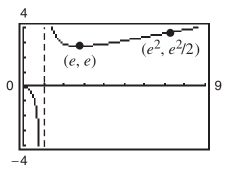
$$y' = \frac{(\ln x)(1) - (x)(1/x)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} = 0 \text{ when } x = e.$$

$$y'' = \frac{(\ln x)^2(1/x) - (\ln x - 1)(2/x)\ln x}{(\ln x)^4}$$

$$= \frac{2 - \ln x}{x(\ln x)^3} = 0 \text{ when } x = e^2.$$

Relative minimum: (e, e)

Point of inflection: $\left(e^2, \dfrac{e^2}{2}\right)$



92. $y = x^2 \ln \dfrac{x}{4}$, Domain: $x > 0$

$$y' = x^2\left(\frac{1}{x}\right) + 2x \ln \frac{x}{4} = x\left(1 + 2 \ln \frac{x}{4}\right) = 0 \text{ when:}$$

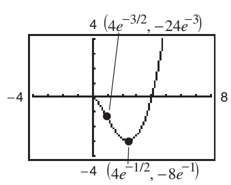
$$-1 = 2 \ln \frac{x}{4} \Rightarrow \ln \frac{x}{4} = -\frac{1}{2} \Rightarrow x = 4e^{-1/2}$$

$$y'' = 1 + 2 \ln \frac{x}{4} + 2x\left(\frac{1}{x}\right) = 3 + 2 \ln \frac{x}{4}$$

$y'' = 0$ when $x = 4e^{-3/2}$

Relative minimum: $\left(4e^{-1/2}, -8e^{-1}\right)$

Point of inflection: $\left(4e^{-3/2}, -24e^{-3}\right)$



93. Find x such that $\ln x = -x$.

$$f(x) = \ln x + x = 0$$

$$f'(x) = \frac{1}{x} + 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n\left[\frac{1 - \ln x_n}{1 + x_n}\right]$$

n	1	2	3
x_n	0.5	0.5644	0.5671
$f(x_n)$	–0.1931	–0.0076	–0.0001

Approximate root: $x \approx 0.567$

94. Find x such that $\ln x = 3 - x$.

$$f(x) = x + (\ln x) - 3 = 0$$

$$f'(x) = 1 + \frac{1}{x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n\left[\frac{4 - \ln x_n}{1 + x_n}\right]$$

n	1	2	3
x_n	2	2.2046	2.2079
$f(x_n)$	–0.3069	–0.0049	0.0000

Approximate root: $x \approx 2.208$

95. $g(x) = \ln f(x), \quad f(x) > 0$

$$g'(x) = \frac{f'(x)}{f(x)}$$

Yes. If the graph of g is increasing, then $g'(x) > 0$. Because $f(x) > 0$, you know that $f'(x) = g'(x)f(x)$ and so, $f'(x) > 0$. Therefore, the graph of f is increasing.

96. No. From Exercise 95, let $f(x) = x^2 + 1$ (positive and concave upward). $g(x) = \ln(x^2 + 1)$ is not concave upward.

97. No. For example, $(\ln 2)(\ln 3) \approx 0.76$, whereas $\ln(2 \cdot 3) = \ln 6 \approx 1.79$.

The correct formula is $\ln xy = \ln x + \ln y$.

98. (a) $\displaystyle\lim_{h \to \infty} T = 20$

The temperature of the object seems to approach 20°C, which is the temperature of the surrounding medium.

(b) The temperature changes most rapidly when it is first removed from the furnace. The slope is steepest at $h = 0$.

99. True. $\ln\left(a^{n+m}\right) = (n + m)\ln a = n \ln a + m \ln a$

100. True. $\ln(cx) = \ln c + \ln x$ and

$$\frac{d}{dx}[\ln(cx)] = 0 + \frac{1}{x} = \frac{d}{dx}[\ln x].$$

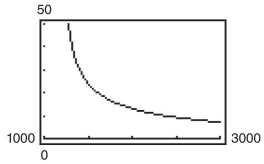
101. False; π is a constant.

$$\frac{d}{dx}[\ln \pi] = 0$$

102. False. If $y = \ln e = 1$, then $y' = 0$.

103. $t = 13.375 \ln\left(\dfrac{x}{x - 1250}\right), \quad x > 1250$

(a)



(b) When $x = 1398.43$: $t \approx 30$ years

Total amount paid $= (1398.43)(30)(12) = \$503{,}434.80$

(c) When $x = 1611.19$: $t \approx 20$ years

Total amount paid $= (1611.19)(20)(12) = \$386{,}685.60$

(d) $\dfrac{dt}{dx} = \dfrac{d}{dx}\left[13.375(\ln x - \ln(x - 1250))\right] = 13.375\left[\dfrac{1}{x} - \dfrac{1}{x - 1250}\right] = \dfrac{-16718.75}{x(x - 1250)}$

When $x = 1398.43$: $\dfrac{dt}{dx} \approx -0.0805$

When $x = 1611.19$: $\dfrac{dt}{dx} \approx -0.0287$

(e) The benefits include a shorter term, and a lower total amount paid.

104. (a) $\beta = \dfrac{10}{\ln 10} \ln\left(\dfrac{I}{10^{-16}}\right)$

$= \dfrac{10}{\ln 10}\left[\ln I - \ln 10^{-16}\right]$

$= \dfrac{10}{\ln 10}\left[\ln I + 16 \ln 10\right]$

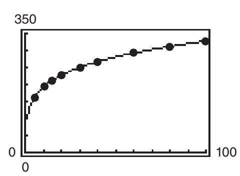
$= \dfrac{10}{\ln 10} \ln I + 160$

$= 10 \log_{10} I + 160$

(b) $\beta(10^{-5}) = \dfrac{10}{\ln 10} \ln 10^{-5} + 160$

$= -50 + 160 = 110$ decibels

105. (a)

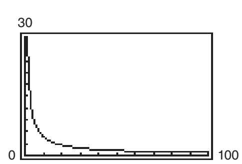


(b) $T'(p) = \dfrac{34.96}{p} + \dfrac{3.955}{\sqrt{p}}$

$T'(10) \approx 4.75$ deg/ lb / in.^{2}

$T'(70) \approx 0.97$ deg/ lb / in.^{2}

(c)



$\lim\limits_{p \to \infty} T'(p) = 0$

Answers will vary. *Sample answer*: As the pounds per square inch approach infinity, the temperature will not change.

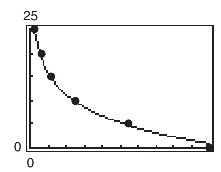
106. (a) You get an error message because $\ln h$ does not exist for $h = 0$.

(b) Reversing the data, you obtain

$h = 0.8627 - 6.4474 \ln p$.

[**Note:** Fit a line to the data $(x, y) = (\ln p, h)$.]

(c)



(d) If $p = 0.75$, $h \approx 2.72$ km.

(e) If $h = 13$ km, $p \approx 0.15$ atmosphere.

(f) $h = 0.8627 - 6.4474 \ln p$

$1 = -6.4474 \dfrac{1}{p} \dfrac{dp}{dh}$ (implicit differentiation)

$\dfrac{dp}{dh} = \dfrac{p}{-6.4474}$

For $h = 5$, $p = 0.5264$ and

$dp/dh = -0.0816$ atmos/km.

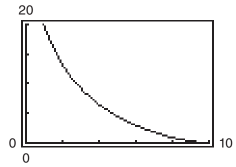
For $h = 20$, $p = 0.0514$ and

$dp/dh = -0.0080$ atmos/km.

As the altitude increases, the rate of change of pressure decreases.

107. $y = 10\ln\left(\dfrac{10 + \sqrt{100 - x^2}}{x}\right) - \sqrt{100 - x^2} = 10\left[\ln\left(10 + \sqrt{100 - x^2}\right) - \ln x\right] - \sqrt{100 - x^2}$

(a)



(b) $$\frac{dy}{dx} = 10\left[\frac{-x}{\sqrt{100 - x^2}\left(10 + \sqrt{100 - x^2}\right)} - \frac{1}{x}\right] + \frac{x}{\sqrt{100 - x^2}} = \frac{x}{\sqrt{100 - x^2}}\left[\frac{-10}{10 + \sqrt{100 - x^2}}\right] - \frac{10}{x} + \frac{x}{\sqrt{100 - x^2}}$$

$$= \frac{x}{\sqrt{100 - x^2}}\left[\frac{-10}{10 + \sqrt{100 - x^2}} + 1\right] - \frac{10}{x} = \frac{x}{\sqrt{100 - x^2}}\left[\frac{\sqrt{100 - x^2}}{10 + \sqrt{100 - x^2}}\right] - \frac{10}{x}$$

$$= \frac{x}{10 + \sqrt{100 - x^2}} - \frac{10}{x} = \frac{x\left(10 - \sqrt{100 - x^2}\right)}{x^2} - \frac{10}{x} = -\frac{\sqrt{100 - x^2}}{x}$$

When $x = 5$, $dy/dx = -\sqrt{3}$.

When $x = 9$, $dy/dx = -\sqrt{19}/9$.

(c) $\displaystyle\lim_{x \to 10^-} \frac{dy}{dx} = 0$

108. $p(x) = \dfrac{x}{\ln x}$

$$p'(x) = \frac{(\ln x)(1) - x\left(\frac{1}{x}\right)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$$

(a) $p'(1000) = \dfrac{\ln 1000 - 1}{(\ln 1000)^2} \approx 0.1238$

About 12.4 primes per 100 integers

(b) $p'(1{,}000{,}000) = \dfrac{\ln(1{,}000{,}000) - 1}{(\ln 1{,}000{,}000)^2} \approx 0.0671$

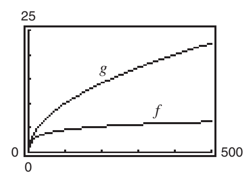
About 6.7 primes per 100 integers

(c) $p'(1{,}000{,}000{,}000) = \dfrac{\ln(1{,}000{,}000{,}000) - 1}{(\ln 1{,}000{,}000{,}000)^2}$

≈ 0.0459

About 4.6 primes per 100 integers

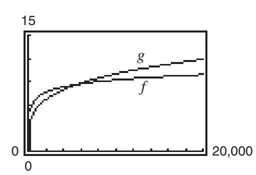
109. (a) $f(x) = \ln x$, $g(x) = \sqrt{x}$



$f'(x) = \dfrac{1}{x}$, $g'(x) = \dfrac{1}{2\sqrt{x}}$

For $x > 4$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for "large" values of x.

(b) $f(x) = \ln x$, $g(x) = \sqrt[4]{x}$



$f'(x) = \dfrac{1}{x}$, $g'(x) = \dfrac{1}{4\sqrt[4]{x^3}}$

For $x > 256$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for "large" values of x. $f(x) = \ln x$ increases very slowly for "large" values of x.

Section 5.2 The Natural Logarithmic Function: Integration

1. No. To use the Log Rule, look for quotients in which the numerator is the derivative of the denominator, with rewriting in mind. This integral requires the General Power Rule:

$$\int \frac{x}{\left(x^2 - 4\right)^3}\,dx = \frac{1}{2}\int \left(x^2 - 4\right)^{-3} 2x\,dx.$$

2. If a rational function has a numerator of degree greater than or equal to that of the denominator, long division may help.

3. Some ways to alter an integrand include rewriting using a trigonometric identity, multiplying and dividing by the same quantity, adding and subtracting the same quantity, and using long division.

4. The integral of the cotangent function:

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln\left|\sin x\right| + C$$

5. $\displaystyle\int \frac{5}{x}\,dx = 5\int \frac{1}{x}\,dx = 5\ln|x| + C$

6. $u = x - 5,\ du = dx$

$$\int \frac{1}{x - 5}\,dx = \ln|x - 5| + C$$

7. $u = 2x + 5,\ du = 2\,dx$

$$\int \frac{1}{2x + 5}\,dx = \frac{1}{2}\int \frac{1}{2x + 5}(2)\,dx = \frac{1}{2}\ln|2x + 5| + C$$

8. $u = 5 - 4x,\ du = -4\,dx$

$$\int \frac{9}{5 - 4x}\,dx = -\frac{9}{4}\int \frac{1}{5 - 4x}(-4dx) = -\frac{9}{4}\ln|5 - 4x| + C$$

9. $u = x^2 - 3,\ du = 2x\,dx$

$$\int \frac{x}{x^2 - 3}\,dx = \frac{1}{2}\int \frac{1}{x^2 - 3}(2x)\,dx = \frac{1}{2}\ln\left|x^2 - 3\right| + C$$

10. $u = 5 - x^3,\ du = -3x^2\,dx$

$$\int \frac{x^2}{5 - x^3}\,dx = -\frac{1}{3}\int \frac{1}{5 - x^3}\left(-3x^2\right)dx = -\frac{1}{3}\ln\left|5 - x^3\right| + C$$

11. $u = x^4 + 3x,\ du = \left(4x^3 + 3\right)dx$

$$\int \frac{4x^3 + 3}{x^4 + 3x}\,dx = \int \frac{1}{x^4 + 3x}\left(4x^3 + 3\right)dx = \ln\left|x^4 + 3x\right| + C$$

12. $u = x^3 - 3x^2,\ du = \left(3x^2 - 6x\right)dx = 3\left(x^2 - 2x\right)dx$

$$\int \frac{x^2 - 2x}{x^3 - 3x^2}\,dx = \frac{1}{3}\int \frac{1}{x^3 - 3x^2}\left(3x^2 - 6x\right)dx = \frac{1}{3}\ln\left|x^3 - 3x^2\right| + C$$

13. $\displaystyle\int \frac{x^2 - 7}{7x}\,dx = \frac{1}{7}\int x\,dx - \int \frac{1}{x}\,dx = \frac{1}{14}x^2 - \ln|x| + C$

14. $\displaystyle\int \frac{x^3 - 8x}{x^2}\,dx = \int\left(x - \frac{8}{x}\right)dx = \frac{x^2}{2} - 8\ln|x| + C$

15. $u = x^3 + 3x^2 + 9x,\ du = 3\left(x^2 + 2x + 3\right)dx$

$$\int \frac{x^2 + 2x + 3}{x^3 + 3x^2 + 9x}\,dx = \frac{1}{3}\int \frac{3\left(x^2 + 2x + 3\right)}{x^3 + 3x^2 + 9x}\,dx = \frac{1}{3}\ln\left|x^3 + 3x^2 + 9x\right| + C$$

16. $u = x^3 + 6x^2 + 5,\ du = \left(3x^2 + 12x\right)dx = 3\left(x^2 + 4x\right)dx$

$$\int \frac{x^2 + 4x}{x^3 + 6x^2 + 5}\,dx = \frac{1}{3}\int \frac{1}{x^3 + 6x^2 + 5}3\left(x^2 + 4x\right)dx = \frac{1}{3}\ln\left|x^3 + 6x^2 + 5\right| + C$$

17. $\displaystyle\int \frac{x^2 - 3x + 2}{x + 1}\,dx = \int\left(x - 4 + \frac{6}{x + 1}\right)dx = \frac{x^2}{2} - 4x + 6\ln|x + 1| + C$

18. $\displaystyle\int \frac{2x^2 + 7x - 3}{x - 2}\,dx = \int\left(2x + 11 + \frac{19}{x - 2}\right)dx = x^2 + 11x + 19\ln|x - 2| + C$

19. $\int \frac{x^3 - 3x^2 + 5}{x - 3}\,dx = \int\left(x^2 + \frac{5}{x-3}\right)dx$

$= \frac{x^3}{3} + 5\ln|x - 3| + C$

20. $\int \frac{x^3 - 6x - 20}{x+5}\,dx = \int\left(x^2 - 5x + 19 - \frac{115}{x+5}\right)dx$

$= \frac{x^3}{3} - \frac{5x^2}{2} + 19x - 115\ln|x+5| + C$

21. $\int \frac{x^4 + x - 4}{x^2 + 2}\,dx = \int\left(x^2 - 2 + \frac{x}{x^2+2}\right)dx$

$= \frac{x^3}{3} - 2x + \frac{1}{2}\ln\left(x^2 + 2\right) + C$

$= \frac{x^3}{3} - 2x + \ln\sqrt{x^2 + 2} + C$

22. $\int \frac{x^3 - 4x^2 - 4x + 20}{x^2 - 5}\,dx = \int\left(x - 4 + \frac{x}{x^2 - 5}\right)dx$

$= \frac{x^2}{2} - 4x + \frac{1}{2}\ln\left|x^2 - 5\right| + C$

23. $u = \ln x,\ du = \frac{1}{x}dx$

$\int \frac{(\ln x)^2}{x}\,dx = \frac{1}{3}(\ln x)^3 + C$

24. $u = \ln x^2,\ du = \frac{1}{x^2}(2x)\,dx = \frac{2}{x}\,dx$

$\int \frac{dx}{x\left(\ln x^2\right)^3} = \frac{1}{2}\int\left(\ln x^2\right)^{-3}\frac{2}{x}\,dx$

$= \frac{1}{2}\frac{\left(\ln x^2\right)^{-2}}{(-2)} + C$

$= \frac{-1}{4\left(\ln x^2\right)^2} + C$

or, $\int \frac{dx}{x\left(\ln x^2\right)^3} = \int \frac{dx}{x(2\ln x)^3} = \frac{1}{8}\int(\ln x)^{-3}\frac{1}{x}\,dx$

$= \frac{1}{8}\frac{(\ln x)^{-2}}{(-2)} + C = \frac{-1}{16\left[\ln(x)\right]^2} + C$

The answers are the same.

25. $u = 1 - 3\sqrt{x},\ du = \frac{-3}{2\sqrt{x}}$

$\int \frac{1}{\sqrt{x}\left(1 - 3\sqrt{x}\right)}\,dx = -\frac{2}{3}\int \frac{1}{1 - 3\sqrt{x}}\left(\frac{-3}{2\sqrt{x}}\right)dx$

$= -\frac{2}{3}\ln\left|1 - 3\sqrt{x}\right| + C$

26. $u = 1 + x^{1/3},\ du = \frac{1}{3x^{2/3}}\,dx$

$\int \frac{1}{x^{2/3}\left(1 + x^{1/3}\right)}\,dx = 3\int \frac{1}{1 + x^{1/3}}\left(\frac{1}{3x^{2/3}}\right)dx$

$= 3\ln\left|1 + x^{1/3}\right| + C$

27. $\int \frac{6x}{(x-5)^2}\,dx = 6\int \frac{x}{(x-5)^2}\,dx$

$= 6\int \frac{x-5}{(x-5)^2}\,dx + 6\int \frac{5}{(x-5)^2}\,dx$

$= 6\int \frac{1}{x-5}\,dx + 30\int (x-5)^2\,dx$

$= 6\ln|x-5| - 30(x-5)^{-1} + C$

$= 6\ln|x-5| - \frac{30}{x-5} + C$

28. $\int \frac{x(x-2)}{(x-1)^3}\,dx = \int \frac{x^2 - 2x + 1 - 1}{(x-1)^3}\,dx$

$= \int \frac{(x-1)^2}{(x-1)^3}\,dx - \int \frac{1}{(x-1)^3}\,dx$

$= \int \frac{1}{x-1}\,dx - \int \frac{1}{(x-1)^3}\,dx$

$= \ln|x-1| + \frac{1}{2(x-1)^2} + C$

29. $u = 1 + \sqrt{2x},\ du = \frac{1}{\sqrt{2x}}\,dx \Rightarrow (u-1)\,du = dx$

$\int \frac{1}{1 + \sqrt{2x}}\,dx = \int \frac{(u-1)}{u}\,du = \int\left(1 - \frac{1}{u}\right)du$

$= u - \ln|u| + C_1$

$= \left(1 + \sqrt{2x}\right) - \ln\left|1 + \sqrt{2x}\right| + C_1$

$= \sqrt{2x} - \ln\left(1 + \sqrt{2x}\right) + C$

where $C = C_1 + 1$.

30. $u = 1 + \sqrt{5x}, du = \frac{5}{2\sqrt{5x}}\,dx \Rightarrow dx = \frac{2\sqrt{5x}}{5}\,du \Rightarrow dx = \frac{2}{5}(u - 1)\,du$

$$\begin{aligned}\int \frac{4}{1 + \sqrt{5x}}\,dx &= \int \frac{4}{u}\left[\frac{2}{5}(u - 1)\right] du\\ &= \frac{8}{5}\int\left(1 - \frac{1}{u}\right) du\\ &= \frac{8}{5}u - \frac{8}{5}\ln|u| + C\\ &= \frac{8}{5}\left[1 + \sqrt{5x} - \ln\left(1 + \sqrt{5x}\right)\right] + C\\ &= \frac{8}{5}\left[\sqrt{5x} - \ln\left(1 + \sqrt{5x}\right)\right] + C_1\end{aligned}$$

31. $u = \sqrt{x} - 3, du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2(u + 3)du = dx$

$$\begin{aligned}\int \frac{\sqrt{x}}{\sqrt{x} - 3}\,dx &= 2\int \frac{(u + 3)^2}{u}\,du\\ &= 2\int \frac{u^2 + 6u + 9}{u}\,du = 2\int\left(u + 6 + \frac{9}{u}\right) du\\ &= 2\left[\frac{u^2}{2} + 6u + 9\ln|u|\right] + C_1\\ &= u^2 + 12u + 18\ln|u| + C_1\\ &= \left(\sqrt{x} - 3\right)^2 + 12\left(\sqrt{x} - 3\right) + 18\ln\left|\sqrt{x} - 3\right| + C_1\\ &= x + 6\sqrt{x} + 18\ln\left|\sqrt{x} - 3\right| + C\end{aligned}$$

where $C = C_1 - 27$.

32. $u = x^{1/3} - 1, du = \frac{1}{3x^{2/3}}\,dx \Rightarrow dx = 3(u + 1)^2\,du$

$$\begin{aligned}\int \frac{\sqrt[3]{x}}{\sqrt[3]{x} - 1}\,dx &= \int \frac{u + 1}{u}3(u + 1)^2 du\\ &= 3\int \frac{u + 1}{u}\left(u^2 + 2u + 1\right) du\\ &= 3\int\left(u^2 + 3u + 3 + \frac{1}{u}\right) du\\ &= 3\left[\frac{u^3}{3} + \frac{3u^2}{2} + 3u + \ln|u|\right] + C\\ &= 3\left[\frac{\left(x^{1/3} - 1\right)^3}{3} + \frac{3\left(x^{1/3} - 1\right)^2}{2} + 3\left(x^{1/3} - 1\right) + \ln\left|x^{1/3} - 1\right|\right] + C\\ &= 3\ln\left|x^{1/3} - 1\right| + \frac{3x^{2/3}}{2} + 3x^{1/3} + x + C_1\end{aligned}$$

33.
$$\begin{aligned}\int \cot\left(\frac{\theta}{3}\right) d\theta &= 3\int \cot\left(\frac{\theta}{3}\right)\left(\frac{1}{3}\right) d\theta\\ &= 3\ln\left|\sin\frac{\theta}{3}\right| + C\end{aligned}$$

34.
$$\begin{aligned}\int \theta\tan\left(2\theta^2\right) d\theta &= \frac{1}{4}\int \tan\left(2\theta^2\right)(4\theta\,d\theta)\\ &= \frac{-1}{4}\ln\left|\cos\left(2\theta^2\right)\right| + C\end{aligned}$$

35. $\int \csc 2x\,dx = \frac{1}{2}\int (\csc 2x)(2)\,dx$

$= -\frac{1}{2}\ln\left|\csc 2x + \cot 2x\right| + C$

36. $\int \sec \frac{x}{2}\,dx = 2\int \sec \frac{x}{2}\left(\frac{1}{2}\right)dx$

$= 2\ln\left|\sec \frac{x}{2} + \tan \frac{x}{2}\right| + C$

37. $\int (5 - \cos 3\theta)\,d\theta = \int 5d\theta - \frac{1}{3}\int \cos 3\theta(3\,d\theta)$

$= 5\theta - \frac{1}{3}\sin 3\theta + C$

38. $\int\left(2 - \tan\frac{\theta}{4}\right)d\theta = \int 2d\theta - 4\int \tan\frac{\theta}{4}\left(\frac{1}{4}\right)d\theta$

$= 2\theta + 4\ln\left|\cos\frac{\theta}{4}\right| + C$

39. $u = 1 + \sin t,\ du = \cos t\,dt$

$\int \frac{\cos t}{1 + \sin t}\,dt = \ln\left|1 + \sin t\right| + C$

40. $u = \cot t,\ du = -\csc^2 t\,dt$

$\int \frac{\csc^2 t}{\cot t}\,dt = -\ln\left|\cot t\right| + C$

41. $u = \sec x - 1,\ du = \sec x \tan x\,dx$

$\int \frac{\sec x \tan x}{\sec x - 1}\,dx = \ln\left|\sec x - 1\right| + C$

42. $\int(\sec 2x + \tan 2x)\,dx = \frac{1}{2}\int(\sec 2x + \tan 2x)(2)\,dx = \frac{1}{2}\ln\left|\sec 2x + \tan 2x\right| - \ln\left|\cos 2x\right| + C$

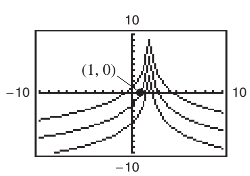
43. $y = \int \frac{3}{2 - x}\,dx$

$= -3\int \frac{1}{x - 2}\,dx$

$= -3\ln\left|x - 2\right| + C$

$(1, 0)$: $0 = -3\ln\left|1 - 2\right| + C \Rightarrow C = 0$

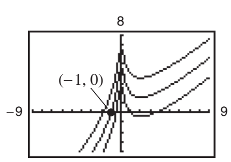
$y = -3\ln\left|x - 2\right|$



44. $y = \int \frac{x - 2}{x}\,dx = \int\left(1 - \frac{2}{x}\right)dx = x - 2\ln\left|x\right| + C$

$(-1, 0)$: $0 = -1 - 2\ln\left|-1\right| + C = -1 + C \Rightarrow C = 1$

$y = x - 2\ln\left|x\right| + 1$

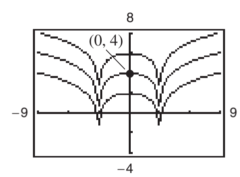


45. $y = \int \frac{2x}{x^2 - 9}\,dx$

$= \ln\left|x^2 - 9\right| + C$

$(0, 4)$: $4 = \ln\left|0 - 9\right| + C \Rightarrow C = 4 - \ln 9$

$y = \ln\left|x^2 - 9\right| + 4 - \ln 9$

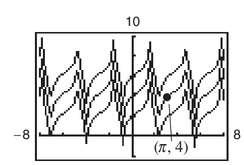


46. $r = \int \frac{\sec^2 t}{\tan t + 1}\,dt$

$= \ln\left|\tan t + 1\right| + C$

$(\pi, 4)$: $4 = \ln\left|0 + 1\right| + C \Rightarrow C = 4$

$r = \ln\left|\tan t + 1\right| + 4$



47. $f''(x) = \dfrac{2}{x^2} = 2x^{-2}, \quad x > 0$

$f'(x) = \dfrac{-2}{x} + C$

$f'(1) = 1 = -2 + C \Rightarrow C = 3$

$f'(x) = \dfrac{-2}{x} + 3$

$f(x) = -2 \ln x + 3x + C_1$

$f(1) = 1 = -2(0) + 3 + C_1 \Rightarrow C_1 = -2$

$f(x) = -2 \ln x + 3x - 2$

48. $f''(x) = \dfrac{-4}{(x-1)^2} - 2 = -4(x-1)^{-2} - 2, \quad x > 1$

$f'(x) = \dfrac{4}{(x-1)} - 2x + C$

$f'(2) = 0 = 4 - 4 + C \Rightarrow C = 0$

$f'(x) = \dfrac{4}{x-1} - 2x$

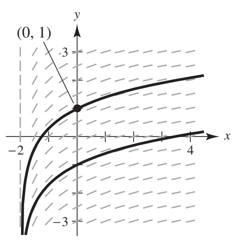
$f(x) = 4 \ln(x-1) - x^2 + C_1$

$f(2) = 3 = 4(0) - 4 + C_1 \Rightarrow C_1 = 7$

$f(x) = 4 \ln(x-1) - x^2 + 7$

49. $\dfrac{dy}{dx} = \dfrac{1}{x+2}, (0, 1)$

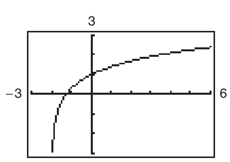
(a)



(b) $y = \displaystyle\int \frac{1}{x+2}\, dx = \ln|x+2| + C$

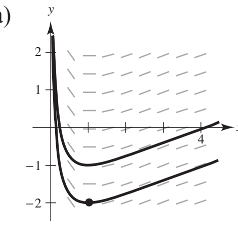
$y(0) = 1 \Rightarrow 1 = \ln 2 + C \Rightarrow C = 1 - \ln 2$

So, $y = \ln|x+2| + 1 - \ln 2 = \ln\left(\dfrac{x+2}{2}\right) + 1.$



50. $\dfrac{dy}{dx} = \dfrac{\ln x}{x}, (1, -2)$

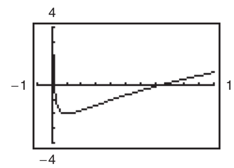
(a)



(b) $y = \displaystyle\int \frac{\ln x}{x}\, dx = \frac{(\ln x)^2}{2} + C$

$y(1) = -2 \Rightarrow -2 = \dfrac{(\ln 1)^2}{2} + C \Rightarrow C = -2$

So, $y = \dfrac{(\ln x)^2}{2} - 2.$



51. $\displaystyle\int_0^4 \frac{5}{3x+1}\, dx = \left[\frac{5}{3} \ln|3x+1|\right]_0^4 = \frac{5}{3} \ln 13 \approx 4.275$

52. $\displaystyle\int_{-1}^1 \frac{1}{2x+3}\, dx = \frac{1}{2}\Big[\ln|2x+3|\Big]_{-1}^1$

$= \dfrac{1}{2}[\ln 5 - \ln 1] = \dfrac{1}{2} \ln 5 \approx 0.805$

53. $u = 1 + \ln x, du = \dfrac{1}{x}\, dx$

$\displaystyle\int_1^e \frac{(1 + \ln x)^2}{x}\, dx = \left[\frac{1}{3}(1 + \ln x)^3\right]_1^e = \frac{7}{3}$

54. $u = \ln x, du = \dfrac{1}{x} dx$

$\displaystyle\int_e^{e^2} \frac{1}{x \ln x}\, dx = \int_e^{e^2} \left(\frac{1}{\ln x}\right)\frac{1}{x}\, dx = \Big[\ln|\ln|x||\Big]_e^{e^2} = \ln 2$

≈ 0.693

55. $\displaystyle\int_0^2 \frac{x^2 - 2}{x+1}\, dx = \int_0^2 \left(x - 1 - \frac{1}{x+1}\right) dx$

$= \left[\dfrac{1}{2}x^2 - x - \ln|x+1|\right]_0^2 = -\ln 3$

≈ -1.099

56. $\displaystyle\int_0^1 \frac{x-1}{x+1}\, dx = \int_0^1 1\, dx + \int_0^1 \frac{-2}{x+1}\, dx$

$= \Big[x - 2 \ln|x+1|\Big]_0^1 = 1 - 2 \ln 2$

≈ -0.386

57. $\displaystyle\int_1^2 \frac{1-\cos\theta}{\theta-\sin\theta}\,d\theta = \Big[\ln|\theta-\sin\theta|\Big]_1^2 = \ln\left|\frac{2-\sin 2}{1-\sin 1}\right| \approx 1.929$

58. $u = 2\theta,\ du = 2\,d\theta,\ \theta = \dfrac{\pi}{8} \Rightarrow u = \dfrac{\pi}{4},\ \theta = \dfrac{\pi}{4} \Rightarrow u = \dfrac{\pi}{2}$

$$\begin{aligned}\int_{\pi/8}^{\pi/4}(\csc 2\theta - \cot 2\theta)\,d\theta &= \frac{1}{2}\int_{\pi/4}^{\pi/2}(\csc u - \cot u)\,du\\ &= \frac{1}{2}\Big[-\ln|\csc u + \cot u| - \ln|\sin u|\Big]_{\pi/4}^{\pi/2}\\ &= \frac{1}{2}\left[-\ln(1+0) - \ln(1) + \ln\left(\sqrt{2}+1\right) + \ln\frac{\sqrt{2}}{2}\right]\\ &= \frac{1}{2}\left[\ln\left(\sqrt{2}+1\right) + \ln\frac{\sqrt{2}}{2}\right]\\ &= \frac{1}{2}\ln\left(1+\frac{\sqrt{2}}{2}\right)\end{aligned}$$

59. $\displaystyle\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx = 4\sqrt{x} - x - 4\ln\left(1+\sqrt{x}\right) + C$

60. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{\sin^2 x - \cos^2 x}{\cos x}\,dx = \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) - 2\sqrt{2} \approx -1.066$

Note: In Exercises 61–64, you can use the Second Fundamental Theorem of Calculus or integrate the function.

61. $F(x) = \displaystyle\int_1^x \frac{1}{t}\,dt$

$F'(x) = \dfrac{1}{x}$

62. $F(x) = \displaystyle\int_0^x \tan t\,dt$

$F'(x) = \tan x$

63.
$$\begin{aligned}F(x) &= \int_1^{4x} \cot t\,dt\\ &= \Big[\ln|\sin t|\Big]_1^{4x}\\ &= \ln|\sin 4x| - \ln|\sin(1)|\end{aligned}$$

$F'(x) = \dfrac{1}{\sin 4x}(\cos 4x)(4) = 4\cot 4x$

Alternate Solution:

Using the Second Fundamental Theorem of Calculus:

$F'(x) = \cot(4x)\dfrac{d}{dx}(4x) = 4\cot 4x$

64. Using the Second Fundamental Theorem of Calculus:

$F(x) = \displaystyle\int_0^{x^2} \frac{3}{t+1}\,dt$

$F'(x) = \dfrac{3}{\left(x^2\right)+1}\dfrac{d}{dx}\left(x^2\right) = \dfrac{6x}{x^2+1}$

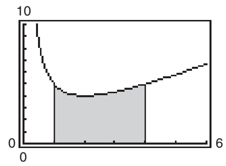
65. $A = \displaystyle\int_1^3 \frac{6}{x}\,dx = \Big[6\ln|x|\Big]_1^3 = 6\ln 3$

66.
$$\begin{aligned}A &= \int_1^2 \frac{1+\ln x^3}{x}\,dx = \int_1^2 \frac{1+3\ln x}{x}\,dx\\ &= \left[\ln x + \frac{3}{2}(\ln x)^2\right]_1^2\\ &= \ln 2 + \frac{3}{2}(\ln 2)^2\end{aligned}$$

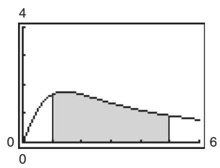
67.
$$\begin{aligned}A &= \int_0^1 \csc(x+1)\,dx\\ &= \Big[-\ln|\csc(x+1) + \cot(x+1)|\Big]_0^1\\ &= -\ln|\csc 2 + \cot 2| + \ln|\csc 1 + \cot 1|\\ &\approx 1.048\end{aligned}$$

68.
$$\begin{aligned}A &= \int_{\pi/4}^{3\pi/4} \frac{\sin x}{1+\cos x}\,dx\\ &= -\ln|1+\cos x|\Big]_{\pi/4}^{3\pi/4}\\ &= -\ln\left(1-\frac{\sqrt{2}}{2}\right) + \ln\left(1+\frac{\sqrt{2}}{2}\right)\\ &= \ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)\\ &= \ln\left(3+2\sqrt{2}\right)\end{aligned}$$

69. $A = \int_1^4 \frac{x^2 + 4}{x}\,dx = \int_1^4 \left(x + \frac{4}{x}\right) dx = \left[\frac{x^2}{2} + 4 \ln x\right]_1^4 = (8 + 4 \ln 4) - \frac{1}{2} = \frac{15}{2} + 8 \ln 2 \approx 13.045$

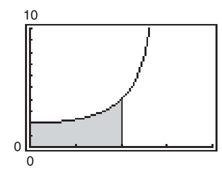


70. $A = \int_1^5 \frac{5x}{x^2 + 2}\,dx = \frac{5}{2}\int_1^5 \frac{1}{x^2 + 2}(2x\,dx) = \left[\frac{5}{2}\ln\left|x^2 + 2\right|\right]_1^5 = \frac{5}{2}(\ln 27 - \ln 3) = \frac{5}{2}\ln 9 = 5 \ln 3 \approx 5.4931$

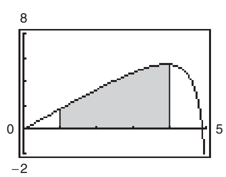


71. $\int_0^2 2 \sec \frac{\pi x}{6}\,dx = \frac{12}{\pi}\int_0^2 \sec\left(\frac{\pi x}{6}\right)\frac{\pi}{6}\,dx = \frac{12}{\pi}\left[\ln\left|\sec \frac{\pi x}{6} + \tan \frac{\pi x}{6}\right|\right]_0^2$

$= \frac{12}{\pi}\left(\ln\left|\sec \frac{\pi}{3} + \tan \frac{\pi}{3}\right| - \ln|1 + 0|\right) = \frac{12}{\pi}\ln\left(2 + \sqrt{3}\right) \approx 5.03041$



72. $\int_1^4 \left(2x - \tan(0.3x)\right) dx = \left[x^2 + \frac{10}{3}\ln\left|\cos(0.3x)\right|\right]_1^4 = \left[16 + \frac{10}{3}\ln\cos(1.2)\right] - \left[1 + \frac{10}{3}\ln\cos(0.3)\right] \approx 11.7686$



73. Average value $= \frac{1}{4 - 2}\int_2^4 \frac{8}{x^2}\,dx$

$= 4\int_2^4 x^{-2}\,dx$

$= \left[-4\frac{1}{x}\right]_2^4$

$= -4\left(\frac{1}{4} - \frac{1}{2}\right) = 1$

74. Average value $= \frac{1}{4 - 2}\int_2^4 \frac{4(x + 1)}{x^2}\,dx$

$= 2\int_2^4 \left(\frac{1}{x} + \frac{1}{x^2}\right) dx$

$= 2\left[\ln x - \frac{1}{x}\right]_2^4$

$= 2\left[\ln 4 - \frac{1}{4} - \ln 2 + \frac{1}{2}\right]$

$= 2\left[\ln 2 + \frac{1}{4}\right] = \ln 4 + \frac{1}{2} \approx 1.8863$

75. Average value $= \frac{1}{e - 1}\int_1^e \frac{2 \ln x}{x}\,dx$

$= \frac{2}{e - 1}\left[\frac{(\ln x)^2}{2}\right]_1^e$

$= \frac{1}{e - 1}(1 - 0)$

$= \frac{1}{e - 1} \approx 0.582$

76. Average value $= \frac{1}{2 - 0}\int_0^2 \sec \frac{\pi x}{6}\,dx$

$= \left[\frac{1}{2}\left(\frac{6}{\pi}\right)\ln\left|\sec \frac{\pi x}{6} + \tan \frac{\pi x}{6}\right|\right]_0^2$

$= \frac{3}{\pi}\left[\ln\left(2 + \sqrt{3}\right) - \ln(1 + 0)\right]$

$= \frac{3}{\pi}\ln\left(2 + \sqrt{3}\right)$

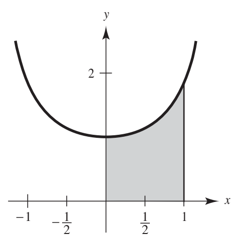
77. $n = 4, \Delta x = \frac{3-1}{4} = \frac{1}{2}$

Midpoint approximation: $\int_1^3 \frac{12}{x}\,dx \approx \frac{1}{2}\left[f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right)\right] \approx \frac{1}{2}[9.6 + 6.8571 + 5.3333 + 4.3636] \approx 13.077$

78. $n = 4, \Delta x = \frac{\pi/4}{4} = \frac{\pi}{16}$

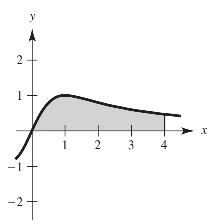
Midpoint approximation: $\int_0^{\pi/4} \sec x\,dx \approx \frac{\pi}{16}\left[f\left(\frac{\pi}{32}\right) + f\left(\frac{3\pi}{32}\right) + f\left(\frac{5\pi}{32}\right) + f\left(\frac{7\pi}{32}\right)\right] \approx 0.8791$

79.



$A \approx 1.25$; Matches (d)

80.



$A \approx 3$; Matches (a)

81. Let $f(t) = \ln t$ on $[x, y]$, $0 < x < y$.

By the Mean Value Theorem,

$$\frac{f(y) - f(x)}{y - x} = f'(c), \quad x < c < y,$$

$$\frac{\ln y - \ln x}{y - x} = \frac{1}{c}.$$

Because $0 < x < c < y$, $\frac{1}{x} > \frac{1}{c} > \frac{1}{y}$. So,

$$\frac{1}{y} < \frac{\ln y - \ln x}{y - x} < \frac{1}{x}.$$

82. The function is constant on $(0, \infty)$:

$$F(x) = \int_x^{2x} \frac{1}{t}\,dt = \left[\ln t\right]_x^{2x} = \ln 2x - \ln x$$

$$= \ln\left(\frac{2x}{x}\right) = \ln 2$$

83. $\int_1^x \frac{3}{t}\,dt = \int_{1/4}^x \frac{1}{t}\,dt$

$$\left[3\ln|t|\right]_1^x = \left[\ln|t|\right]_{1/4}^x$$

$$3\ln x = \ln x - \ln\left(\frac{1}{4}\right)$$

$$2\ln x = -\ln\left(\frac{1}{4}\right) = \ln 4$$

$$\ln x = \frac{1}{2}\ln 4 = \ln 2$$

$$x = 2$$

84. $\int_1^x \frac{1}{t}\,dt = \left[\ln|t|\right]_1^x = \ln x \qquad (\text{assume } x > 0)$

(a) $\ln x = \ln 5 \Rightarrow x = 5$

(b) $\ln x = 1 \Rightarrow x = e$

85. $\int \cot u\,du = \int \frac{\cos u}{\sin u}\,du = \ln|\sin u| + C$

Alternate solution:

$$\frac{d}{du}\left[\ln|\sin u| + C\right] = \frac{1}{\sin u}\cos u + C = \cot u + C$$

86. $\int \csc u\,du = \int \csc u\left(\frac{\csc u + \cot u}{\csc u + \cot u}\right)du = -\int \frac{1}{\csc u + \cot u}\left(-\csc u \cot u - \csc^2 u\right)du = -\ln|\csc u + \cot u| + C$

Alternate solution:

$$\frac{d}{du}\left[-\ln|\csc u + \cot u| + C\right] = -\frac{1}{\csc u + \cot u}\left(-\csc u \cot u - \csc^2 u\right) = \frac{\csc u(\cot u + \csc u)}{\csc u + \cot u} = \csc u$$

87. $-\ln|\cos x| + C = \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C$

88. $\ln|\sin x| + C = \ln\left|\frac{1}{\csc x}\right| + C = -\ln|\csc x| + C$

89. $\ln|\sec x + \tan x| + C = \ln\left|\dfrac{(\sec x + \tan x)(\sec x - \tan x)}{(\sec x - \tan x)}\right| + C = \ln\left|\dfrac{\sec^2 x - \tan^2 x}{\sec x - \tan x}\right| + C$

$= \ln\left|\dfrac{1}{\sec x - \tan x}\right| + C = -\ln|\sec x - \tan x| + C$

90. $-\ln|\csc x + \cot x| + C = -\ln\left|\dfrac{(\csc x + \cot x)(\csc x - \cot x)}{(\csc x - \cot x)}\right| + C = -\ln\left|\dfrac{\csc^2 x - \cot^2 x}{\csc x - \cot x}\right| + C$

$= -\ln\left|\dfrac{1}{\csc x - \cot x}\right| + C = \ln|\csc x - \cot x| + C$

91. $P(t) = \int \dfrac{3000}{1 + 0.25t}\,dt = (3000)(4)\int \dfrac{0.25}{1 + 0.25t}\,dt$

$= 12{,}000 \ln|1 + 0.25t| + C$

$P(0) = 12{,}000 \ln|1 + 0.25(0)| + C = 1000$

$C = 1000$

$P(t) = 12{,}000 \ln|1 + 0.25t| + 1000$

$= 1000[12 \ln|1 + 0.25t| + 1]$

$P(3) = 1000[12(\ln 1.75) + 1] \approx 7715$

92. $\dfrac{dS}{dt} = \dfrac{k}{t}$

$S(t) = \int \dfrac{k}{t}\,dt = k \ln|t| + C = k \ln t + C$ because $t > 1$.

$S(2) = k \ln 2 + C = 200$

$S(4) = k \ln 4 + C = 300$

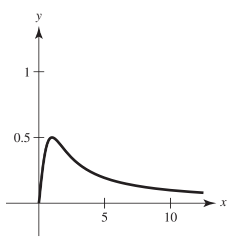
Solving this system yields $k = 100/\ln 2$ and $C = 100$.

So, $S(t) = \dfrac{100 \ln t}{\ln 2} + 100 = 100\left[\dfrac{\ln t}{\ln 2} + 1\right]$.

93. $t = \dfrac{10}{\ln 2}\int_{250}^{300} \dfrac{1}{T - 100}\,dT = \dfrac{10}{\ln 2}\Big[\ln(T - 100)\Big]_{250}^{300} = \dfrac{10}{\ln 2}[\ln 200 - \ln 150] = \dfrac{10}{\ln 2}\left[\ln\left(\dfrac{4}{3}\right)\right] \approx 4.1504$ min

94. $\dfrac{1}{50 - 40}\int_{40}^{50} \dfrac{90{,}000}{400 + 3x}\,dx = \Big[3000 \ln|400 + 3x|\Big]_{40}^{50} \approx \$168.27$

95. $f(x) = \dfrac{x}{1 + x^2}$



(a) $y = \dfrac{1}{2}x$ intersects $f(x) = \dfrac{x}{1 + x^2}$:

$\dfrac{1}{2}x = \dfrac{x}{1 + x^2}$

$1 + x^2 = 2$

$x = 1$

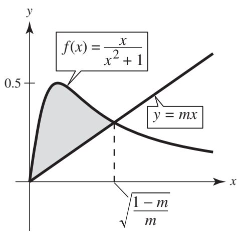
$A = \int_0^1 \left(\left[\dfrac{x}{1 + x^2}\right] - \dfrac{1}{2}x\right) dx = \left[\dfrac{1}{2}\ln(x^2 + 1) - \dfrac{x^2}{4}\right]_0^1 = \dfrac{1}{2}\ln 2 - \dfrac{1}{4}$

(b) $f'(x) = \dfrac{(1 + x^2) - x(2x)}{(1 + x^2)^2} = \dfrac{1 - x^2}{(1 + x^2)^2}$

$f'(0) = 1$

So, for $0 < m < 1$, the graphs of f and $y = mx$ enclose a finite region.

(c)



$f(x) = \dfrac{x}{x^2 + 1}$ intersects $y = mx$:

$$\frac{x}{1 + x^2} = mx$$

$$1 = m + mx^2$$

$$x^2 = \frac{1 - m}{m}$$

$$x = \sqrt{\frac{1 - m}{m}}$$

$$A = \int_0^{\sqrt{(1-m)/m}} \left(\frac{x}{1 + x^2} - mx\right) dx, \quad 0 < m < 1 = \left[\frac{1}{2}\ln\left(1 + x^2\right) - \frac{mx^2}{2}\right]_0^{\sqrt{(1-m)/m}} = \frac{1}{2}\ln\left(1 + \frac{1 - m}{m}\right) - \frac{1}{2}m\left(\frac{1 - m}{m}\right)$$

$$= \frac{1}{2}\ln\left(\frac{1}{m}\right) - \frac{1}{2}(1 - m) = \frac{1}{2}\left[m - \ln(m) - 1\right]$$

96. (a) At $x = -1$, $f'(-1) \approx \dfrac{1}{2}$.

The slope of f at $x = -1$ is approximately $\dfrac{1}{2}$.

(b) Since the slope is positive for $x > -2$, f is increasing on $(-2, \infty)$. Similarly, f is decreasing on $(-\infty, -2)$.

97. True

98. False. For example, let $\theta = \sqrt{\pi}$.

$\ln\left|\cos \theta^2\right| = \ln\left|\cos \pi\right| = \ln\left|-1\right| = \ln 1 = 0,$ whereas $\ln\left(\cos \theta^2\right)$ is undefined.

99. True

$$\int \frac{1}{x}\,dx = \ln|x| + C_1 = \ln|x| + \ln|C| = \ln|Cx|, C \neq 0$$

100. False. The integrand has a nonremovable discontinuity at $x = 0$.

101. First, note that the following function g satisfies the hypothesis for f.

$$g(x) = \begin{cases} 1, & 1 \le x \le 2 \\ -1, & 2 \le x \le 3 \end{cases}$$

Furthermore, $\displaystyle\int_1^3 \frac{g(x)}{x}\,dx = \int_1^2 \frac{1}{x}\,dx + \int_2^3 \frac{-1}{x}\,dx = \ln 2 - (\ln 3 - \ln 2) = \ln\frac{4}{3}$.

You claim that this is the largest value that $\displaystyle\int_1^3 \frac{f(x)}{x}\,dx$ can be. To see this, define

$h(x) = g(x) - f(x)$. Then $\displaystyle\int_1^3 h(x)\,dx = \int_1^3 g(x)\,dx - \int_1^3 f(x)\,dx = 0$. Also, $h(x) \ge 0$ for $1 \le x \le 2$, and $h(x) \le 0$ for $2 \le x \le 3$. Thus,

$$\int_1^3 \frac{h(x)}{x}\,dx = \int_1^2 \frac{|h(x)|}{x}\,dx - \int_2^3 \frac{|h(x)|}{x}\,dx \ge \int_1^2 \frac{|h(x)|}{2}\,dx - \int_2^3 \frac{|h(x)|}{2}\,dx = 0.$$

Hence, $\displaystyle\int_1^3 \frac{g(x)}{x}\,dx - \int_1^3 \frac{f(x)}{x}\,dx \ge 0$. Finally, you have $\displaystyle\int_1^3 \frac{f(x)}{x}\,dx \le \int_1^3 \frac{g(x)}{x}\,dx = \ln\frac{4}{3}$.

So, $\ln\dfrac{4}{3}$ is the largest value.

Section 5.3 Inverse Functions

1. The functions f and g have the effect of "undoing" each other. That is,

$$f(g(x)) = x \text{ and } g(f(x)) = x.$$

2. The graphs of f and f^{-1} are mirror images with respect to the line $y = x$.

3. No. The domain of f^{-1} is the range of f.

4. f^{-1} is also decreasing by Theorem 5.8.

5. Matches (c)

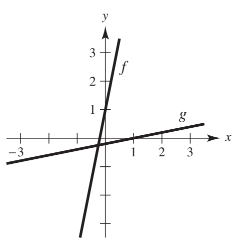
6. Matches (b)

7. Matches (a)

8. Matches (d)

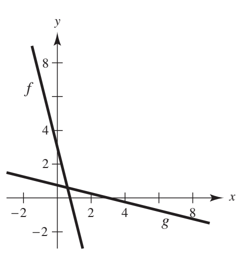
9. (a)
$$f(x) = 5x + 1$$
$$g(x) = \frac{x - 1}{5}$$
$$f(g(x)) = f\left(\frac{x - 1}{5}\right) = 5\left(\frac{x - 1}{5}\right) + 1 = x$$
$$g(f(x)) = g(5x + 1) = \frac{(5x - 1) - 1}{5} = x$$

(b)



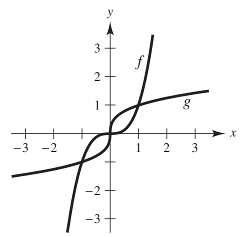
10. (a)
$$f(x) = 3 - 4x$$
$$g(x) = \frac{3 - x}{4}$$
$$f(g(x)) = f\left(\frac{3 - x}{4}\right) = 3 - 4\left(\frac{3 - x}{4}\right) = x$$
$$g(f(x)) = g(3 - 4x) = \frac{3 - (3 - 4x)}{4} = x$$

(b)



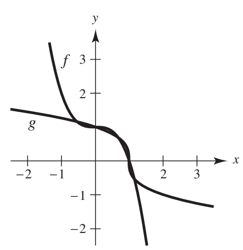
11. (a)
$$f(x) = x^3$$
$$g(x) = \sqrt[3]{x}$$
$$f(g(x)) = f\left(\sqrt[3]{x}\right) = \left(\sqrt[3]{x}\right)^3 = x$$
$$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$$

(b)



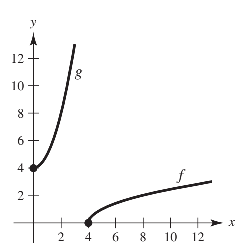
12. (a)
$$f(x) = 1 - x^3$$
$$g(x) = \sqrt[3]{1 - x}$$
$$f(g(x)) = f\left(\sqrt[3]{1 - x}\right) = 1 - \left(\sqrt[3]{1 - x}\right)^3$$
$$= 1 - (1 - x) = x$$
$$g(f(x)) = g(1 - x^3)$$
$$= \sqrt[3]{1 - (1 - x^3)} = \sqrt[3]{x^3} = x$$

(b)



13. (a)
$$f(x) = \sqrt{x - 4}$$
$$g(x) = x^2 + 4, \quad x \geq 0$$
$$f(g(x)) = f(x^2 + 4)$$
$$= \sqrt{(x^2 + 4) - 4} = \sqrt{x^2} = x$$
$$g(f(x)) = g\left(\sqrt{x - 4}\right)$$
$$= \left(\sqrt{x - 4}\right)^2 + 4 = x - 4 + 4 = x$$

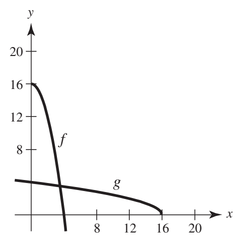
(b)



14. (a)

$$f(x) = 16 - x^2, \quad x \ge 0$$
$$g(x) = \sqrt{16 - x}$$
$$f(g(x)) = f\left(\sqrt{16-x}\right) = 16 - \left(\sqrt{16-x}\right)^2$$
$$= 16 - (16 - x) = x$$
$$g(f(x)) = g\left(16 - x^2\right) = \sqrt{16 - \left(16 - x^2\right)}$$
$$= \sqrt{x^2} = x$$

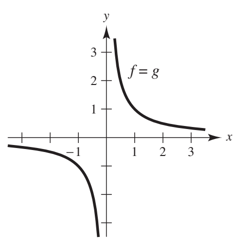
(b)



15. (a)

$$f(x) = \frac{1}{x}$$
$$g(x) = \frac{1}{x}$$
$$f(g(x)) = \frac{1}{1/x} = x$$
$$g(f(x)) = \frac{1}{1/x} = x$$

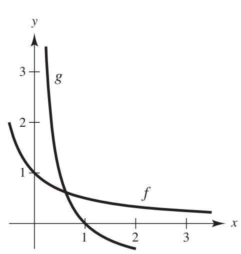
(b)



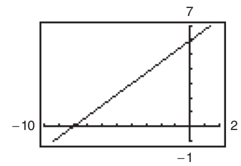
16. (a)

$$f(x) = \frac{1}{1+x}, \qquad x \ge 0$$
$$g(x) = \frac{1-x}{x}, \qquad 0 < x \le 1$$
$$f(g(x)) = f\left(\frac{1-x}{x}\right) = \frac{1}{1 + \dfrac{1-x}{x}} = \frac{1}{\dfrac{1}{x}} = x$$
$$g(f(x)) = g\left(\frac{1}{1+x}\right) = \frac{1 - \dfrac{1}{1+x}}{\dfrac{1}{1+x}} = \frac{x}{1+x} \cdot \frac{1+x}{1} = x$$

(b)

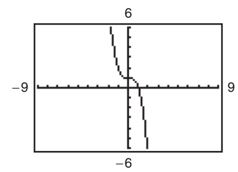


17. $f(x) = \frac{3}{4}x + 6$



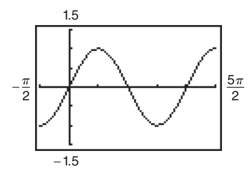
One-to-one; has an inverse

18. $f(x) = 1 - x^3$



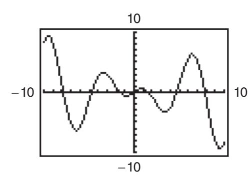
One-to-one; has an inverse

19. $f(\theta) = \sin \theta$



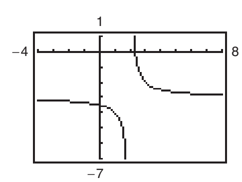
Not one-to-one; does not have an inverse

20. $f(x) = x \cos x$



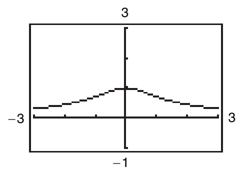
Not one-to-one; does not have an inverse

21. $h(s) = \dfrac{1}{s - 2} - 3$



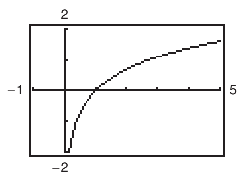
One-to-one; has an inverse

22. $g(t) = \dfrac{1}{\sqrt{t^2 + 1}}$



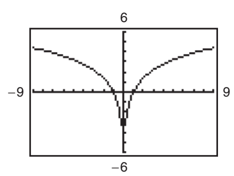
Not one-to-one; does not have an inverse

23. $f(x) = \ln x$



One-to-one; has an inverse

24. $h(x) = \ln x^2$



Not one-to-one; does not have an inverse

25. $f(x) = 2 - x - x^3$

$f'(x) = -1 - 3x^2 < 0$ for all x

f is decreasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

26. $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12 = 3(x - 2)^2 \geq 0$ for all x

f is increasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

27. $f(x) = 8x^3 + x^2 - 1$

$f'(x) = 24x^2 + 2x$

f is not strictly monotonic on $(-\infty, \infty)$. Therefore, f does not have an inverse.

(**Note:** f is increasing on $\left(-\infty, -\dfrac{1}{12}\right)$ and $(0, \infty)$, and decreasing on $\left(-\dfrac{1}{12}, 0\right)$.)

28. $f(x) = 1 - x^3 - 6x^5$

$f'(x) = -3x^2 - 30x^4 \leq 0$ for all x

f is decreasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

29. $f(x) = \ln(x - 3), \quad x > 3$

$f'(x) = \dfrac{1}{x - 3} > 0$ for $x > 3$

f is increasing on $(3, \infty)$. Therefore, f is strictly monotonic and has an inverse.

30. $f(x) = \cos \dfrac{3x}{2}$

$f'(x) = -\dfrac{3}{2} \sin \dfrac{3x}{2} = 0$ when $x = 0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \cdots$

f is not strictly monotonic on $(-\infty, \infty)$. Therefore, f does not have an inverse.

31. $f(x) = (x - 4)^2$ on $[4, \infty)$

$f'(x) = 2(x - 4) > 0$ on $[4, \infty)$

f is increasing on $[4, \infty)$. Therefore, f is strictly monotonic and has an inverse.

32. $f(x) = |x + 2|$ on $[-2, \infty)$

$f'(x) = \dfrac{|x + 2|}{x + 2}(1) = 1 > 0$ on $[-2, \infty)$

f is increasing on $[-2, \infty)$. Therefore, f is strictly monotonic and has an inverse.

33. $f(x) = \cot x$ on $(0, \pi)$

$f'(x) = -\csc^2 x < 0$ on $(0, \pi)$

f is decreasing on $(0, \pi)$. Therefore, f is strictly monotonic and has an inverse.

34. $f(x) = \sec x$ on $\left[0, \dfrac{\pi}{2}\right)$

$f'(x) = \sec x \tan x > 0$ on $\left(0, \dfrac{\pi}{2}\right)$

f is increasing on $[0, \pi/2)$. Therefore, f is strictly monotonic and has an inverse.

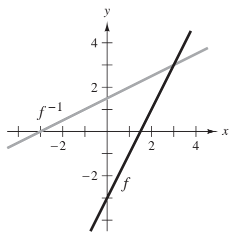
35. (a) $f(x) = 2x - 3 = y$

$$x = \frac{y+3}{2}$$

$$y = \frac{x+3}{2}$$

$$f^{-1}(x) = \frac{x+3}{2}$$

(b)



(c) The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

(d) Domain of f: all real numbers
Range of f: all real numbers
Domain of f^{-1}: all real numbers
Range of f^{-1}: all real numbers

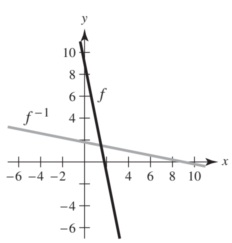
36. (a) $f(x) = 9 - 5x = y$

$$x = \frac{9-y}{5}$$

$$y = \frac{9-x}{5}$$

$$f^{-1}(x) = \frac{1}{5}(9 - x)$$

(b)



(c) The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

(d) Domain of f: all real numbers
Range of f: all real numbers
Domain of f^{-1}: all real numbers
Range of f^{-1}: all real numbers

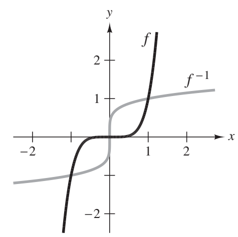
37. (a) $f(x) = x^5 = y$

$$x = \sqrt[5]{y}$$

$$y = \sqrt[5]{x}$$

$$f^{-1}(x) = \sqrt[5]{x} = x^{1/5}$$

(b)



(c) The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

(d) Domain of f: all real numbers
Range of f: all real numbers
Domain of f^{-1}: all real numbers
Range of f^{-1}: all real numbers

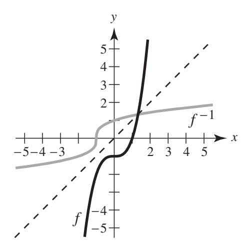
38. (a) $f(x) = x^3 - 1 = y$

$$x = \sqrt[3]{y+1}$$

$$y = \sqrt[3]{x+1}$$

$$f^{-1}(x) = \sqrt[3]{x+1} = (x+1)^{1/3}$$

(b)



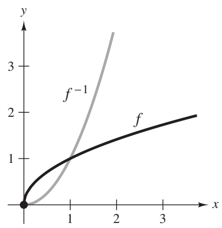
(c) The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

(d) Domain of f: all real numbers
Range of f: all real numbers
Domain of f^{-1}: all real numbers
Range of f^{-1}: all real numbers

39. (a)

$$f(x) = \sqrt{x} = y$$
$$x = y^2$$
$$y = x^2$$
$$f^{-1}(x) = x^2, \quad x \ge 0$$

(b)



(c) The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

(d) Domain of f: $x \ge 0$

Range of f: $y \ge 0$

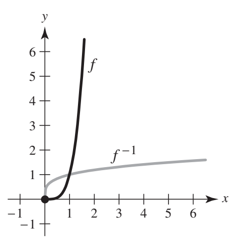
Domain of f^{-1}: $x \ge 0$

Range of f^{-1}: $y \ge 0$

40. (a)

$$f(x) = x^4, x \ge 0$$
$$y = x^4$$
$$x = y^{1/4}$$
$$f^{-1}(x) = x^{1/4} = \sqrt[4]{x}$$

(b)



(c) The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

(d) Domain of f: $x \ge 0$

Range of f: $y \ge 0$

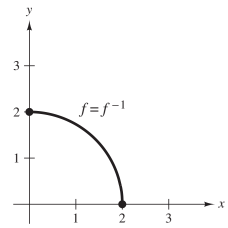
Domain of f^{-1}: $x \ge 0$

Range of f^{-1}: $y \ge 0$

41. (a)

$$f(x) = \sqrt{4 - x^2} = y, \quad 0 \le x \le 2$$
$$4 - x^2 = y^2$$
$$x^2 = 4 - y^2$$
$$x = \sqrt{4 - y^2}$$
$$y = \sqrt{4 - x^2}$$
$$f^{-1}(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2$$

(b)



(c) The graphs of f and f^{-1} are reflections of each other across the line $y = x$. In fact, the graphs are identical.

(d) Domain of f: $0 \le x \le 2$

Range of f: $0 \le y \le 2$

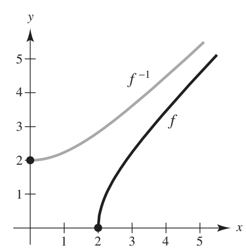
Domain of f^{-1}: $0 \le x \le 2$

Range of f^{-1}: $0 \le y \le 2$

42. (a)

$$f(x) = \sqrt{x^2 - 4} = y, \quad x \ge 2$$
$$x^2 = y^2 + 4$$
$$x = \sqrt{y^2 + 4}$$
$$y = \sqrt{x^2 + 4}$$
$$f^{-1}(x) = \sqrt{x^2 + 4}, \quad x \ge 0$$

(b)



(c) The graphs of f and f^{-1} are reflections of each other across the line $y = x$

(d) Domain of f: $x \ge 2$

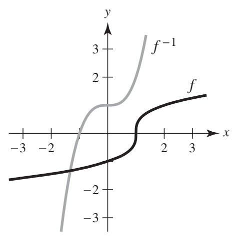
Range of f: $y \ge 0$

Domain of f^{-1}: $x \ge 0$

Range of f^{-1}: $y \ge 2$

43. (a) $$\begin{aligned} f(x) &= \sqrt[3]{x-1} = y \\ x - 1 &= y^3 \\ x &= y^3 + 1 \\ y &= x^3 + 1 \\ f^{-1}(x) &= x^3 + 1 \end{aligned}$$

(b)

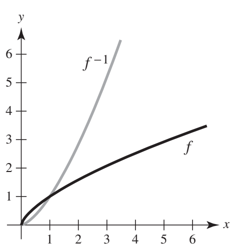


(c) The graphs of f and f^{-1} are reflections of each other across the line $y = x$

(d) Domain of f: all real numbers
Range of f: all real numbers
Domain of f^{-1}: all real numbers
Range of f^{-1}: all real numbers

44. (a) $$\begin{aligned} f(x) &= x^{2/3} = y, \quad x \ge 0 \\ x &= y^{3/2} \\ y &= x^{3/2} \\ f^{-1}(x) &= x^{3/2}, \ x \ge 0 \end{aligned}$$

(b)

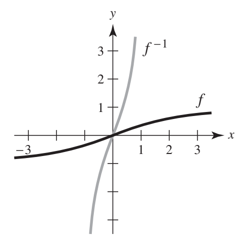


(c) The graphs of f and f^{-1} are reflections of each other across the line $y = x$

(d) Domain of f: $x \ge 0$
Range of f: $y \ge 0$
Domain of f^{-1}: $x \ge 0$
Range of f^{-1}: $y \ge 0$

45. (a) $$\begin{aligned} f(x) &= \frac{x}{\sqrt{x^2 + 7}} = y \\ x &= y\sqrt{x^2 + 7} \\ x^2 &= y^2(x^2 + 7) = y^2x^2 + 7y^2 \\ x^2(1 - y^2) &= 7y^2 \\ x &= \frac{\sqrt{7}y}{\sqrt{1 - y^2}} \\ y &= \frac{\sqrt{7}x}{\sqrt{1 - x^2}} \\ f^{-1}(x) &= \frac{\sqrt{7}x}{\sqrt{1 - x^2}}, \quad -1 < x < 1 \end{aligned}$$

(b)

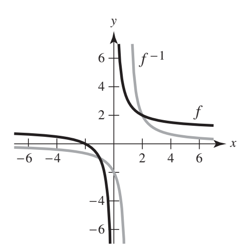


(c) The graphs of f and f^{-1} are reflections of each other in the line $y = x$

(d) Domain of f: all real numbers
Range of f: $-1 < y < 1$
Domain of f^{-1}: $-1 < x < 1$
Range of f^{-1}: all real numbers

46. (a) $$\begin{aligned} f(x) &= \frac{x + 2}{x} = y, \quad x \ne 0 \\ x + 2 &= yx \\ x(1 - y) &= -2 \\ x &= \frac{2}{y - 1} \\ y &= \frac{2}{x - 1} \\ f^{-1}(x) &= \frac{2}{x - 1}, \quad x \ne 1 \end{aligned}$$

(b)



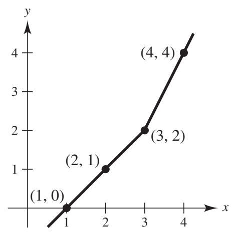
(c) The graphs of f and f^{-1} are reflections of each other in the line $y = x$

(d) Domain of f: all $x \ne 0$
Range of f: all $y \ne 1$
Domain of f^{-1}: all $x \ne 1$
Range of f^{-1}: all $y \ne 0$

47.

x	0	1	2	3
$f(x)$	1	2	3	4

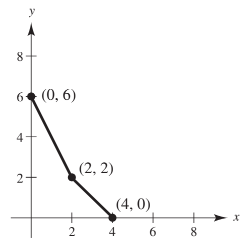
x	1	2	3	4
$f^{-1}(x)$	0	1	2	4



48.

x	0	2	6
$f(x)$	4	2	0

x	0	2	4
$f^{-1}(x)$	6	2	0



49. (a) Let x be the number of pounds of the commodity costing \$1.25 per pound. Because there are 50 pounds total, the amount of the second commodity is 50-x. The total cost is

$$f(x) = y = 1.25x + 2.75(50 - x)$$
$$= -1.5x + 137.5,\ 0 \le x \le 50$$

(b)
$$y = -1.5x + 137.5$$
$$1.5x = 137.5 - y$$
$$x = \frac{(137.5 - y)}{1.5}$$
$$y = f^{-1}(x) = \frac{2}{3}(137.5 - x)$$

x represents the total cost and y represents the number of pounds of the less expensive commodity.

(c) The range of f is $[62.5, 137.5]$, so the domain of f^{-1} is the same. $50(1.25) = 62.5$ gives the total cost when purchasing 50 pounds of the less expensive commodity, and $50(2.75) = 137.5$ gives the total cost when purchasing 50 pounds of the more expensive commodity.

(d) If $x = 73$, then $f^{-1}(73) = 43$ pounds.

50. $C = \frac{5}{9}(F - 32), \quad F \ge -459.6$

(a) $\frac{9}{5}C = F - 32$

$F = 32 + \frac{9}{5}C$

(b) The inverse function gives the temperature F corresponding to the Celsius temperature C.

(c) For $F \ge -459.6$, $C = \frac{5}{9}(F - 32) \ge -273.1\overline{1}$.

Therefore, domain is $C \ge -273.\overline{1} = -273\frac{1}{9}$.

(d) If $C = 22°$, then $F = 32 + \frac{9}{5}(22) = 71.6°\text{F}$.

51. $f(x) = \sqrt{x - 2}$, Domain: $x \ge 2$

$$f'(x) = \frac{1}{2\sqrt{x - 2}} > 0 \text{ for } x > 2$$

f is one-to-one; has an inverse

$$\sqrt{x - 2} = y$$
$$x - 2 = y^2$$
$$x = y^2 + 2$$
$$y = x^2 + 2$$
$$f^{-1}(x) = x^2 + 2, \quad x \ge 0$$

52. $f(x) = -3$

Not one-to-one; does not have an inverse

53. $f(x) = |x - 2|,\ x \le 2$

$$= -(x - 2)$$
$$= 2 - x$$

f is one-to-one; has an inverse

$$2 - x = y$$
$$2 - y = x$$
$$f^{-1}(x) = 2 - x, \quad x \ge 0$$

54. $f(x) = ax + b$

f is one-to-one; has an inverse

$$ax + b = y$$
$$x = \frac{y - b}{a}$$
$$y = \frac{x - b}{a}$$
$$f^{-1}(x) = \frac{x - b}{a}, \quad a \ne 0$$

55. $f(x) = (x - 3)^2$ is one-to-one for $x \ge 3$.

$$(x - 3)^2 = y$$
$$x - 3 = \sqrt{y}$$
$$x = \sqrt{y} + 3$$
$$y = \sqrt{x} + 3$$
$$f^{-1}(x) = \sqrt{x} + 3, \quad x \ge 0$$

(Answer is not unique.)

56. $f(x) = |x - 3|$ is one-to-one for $x \ge 3$.

$$x - 3 = y$$
$$x = y + 3$$
$$y = x + 3$$
$$f^{-1}(x) = x + 3, \quad x \ge 0$$

(Answer is not unique.)

57. $f(x) = |x + 3|$ is one-to-one for $x \geq -3$.

$$x + 3 = y$$
$$x = y - 3$$
$$y = x - 3$$
$$f^{-1}(x) = x - 3, \quad x \geq 0$$

(Answer is not unique.)

58. $f(x) = 16 - x^4$ is one-to-one for $x \geq 0$.

$$16 - x^4 = y$$
$$16 - y = x^4$$
$$\sqrt[4]{16 - y} = x$$
$$\sqrt[4]{16 - x} = y$$
$$f^{-1}(x) = \sqrt[4]{16 - x}, \quad x \leq 16$$

(Answer is not unique.)

59. Yes, the volume is an increasing function, and therefore one-to-one. The inverse function gives the time t corresponding to the volume V.

60. No, there could be two times $t_1 \neq t_2$ for which $h(t_1) = h(t_2)$.

61. No, $C(t)$ is not one-to-one because long distance costs are step functions. A call lasting 2.1 minutes costs the same as one lasting 2.2 minutes.

62. Yes, the area function is increasing and therefore one-to-one. The inverse function gives the radius r corresponding to the area A.

63. $f(x) = 5 - 2x^3, \quad a = 7$

$$f'(x) = -6x^2$$

f is monotonic (decreasing) on $(-\infty, \infty)$. Therefore, f has an inverse.

$$f(-1) = 7 \Rightarrow f^{-1}(7) = -1$$
$$(f^{-1})'(7) = \frac{1}{f'(f^{-1}(7))} = \frac{1}{f'(-1)} = \frac{1}{-6(-1)^2} = \frac{-1}{6}$$

64. $f(x) = x^3 + 3x - 1, a = -5$

$$f'(x) = 3x^2 + 3 > 0$$

f is monotonic (increasing) on $(-\infty, \infty)$. Therefore, f has an inverse.

$$f(-1) = -5 \Rightarrow f^{-1}(-5) = -1$$
$$(f^{-1})'(-5) = \frac{1}{f'(f^{-1}(-5))} = \frac{1}{f'(-1)} = \frac{1}{6}$$

65. $f(x) = \frac{1}{27}(x^5 + 2x^3), \quad a = -11$

$$f'(x) = \frac{1}{27}(5x^4 + 6x^2)$$

f is monotonic (increasing) on $(-\infty, \infty)$. Therefore, f has an inverse.

$$f(-3) = \frac{1}{27}(-243 - 54) = -11 \Rightarrow f^{-1}(-11) = -3$$
$$(f^{-1})'(-11) = \frac{1}{f'(f^{-1}(-11))} = \frac{1}{f'(-3)}$$
$$= \frac{1}{\frac{1}{27}(5(-3)^4 + 6(-3)^2)} = \frac{1}{\frac{1}{27}(459)} = \frac{1}{17}$$

66. $f(x) = \sqrt{x - 4}, \quad a = 2, \quad x \geq 4$

$$f'(x) = \frac{1}{2\sqrt{x - 4}} > 0 \text{ on } (4, \infty)$$

f is monotonic (increasing) on $[4, \infty)$. Therefore, f has an inverse.

$$f(8) = 2 \Rightarrow f^{-1}(2) = 8$$
$$f'(8) = \frac{1}{2\sqrt{8 - 4}} = \frac{1}{4}$$
$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(8)} = \frac{1}{1/4} = 4$$

67. $f(x) = \sin x, \quad a = 1/2, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$$f'(x) = \cos x > 0 \text{ on } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

f is monotonic (increasing) on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore, f has an inverse.

$$f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2} \Rightarrow f^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$
$$(f^{-1})'\left(\frac{1}{2}\right) = \frac{1}{f'\left(f^{-1}\left(\frac{1}{2}\right)\right)}$$
$$= \frac{1}{f'\left(\frac{\pi}{6}\right)} = \frac{1}{\cos\left(\frac{\pi}{6}\right)} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

68. $f(x) = \cos 2x, \quad a = 1, 0 \le x \le \pi/2$

$f'(x) = -2\sin 2x < 0$ on $(0, \pi/2)$

f is monotonic (decreasing) on $[0, \pi/2]$. Therefore, f has an inverse.

$f(0) = 1 \Rightarrow f^{-1}(1) = 0$

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{-2\sin 0} = \frac{1}{0}$$

So, $(f^{-1})'(1)$ is undefined.

69. $f(x) = \dfrac{x+6}{x-2}, \quad x > 0, a = 3$

$$f'(x) = \frac{(x-2)(1) - (x+6)(1)}{(x-2)^2} = \frac{-8}{(x-2)^2} < 0 \text{ on } (2, \infty)$$

f is monotonic (decreasing) on $(2, \infty)$. Therefore, f has an inverse.

$f(6) = 3 \Rightarrow f^{-1}(3) = 6$

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(6)} = \frac{1}{-8/(6-2)^2} = -2$$

70. $f(x) = \dfrac{x+3}{x+1}, \quad x > -1, \quad a = 2$

$$f'(x) = \frac{(x+1)(1) - (x+3)(1)}{(x+1)^2} = \frac{-2}{(x+1)^2} < 0 \text{ on } (-1, \infty)$$

f is monotonic (decreasing) on $(-1, \infty)$. Therefore, f has an inverse.

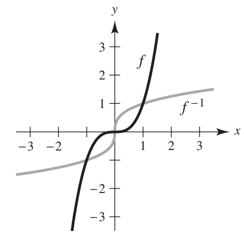
$f(1) = 2 \Rightarrow f^{-1}(2) = 1$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{(-2)/(1+1)^2} = -2$$

71. (a) Domain f = Domain $f^{-1} = (-\infty, \infty)$

(b) Range f = Range $f^{-1} = (-\infty, \infty)$

(c)



(d) $f(x) = x^3, \quad \left(\frac{1}{2}, \frac{1}{8}\right)$

$f'(x) = 3x^2$

$f'\left(\frac{1}{2}\right) = \frac{3}{4}$

$f^{-1}(x) = \sqrt[3]{x}, \quad \left(\frac{1}{8}, \frac{1}{2}\right)$

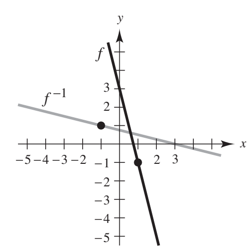
$(f^{-1})'(x) = \dfrac{1}{3\sqrt[3]{x}}$

$(f^{-1})'\left(\frac{1}{8}\right) = \frac{4}{3}$

72. (a) Domain f = Domain $f^{-1} = (-\infty, \infty)$

(b) Range f = Range $f^{-1} = (-\infty, \infty)$

(c)



(d) $f(x) = 3 - 4x, \quad (1, -1)$

$f'(x) = -4$

$f'(1) = -4$

$f^{-1}(x) = \dfrac{3-x}{4}, \quad (-1, 1)$

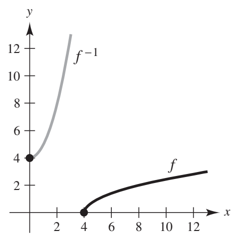
$(f^{-1})'(x) = -\dfrac{1}{4}$

$(f^{-1})'(-1) = -\dfrac{1}{4}$

73. (a) Domain $f = [4, \infty)$, Domain $f^{-1} = [0, \infty)$

(b) Range $f = [0, \infty)$, Range $f^{-1} = [4, \infty)$

(c)



(d) $f(x) = \sqrt{x - 4}, \quad (5, 1)$

$$f'(x) = \frac{1}{2\sqrt{x-4}}$$

$$f'(5) = \frac{1}{2}$$

$$f^{-1}(x) = x^2 + 4, \quad (1, 5)$$

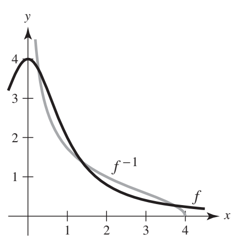
$$\left(f^{-1}\right)'(x) = 2x$$

$$\left(f^{-1}\right)'(1) = 2$$

74. (a) Domain $f = [0, \infty)$, Domain $f^{-1} = (0, 4]$

(b) Range $f = (0, 4]$, Range $f^{-1} = [0, \infty)$

(c)



(d) $f(x) = \dfrac{4}{1 + x^2}$

$$f'(x) = \frac{-8x}{\left(x^2 + 1\right)^2}$$

$$f'(1) = -2$$

$$f^{-1}(x) = \sqrt{\frac{4 - x}{x}}$$

$$\left(f^{-1}\right)'(x) = \frac{-2}{x^2\sqrt{(4 - x)/x}}$$

$$\left(f^{-1}\right)'(2) = -\frac{1}{2}$$

In Exercises 75–78, use the following.

$\boldsymbol{f(x) = \frac{1}{8}x - 3}$ **and** $\boldsymbol{g(x) = x^3}$

$\boldsymbol{f^{-1}(x) = 8(x + 3)}$ **and** $\boldsymbol{g^{-1}(x) = \sqrt[3]{x}}$

75. $\left(f^{-1} \circ g^{-1}\right)(1) = f^{-1}\left(g^{-1}(1)\right) = f^{-1}(1) = 32$

76. $\left(g^{-1} \circ f^{-1}\right)(-3) = g^{-1}\left(f^{-1}(-3)\right) = g^{-1}(0) = 0$

77. $\left(f^{-1} \circ f^{-1}\right)(-2) = f^{-1}\left(f^{-1}(-2)\right) = f^{-1}(8) = 88$

78. $\left(g^{-1} \circ g^{-1}\right)(8) = g^{-1}\left(g^{-1}(8)\right) = g^{-1}(2) = \sqrt[3]{2}$

In Exercises 79–82, use the following.

$\boldsymbol{f(x) = x + 4}$ **and** $\boldsymbol{g(x) = 2x - 5}$

$\boldsymbol{f^{-1}(x) = x - 4}$ **and** $\boldsymbol{g^{-1}(x) = \dfrac{x + 5}{2}}$

79.
$$\begin{aligned}\left(g^{-1} \circ f^{-1}\right)(x) &= g^{-1}\left(f^{-1}(x)\right)\\ &= g^{-1}(x - 4)\\ &= \frac{(x - 4) + 5}{2}\\ &= \frac{x + 1}{2}\end{aligned}$$

80.
$$\begin{aligned}\left(f^{-1} \circ g^{-1}\right)(x) &= f^{-1}\left(g^{-1}(x)\right)\\ &= f^{-1}\left(\frac{x + 5}{2}\right)\\ &= \frac{x + 5}{2} - 4\\ &= \frac{x - 3}{2}\end{aligned}$$

81.
$$\begin{aligned}(f \circ g)(x) &= f\left(g(x)\right)\\ &= f(2x - 5)\\ &= (2x - 5) + 4\\ &= 2x - 1\end{aligned}$$

So, $(f \circ g)^{-1}(x) = \dfrac{x + 1}{2}$.

Note: $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

82.
$$\begin{aligned}(g \circ f)(x) &= g\left(f(x)\right) = g(x + 4)\\ &= 2(x + 4) - 5 = 2x + 3\end{aligned}$$

So, $(g \circ f)^{-1}(x) = \dfrac{x - 3}{2}$.

Note: $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

83. Yes. Functions of the form $f(x) = x^n$, n is odd, are always increasing or always decreasing. So, it is one-to-one and therefore has an inverse function.

84. No. Graphically, adding a constant shifts the graph vertically.

85. f is not one-to-one because many different x-values yield the same y-value.

Example: $f(0) = f(\pi) = 0$

Not continuous at $\frac{(2n-1)\pi}{2}$, where n is an integer.

86. f is not one-to-one because different x-values yield the same y-value.

Example: $f(3) = f\left(-\frac{4}{3}\right) = \frac{3}{5}$

Not continuous at ± 2.

87. $f(x) = k(2 - x - x^3)$ is one-to-one. Because

$f^{-1}(3) = -2$,

$f(-2) = 3 = k\left(2 - (-2) - (-2)^3\right) = 12k \Rightarrow k = \frac{1}{4}$.

88. (a) Since the slope of the tangent line to f at $\left(-1, -\frac{1}{2}\right)$ is $\frac{1}{2}$, the slope of the tangent line to f^{-1} at $\left(-\frac{1}{2}, 1\right)$ is $m = \frac{1}{(1/2)} = 2$.

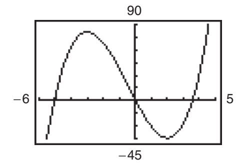
(b) Since the slope of the tangent line to f at $(2, 1)$ is 2, the slope of the tangent line to f^{-1} at $(1, 2)$ is $m = \frac{1}{2}$.

89. False. Let $f(x) = x^2$.

90. True; if f has a y-intercept.

91. (a) $f(x) = 2x^3 + 3x^2 - 36x$

f does not pass the horizontal line test.



(b) $f'(x) = 6x^2 + 6x - 36$

$= 6(x^2 + x - 6) = 6(x + 3)(x - 2)$

$f'(x) = 0$ at $x = 2, -3$

On the interval $(-2, 2)$, f is one-to-one, so, $c = 2$.

92. Let f and g be one-to-one functions.

(a) Let

$$(f \circ g)(x_1) = (f \circ g)(x_2)$$
$$f(g(x_1)) = f(g(x_2))$$
$$g(x_1) = g(x_2) \quad \text{(Because } f \text{ is one-to-one.)}$$
$$x_1 = x_2 \quad \text{(Because } g \text{ is one-to-one.)}$$

So, $f \circ g$ is one-to-one.

(b) Let $(f \circ g)(x) = y$, then $x = (f \circ g)^{-1}(y)$. Also:

$$(f \circ g)(x) = y$$
$$f(g(x)) = y$$
$$g(x) = f^{-1}(y)$$
$$x = g^{-1}(f^{-1}(y))$$
$$x = (g^{-1} \circ f^{-1})(y)$$

So, $(f \circ g)^{-1}(y) = (g^{-1} \circ f^{-1})(y)$ and

$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

93. If f has an inverse, then f and f^{-1} are both one-to-one. Let $(f^{-1})^{-1}(x) = y$ then $x = f^{-1}(y)$ and $f(x) = y$. So, $(f^{-1})^{-1} = f$.

94. Suppose $g(x)$ and $h(x)$ are both inverses of $f(x)$. Then the graph of $f(x)$ contains the point (a, b) if and only if the graphs of $g(x)$ and $h(x)$ contain the point (b, a). Because the graphs of $g(x)$ and $h(x)$ are the same, $g(x) = h(x)$. Therefore, the inverse of $f(x)$ is unique.

95. If f has an inverse and $f(x_1) = f(x_2)$, then $f^{-1}(f(x_1)) = f^{-1}(f(x_2)) \Rightarrow x_1 = x_2$. Therefore, f is one-to-one. If $f(x)$ is one-to-one, then for every value b in the range, there corresponds exactly one value a in the domain. Define $g(x)$ such that the domain of g equals the range of f and $g(b) = a$. By the reflexive property of inverses, $g = f^{-1}$.

96. Not true.

Let $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 1 - x, & 1 < x \le 2 \end{cases}$.

f is one-to-one, but not strictly monotonic.

97. $f(x) = \int_2^x \sqrt{1 + t^2}\, dt, f(2) = 0$

$f'(x) = \sqrt{1 + x^2}$

$f'(x) > 0$ for all $x \Rightarrow f$ increasing on $(-\infty, \infty) \Rightarrow f$ is one-to-one.

$$(f^{-1})'(0) = \frac{1}{f'(2)} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

98. $f(x) = \int_2^x \frac{dt}{\sqrt{1 + t^4}}, f(2) = 0$

$$f'(x) = \frac{1}{\sqrt{1 + x^4}}$$

$$(f^{-1})'(0) = \frac{1}{f'(2)} = \frac{1}{1/\sqrt{17}} = \sqrt{17}$$

99.

$$y = \frac{x - 2}{x - 1}$$

$$x = \frac{y - 2}{y - 1}$$

$$xy - x = y - 2$$

$$xy - y = x - 2$$

$$y = \frac{x - 2}{x - 1}$$

So, if $f(x) = \dfrac{x - 2}{x - 1}$, then $f^{-1}(x) = f(x)$.

The graph of f is symmetric about the line $y = x$.

100. $f(x) = \dfrac{ax + b}{cx + d}$

(a) Assume $bc - ad \ne 0$ and $f(x_1) = f(x_2)$. Then

$$\frac{ax_1 + b}{cx_1 + d} = \frac{ax_2 + b}{cx_2 + d}$$

$$acx_1x_2 + bcx_2 + adx_1 + bd = acx_1x_2 + adx_2 + bcx_1 + bd$$

$$(ad - bc)x_1 = (ad - bc)x_2$$

$$x_1 = x_2 \qquad (\text{because } ad - bc \ne 0)$$

So, f is one-to-one.

Now assume f is one-to-one. Suppose, on the contrary, that $ad = bc$. If $d = 0$, then either $b = 0$ or $c = 0$. In both cases, f is not one-to-one. Similarly, if $b = 0$, then $a = 0$ or $d = 0$ and f is not one-to-one. So consider

$$f(x) = \frac{ax + b}{cx + d} = \frac{adx + bd}{bcx + bd} \cdot \frac{b}{d} = \frac{bcx + bd}{bcx + bd} \cdot \frac{b}{d} = \frac{b}{d},$$

which is not one-to-one.

Alternate Solution:

$$f(x) = \frac{ax + b}{cx + d} \Rightarrow f'(x) = \frac{ad - bc}{(cx + d)^2}$$

f is monotonic (and therefore one-to-one) if and only if $ad - bc \ne 0$.

(b)

$$y = \frac{ax + b}{cx + d}$$

$$cyx + dy = ax + b$$

$$(cy - a)x = b - dy$$

$$x = \frac{b - dy}{cy - a}$$

$$f^{-1}(x) = y = \frac{b - dx}{cx - a}, \quad bc - ad \ne 0$$

(c)

$$\frac{ax + b}{cx + d} = \frac{b - dx}{cx - a}$$

$$acx^2 + bcx - a^2x - ab = bcx - cdx^2 + bd - d^2x$$

$$(ac + cd)x^2 + (d^2 - a^2)x - bd - ab = 0$$

$$c(a + d)x^2 + (d - a)(d + a)x - b(a + d) = 0$$

So, $f = f^{-1}$ if $a = -d$, or if $c = b = 0$ and $a = d$.

Section 5.4 Exponential Functions: Differentiation and Integration

1. The graph of $f(x) = e^x$ is continuous, increasing, and concave upward on its domain $(-\infty, \infty)$. The graph has a horizontal asymptote, $y = 0$ (to the left). Furthermore, $f(x) > 0$ for all x.

2. Yes. $f(x) = Ce^x$, C a constant.

3. $e^{\ln x} = 4$
$x = 4$

4. $e^{\ln 3x} = 24$
$3x = 24$
$x = 8$

5. $e^x = 12$
$x = \ln 12 \approx 2.485$

6. $5e^x = 36$
$e^x = \frac{36}{5}$
$x = \ln\left(\frac{36}{5}\right) \approx 1.974$

7. $9 - 2e^x = 7$
$2e^x = 2$
$e^x = 1$
$x = 0$

8. $8e^x - 12 = 7$
$8e^x = 19$
$e^x = \frac{19}{8}$
$x = \ln\left(\frac{19}{8}\right)$
≈ 0.865

9. $50e^{-x} = 30$
$e^{-x} = \frac{3}{5}$
$-x = \ln\left(\frac{3}{5}\right)$
$x = \ln\left(\frac{5}{3}\right)$
≈ 0.511

10. $100e^{-2x} = 35$
$e^{-2x} = \frac{35}{100} = \frac{7}{20}$
$-2x = \ln\left(\frac{7}{20}\right)$
$x = -\frac{1}{2}\ln\left(\frac{7}{20}\right) = \frac{1}{2}\ln\left(\frac{20}{7}\right)$
≈ 0.525

11. $\dfrac{800}{100 - e^{x/2}} = 50$
$\dfrac{800}{50} = 100 - e^{x/2}$
$84 = e^{x/2}$
$\ln 84 = \dfrac{x}{2}$
$x = 2\ln 84 \approx 8.862$

12. $\dfrac{5000}{1 + e^{2x}} = 2$
$\dfrac{5000}{2} = 1 + e^{2x}$
$2499 = e^{2x}$
$\ln 2499 = 2x$
$x = \dfrac{1}{2}\ln 2499 \approx 3.912$

13. $\ln x = 2$
$x = e^2 \approx 7.389$

14. $\ln x^2 = -8$
$x^2 = e^{-8}$
$x = \pm\sqrt{e^{-8}} = \pm e^{-4} \approx \pm 0.018$

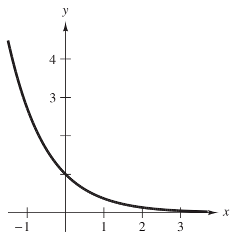
15. $\ln(x - 3) = 2$
$x - 3 = e^2$
$x = 3 + e^2 \approx 10.389$

16. $\ln 4x = 1$
$4x = e^1 = e$
$x = \dfrac{e}{4} \approx 0.680$

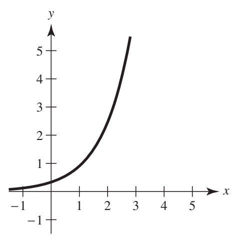
17. $\ln\sqrt{x + 2} = 1$
$\sqrt{x + 2} = e^1 = e$
$x + 2 = e^2$
$x = e^2 - 2 \approx 5.389$

18. $\ln(x - 2)^2 = 12$
$(x - 2)^2 = e^{12}$
$x - 2 = e^6$
$x = 2 + e^6 \approx 405.429$

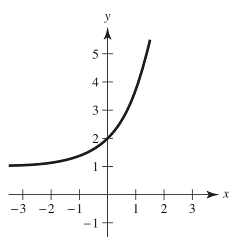
19. $y = e^{-x}$



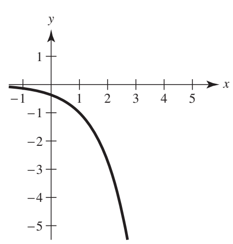
20. $y = \dfrac{1}{3}e^{x}$



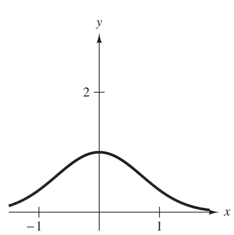
21. $y = e^{x+1}$



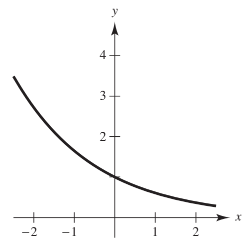
22. $y = -e^{x-1}$



23. $y = e^{-x^2}$



24. $y = e^{-x/2}$



25. $y = Ce^{ax}$

Horizontal asymptote: $y = 0$

Matches (c)

26. $y = Ce^{-ax}$

Horizontal asymptote: $y = 0$

Reflection in the y-axis

Matches (d)

27. $y = C\left(1 - e^{-ax}\right)$

Vertical shift C units

Reflection in both the x- and y-axes

Matches (a)

28. $y = \dfrac{C}{1 + e^{-ax}}$

$\lim\limits_{x\to\infty} \dfrac{C}{1 + e^{-ax}} = C$

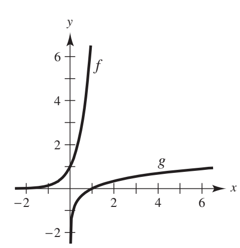
$\lim\limits_{x\to-\infty} \dfrac{C}{1 + e^{-ax}} = 0$

Horizontal asymptotes: $y = C$ and $y = 0$

Matches (b)

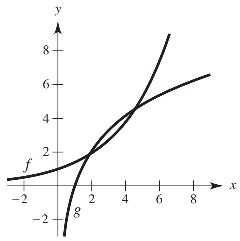
29. $f(x) = e^{2x}$

$g(x) = \ln\sqrt{x} = \dfrac{1}{2}\ln x$



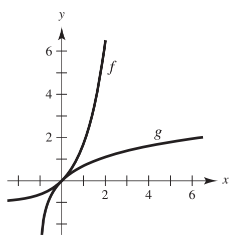
30. $f(x) = e^{x/3}$

$g(x) = \ln x^3 = 3 \ln x$



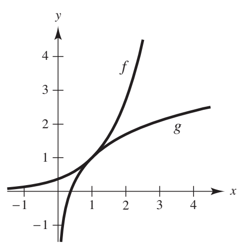
31. $f(x) = e^x - 1$

$g(x) = \ln(x + 1)$



32. $f(x) = e^{x-1}$

$g(x) = 1 + \ln x$



33. $y = e^{5x}$

$y' = 5e^{5x}$

34. $y = e^{-8x}$

$y' = -8e^{-8x}$

35. $y = e^{\sqrt{x}}$

$$\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

36. $y = e^{-2x^3}$

$y' = -6x^2 e^{-2x^3}$

37. $y = e^{x-4}$

$y' = e^{x-4}$

38. $y = 5e^{x^2+5}$

$y' = 5e^{x^2+5}(2x) = 10xe^{x^2+5}$

39. $y = e^x \ln x$

$$y' = e^x\left(\frac{1}{x}\right) + e^x \ln x = e^x\left(\frac{1}{x} + \ln x\right)$$

40. $y = xe^{4x}$

$y' = 4xe^{4x} + e^{4x} = e^{4x}(4x + 1)$

41. $y = (x + 1)^2 e^x$

$$\begin{aligned} y' &= 2(x + 1)e^x + (x + 1)^2 e^x \\ &= (x + 1)e^x(2 + x + 1) \\ &= (x + 1)(x + 3)e^x \end{aligned}$$

42. $y = x^2 e^{-x}$

$y' = x^2(-e^{-x}) + 2xe^{-x} = xe^{-x}(2 - x)$

43. $g(t) = (e^{-t} + e^t)^3$

$g'(t) = 3(e^{-t} + e^t)^2(e^t - e^{-t})$

44. $g(t) = e^{-3/t^2}$

$$g'(t) = e^{-3/t^2}(6t^{-3}) = \frac{6}{t^3 e^{3/t^2}}$$

45. $y = \ln(2 - e^{5x})$

$$\begin{aligned} y' &= \frac{1}{2 - e^{5x}}(-5e^{5x}) \\ &= -\frac{5e^{5x}}{2 - e^{5x}} \end{aligned}$$

46. $y = \ln\left(\frac{1 + e^x}{1 - e^x}\right) = \ln(1 + e^x) - \ln(1 - e^x)$

$$\frac{dy}{dx} = \frac{e^x}{1 + e^x} + \frac{e^x}{1 - e^x} = \frac{2e^x}{1 - e^{2x}}$$

47. $y = \frac{2}{e^x + e^{-x}} = 2(e^x + e^{-x})^{-1}$

$$\frac{dy}{dx} = -2(e^x + e^{-x})^{-2}(e^x - e^{-x}) = \frac{-2(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

48. $y = \frac{e^x - e^{-x}}{2}$

$$\frac{dy}{dx} = \frac{e^x + e^{-x}}{2}$$

49. $y = \dfrac{e^x + 1}{e^x - 1}$

$y' = \dfrac{(e^x - 1)e^x - (e^x + 1)e^x}{(e^x - 1)^2} = \dfrac{-2e^x}{(e^x - 1)^2}$

50. $y = \dfrac{e^{2x}}{e^{2x} + 1}$

$y' = \dfrac{(e^{2x} + 1)2e^{2x} - e^{2x}(2e^{2x})}{(e^{2x} + 1)^2} = \dfrac{2e^{2x}}{(e^{2x} + 1)^2}$

51. $y = e^x(\sin x + \cos x)$

$\dfrac{dy}{dx} = e^x(\cos x - \sin x) + (\sin x + \cos x)(e^x)$

$= e^x(2\cos x) = 2e^x \cos x$

52. $y = e^{2x} \tan 2x$

$y' = e^{2x}[2\sec^2 2x] + 2e^{2x} \tan 2x$

$= 2e^{2x}[\sec^2 2x + \tan 2x]$

53. $F(x) = \int_{\pi}^{\ln x} \cos e^t \, dt$

$F'(x) = \cos(e^{\ln x}) \cdot \dfrac{1}{x} = \dfrac{\cos(x)}{x}$

54. $F(x) = \int_0^{e^{2x}} \ln(t + 1) \, dt$

$F'(x) = \ln(e^{2x} + 1)2e^{2x} = 2e^{2x} \ln(e^{2x} + 1)$

55. $f(x) = e^{3x}, (0, 1)$

$f'(x) = 3e^{3x}, f'(0) = 3$

Tangent line: $y - 1 = 3(x - 0)$

$y = 3x + 1$

56. $f(x) = e^{-x} - 6, (0, -5)$

$f'(x) = -e^{-x}, f'(0) = -1$

Tangent line: $y + 5 = -1(x - 0)$

$y = -x - 5$

57. $y = e^{3x - x^2}, (3, 1)$

$y' = (3 - 2x)e^{3x - x^2}, y'(3) = -3$

Tangent line: $y - 1 = -3(x - 3)$

$y = -3x + 10$

58. $y = e^{-2x + x^2}, \ (2, 1)$

$y' = (2x - 2)e^{-2x + x^2}, \ y'(2) = 2$

Tangent line: $y - 1 = 2(x - 2)$

$y = 2x - 3$

59. $f(x) = e^{-x} \ln x, \ (1, 0)$

$f'(x) = e^{-x}\left(\dfrac{1}{x}\right) - e^{-x} \ln x = e^{-x}\left(\dfrac{1}{x} - \ln x\right)$

$f'(1) = e^{-1}$

Tangent line: $y - 0 = e^{-1}(x - 1)$

$y = \dfrac{1}{e}x - \dfrac{1}{e}$

60. $y = \ln \dfrac{e^x + e^{-x}}{2}, \ (0, 0)$

$y' = \dfrac{1}{[(e^x + e^{-x})/2]}[e^x - e^{-x}]$

$y'(0) = 0$

Tangent line: $y = 0$

61. $y = x^2e^x - 2xe^x + 2e^x, \ (1, e)$

$y' = x^2e^x + 2xe^x - 2xe^x - 2e^x + 2e^x = x^2e^x$

$y'(1) = e$

Tangent line: $y - e = e(x - 1)$

$y = ex$

62. $y = xe^x - e^x, \ (1, 0)$

$y' = xe^x + e^x - e^x = xe^x$

$y'(1) = e$

Tangent line: $y - 0 = e(x - 1)$

$y = ex - e$

63. $xe^y - 10x + 3y = 0$

$xe^y \dfrac{dy}{dx} + e^y - 10 + 3\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx}(xe^y + 3) = 10 - e^y$

$\dfrac{dy}{dx} = \dfrac{10 - e^y}{xe^y + 3}$

64. $$e^{xy} + x^2 - y^2 = 10$$

$$\left(x\frac{dy}{dx} + y\right)e^{xy} + 2x - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(xe^{xy} - 2y\right) = -ye^{xy} - 2x$$

$$\frac{dy}{dx} = -\frac{ye^{xy} + 2x}{xe^{xy} - 2y}$$

65. $$xe^y + ye^x = 1, \quad (0, 1)$$

$$xe^y y' + e^y + ye^x + y'e^x = 0$$

At (0, 1): $e + 1 + y' = 0$

$$y' = -e - 1$$

Tangent line: $y - 1 = (-e - 1)(x - 0)$

$$y = (-e - 1)x + 1$$

66. $1 + \ln(xy) = e^{x-y}, \quad (1, 1)$

$$\frac{1}{xy}[xy' + y] = e^{x-y}[1 - y']$$

At (1, 1): $[y' + 1] = 1 - y'$

$$y' = 0$$

Tangent line: $y - 1 = 0(x - 1)$

$$y = 1$$

67. $f(x) = (3 + 2x)e^{-3x}$

$$f'(x) = (3 + 2x)\left(-3e^{-3x}\right) + 2e^{-3x} = (-7 - 6x)e^{-3x}$$

$$f''(x) = (-7 - 6x)\left(-3e^{-3x}\right) - 6e^{-3x} = 3(6x + 5)e^{-3x}$$

68. $g(x) = \sqrt{x} + e^x \ln x$

$$g'(x) = \frac{1}{2\sqrt{x}} + \frac{e^x}{x} + e^x \ln x$$

$$g''(x) = -\frac{1}{4x^{3/2}} + \frac{xe^x - e^x}{x^2} + \frac{e^x}{x} + e^x \ln x$$

$$= -\frac{1}{4x\sqrt{x}} + \frac{e^x(2x - 1)}{x^2} + e^x \ln x$$

69. $$y = 4e^{-x}$$

$$y' = -4e^{-x}$$

$$y'' = 4e^{-x}$$

$$y'' - y = 4e^{-x} - 4e^{-x} = 0$$

70. $$y = e^{3x} + e^{-3x}$$

$$y' = 3e^{3x} - 3e^{-3x}$$

$$y'' = 9e^{3x} + 9e^{-3x}$$

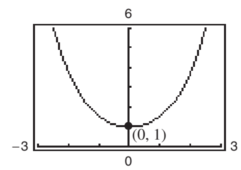
$$y'' - 9y = \left(9e^{3x} + 9e^{-3x}\right) - 9\left(e^{3x} + e^{-3x}\right) = 0$$

71. $f(x) = \dfrac{e^x + e^{-x}}{2}$

$$f'(x) = \frac{e^x - e^{-x}}{2} = 0 \text{ when } x = 0.$$

$$f''(x) = \frac{e^x + e^{-x}}{2} > 0$$

Relative minimum: $(0, 1)$

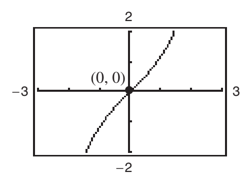


72. $f(x) = \dfrac{e^x - e^{-x}}{2}$

$$f'(x) = \frac{e^x + e^{-x}}{2} > 0$$

$$f''(x) = \frac{e^x - e^{-x}}{2} = 0 \text{ when } x = 0.$$

Point of inflection: $(0, 0)$



73. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-2)^2/2}$

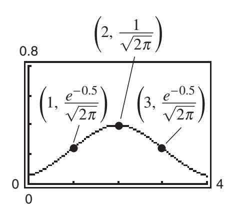
$$g'(x) = \frac{-1}{\sqrt{2\pi}}(x - 2)e^{-(x-2)^2/2}$$

$$g''(x) = \frac{1}{\sqrt{2\pi}}(x - 1)(x - 3)e^{-(x-2)^2/2}$$

Relative maximum: $\left(2, \dfrac{1}{\sqrt{2\pi}}\right) \approx (2, 0.399)$

Points of inflection:

$$\left(1, \frac{1}{\sqrt{2\pi}}e^{-1/2}\right), \left(3, \frac{1}{\sqrt{2\pi}}e^{-1/2}\right) \approx (1, 0.242), (3, 0.242)$$



74. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-3)^2/2}$

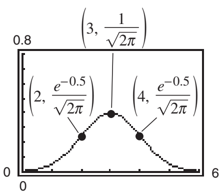
$g'(x) = \dfrac{-1}{\sqrt{2\pi}}(x-3)e^{-(x-3)^2/2}$

$g''(x) = \dfrac{1}{\sqrt{2\pi}}(x-2)(x-4)e^{-(x-3)^2/2}$

Relative maximum: $\left(3, \dfrac{1}{\sqrt{2\pi}}\right) \approx (3, 0.399)$

Points of inflection:

$\left(2, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right), \left(4, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right) \approx (2, 0.242), (4, 0.242)$



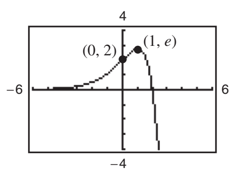
75. $f(x) = (2-x)e^x$

$f'(x) = (2-x)e^x - e^x = e^x(1-x) = 0$ when $x = 1$.

$f''(x) = e^x(1-x) - e^x = -xe^x = 0$ when $x = 0$.

Relative maximum: $(1, e)$

Point of inflection: $(0, 2)$



76. $f(x) = xe^{-x}$

$f'(x) = -xe^{-x} + e^{-x}$

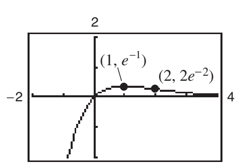
$= e^{-x}(1-x) = 0$ when $x = 1$.

$f''(x) = -e^{-x} + (-e^{-x})(1-x)$

$= e^{-x}(x-2) = 0$ when $x = 2$.

Relative maximum: $(1, e^{-1})$

Point of inflection: $(2, 2e^{-2})$



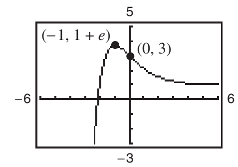
77. $g(t) = 1 + (2+t)e^{-t}$

$g'(t) = -(1+t)e^{-t}$

$g''(t) = te^{-t}$

Relative maximum: $(-1, 1+e) \approx (-1, 3.718)$

Point of inflection: $(0, 3)$



78. $f(x) = -2 + e^{3x}(4-2x)$

$f'(x) = e^{3x}(-2) + 3e^{3x}(4-2x)$

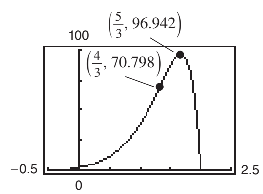
$= e^{3x}(10-6x) = 0$ when $x = \frac{5}{3}$.

$f''(x) = e^{3x}(-6) + 3e^{3x}(10-6x)$

$= e^{3x}(24-18x) = 0$ when $x = \frac{4}{3}$.

Relative maximum: $\left(\frac{5}{3}, 96.942\right)$

Point of inflection: $\left(\frac{4}{3}, 70.798\right)$

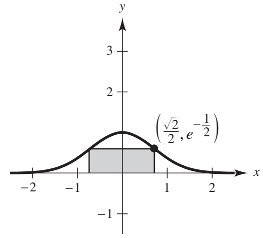


79. $A = (\text{base})(\text{height}) = 2xe^{-x^2}$

$\dfrac{dA}{dx} = -4x^2e^{-x^2} + 2e^{-x^2}$

$= 2e^{-x^2}(1-2x^2) = 0$ when $x = \dfrac{\sqrt{2}}{2}$.

$A = \sqrt{2}e^{-1/2}$

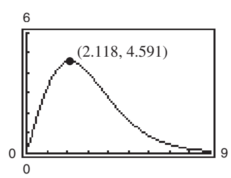


80. (a)
$$f(c) = f(c + x)$$
$$10ce^{-c} = 10(c + x)e^{-(c+x)}$$
$$\frac{c}{e^c} = \frac{c + x}{e^{c+x}}$$
$$ce^{c+x} = (c + x)e^c$$
$$ce^x = c + x$$
$$ce^x - c = x$$
$$c = \frac{x}{e^x - 1}$$

(b) $A(x) = xf(c) = x\left[10\left(\frac{x}{e^x - 1}\right)e^{-\left(x/\left(e^x - 1\right)\right)}\right]$

$$= \frac{10x^2}{e^x - 1}e^{x/\left(1-e^x\right)}$$

(c) $A(x) = \dfrac{10x^2}{e^x - 1}e^{x/\left(1-e^x\right)}$

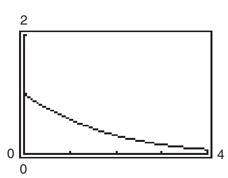


The maximum area is 4.591 for $x = 2.118$ and $f(x) = 2.547$.

(d) $c = \dfrac{x}{e^x - 1}$

$\lim_{x \to 0^+} c = 1$

$\lim_{x \to \infty} c = 0$



Answers will vary. *Sample answer*:

As x approaches 0 from the right, the height of the rectangle approaches 1.
As x approaches ∞, the height of the rectangle approaches 0.

81. $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

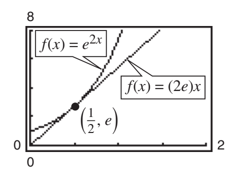
Let $(x, y) = \left(x, e^{2x}\right)$ be the point on the graph where the tangent line passes through the origin. Equating slopes,

$$2e^{2x} = \frac{e^{2x} - 0}{x - 0}$$
$$2 = \frac{1}{x}$$
$$x = \frac{1}{2}, \quad y = e, \quad y' = 2e.$$

Point: $\left(\frac{1}{2}, e\right)$

Tangent line: $y - e = 2e\left(x - \frac{1}{2}\right)$

$$y = 2ex$$

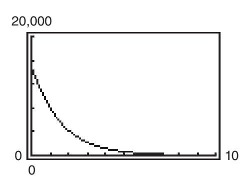


82. (a) f is increasing on $(-\infty, \infty)$. g is decreasing on $(-\infty, \infty)$.

(b) f and g are both concave upward on $(-\infty, \infty)$.

83. $V = 15{,}000e^{-0.6286t}, \quad 0 \le t \le 10$

(a)

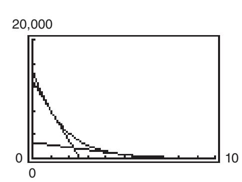


(b) $\dfrac{dV}{dt} = -9429e^{-0.6286t}$

When $t = 1$, $\dfrac{dV}{dt} \approx -5028.84$.

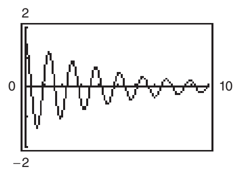
When $t = 5$, $\dfrac{dV}{dt} \approx -406.89$.

(c)



84. $1.56e^{-0.22t}\cos 4.9t \le 0.25$

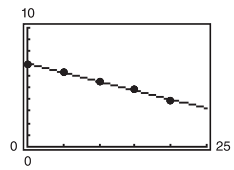
(3 inches equals one-fourth foot.) Using a graphing utility or Newton's Method, you have $t \ge 7.79$ seconds.



85.

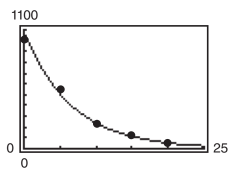
h	0	5	10	15	20
P	1013.2	547.5	233.0	121.6	50.7
$\ln P$	6.921	6.305	5.451	4.801	3.926

(a) Using a graphing utility, $\ln P = -0.1499h + 6.9797$.



(b) $P = e^{-0.1499h+6.9797} = 1074.6e^{-0.1499h}$

(c)



(d) $\dfrac{dP}{dh} = (1074.6)(-0.1499)e^{-0.1499h}$

$= -161.08e^{-0.1499h}$

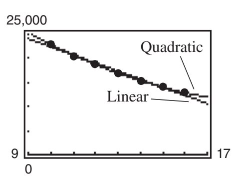
At $h = 5$, $\dfrac{dP}{dh} \approx -76.13$ millibars/km.

At $h = 18$, $\dfrac{dP}{dh} \approx -10.84$ millibars/km.

86. (a) Linear model: $V = -1686.8t + 39{,}309$

Quadratic model:

$V = 109.52t^2 - 4534.4t + 57{,}380$



(b) The slope represents the average loss in value per year.

(c) Exponential model:

$V = 60{,}454.23(0.90724)^t = 60{,}454.23e^{-0.09735t}$

(d) The horizontal asymptote is $V = 0$. As $t \to \infty$, the value of the car approaches \$0.

(e) $V' = (60{,}454.23)(-0.09735)e^{-0.09735t}$

$= -5885.22e^{-0.09735t}$

At $t = 12$, $V' \approx -1829.9$ dollars per year.

At $t = 15$, $V' \approx -1366.4$ dollars per year.

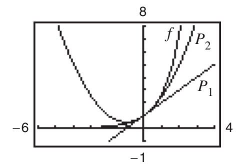
87. $f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad f''(0) = 1$

$P_1(x) = 1 + 1(x - 0) = 1 + x$

$P_2(x) = 1 + 1(x - 0) = \frac{1}{2}(1)(x - 0)^2 = 1 + x + \frac{x^2}{2}$



The values of f, P_1, and P_2 and their first derivatives agree at $x = 0$.

88. $f(x) = e^{x/2}, \qquad f(0) = 1$

$f'(x) = \frac{1}{2}e^{x/2}, \qquad f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{4}e^{x/2}, \qquad f''(0) = \frac{1}{4}$

$P_1(x) = 1 + \frac{1}{2}(x - 0) = \frac{x}{2} + 1, \qquad P_1(0) = 1$

$P_1'(x) = \frac{1}{2}, \qquad P_1'(0) = \frac{1}{2}$

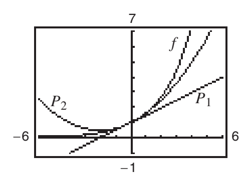
$P_2(x) = 1 + \frac{1}{2}(x - 0) + \frac{1}{8}(x - 0)^2 \qquad P_2(0) = 1$

$= \frac{x^2}{8} + \frac{x}{2} + 1$

$P_2'(x) = \frac{1}{4}x + \frac{1}{2}, \qquad P_2'(0) = \frac{1}{2}$

$P_2''(x) = \frac{1}{4}, \qquad P_2''(0) = \frac{1}{4}$

The values of f, P_1, P_2 and their first derivatives agree at $x = 0$. The values of the second derivatives of f and P_2 agree at $x = 0$.



89. $n = 12$

$12! = 12 \cdot 11 \cdot 10 \cdots 3 \cdot 2 \cdot 1 = 479{,}001{,}600$

Stirlings Formula:

$12! \approx \left(\frac{12}{e}\right)^{12}\sqrt{2\pi(12)} \approx 475{,}687{,}487$

90. $n = 15$

$15! = 15 \cdot 14 \cdots 3 \cdot 2 \cdot 1 = 1{,}307{,}674{,}368{,}000$

Stirlings Formula:

$$15! \approx \left(\frac{15}{e}\right)^{15}\sqrt{2\pi(15)} \approx 1{,}300{,}430{,}722{,}200$$
$$\approx 1.3004 \times 10^{12}$$

91. Let $u = 5x$, $du = 5\,dx$.

$$\int e^{5x}(5)\,dx = e^{5x} + C$$

92. Let $u = -x^4$, $du = -4x^3 dx$.

$$\int e^{-x^4}\left(-4x^3\right)dx = e^{-x^4} + C$$

93. Let $u = 5x - 3$, $du = 5\,dx$.

$$\int e^{5x-3}\,dx = \frac{1}{5}\int e^{5x-3}(5)\,dx = \frac{1}{5}e^{5x-3} + C$$

94. Let $u = 1 - 3x$, $du = -3\,dx$.

$$\int e^{1-3x}\,dx = -\tfrac{1}{3}\int e^{1-3x}(-3)\,dx = -\tfrac{1}{3}e^{1-3x} + C$$

95. Let $u = x^2 + x$, $du = (2x + 1)\,dx$.

$$\int (2x+1)e^{x^2+x}\,dx = e^{x^2+x} + C$$

96. Let $u = e^x + 1$, $du = e^x\,dx$.

$$\int e^x\left(e^x + 1\right)^2 dx = \int\left(e^x + 1\right)^2\left(e^x\right)dx = \frac{\left(e^x + 1\right)^3}{3} + C$$

97. Let $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$.

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)dx = 2e^{\sqrt{x}} + C$$

98. Let $u = \dfrac{1}{x^2}$, $du = \dfrac{-2}{x^3}\,dx$.

$$\int \frac{e^{1/x^2}}{x^3}\,dx = -\frac{1}{2}\int e^{1/x^2}\left(\frac{-2}{x^3}\right)dx = -\frac{1}{2}e^{1/x^2} + C$$

99. Let $u = 1 + e^{-x}$, $du = -e^{-x}\,dx$.

$$\int \frac{e^{-x}}{1 + e^{-x}}\,dx = -\int \frac{-e^{-x}}{1 + e^{-x}}\,dx = -\ln\left(1 + e^{-x}\right) + C$$
$$= \ln\left(\frac{e^x}{e^x + 1}\right) + C$$
$$= x - \ln\left(e^x + 1\right) + C$$

100. Let $u = 1 + e^{2x}$, $du = 2e^{2x}\,dx$.

$$\int \frac{e^{2x}}{1 + e^{2x}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{1 + e^{2x}}\,dx = \frac{1}{2}\ln\left(1 + e^{2x}\right) + C$$

101. Let $u = 1 - e^x$, $du = -e^x\,dx$.

$$\int e^x\sqrt{1 - e^x}\,dx = -\int\left(1 - e^x\right)^{1/2}\left(-e^x\right)dx$$
$$= -\tfrac{2}{3}\left(1 - e^x\right)^{3/2} + C$$

102. Let $u = e^x + e^{-x}$, $du = \left(e^x - e^{-x}\right)dx$.

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \ln\left(e^x + e^{-x}\right) + C$$

103. Let $u = e^x - e^{-x}$, $du = \left(e^x + e^{-x}\right)dx$.

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \ln\left|e^x - e^{-x}\right| + C$$

104. Let $u = e^x + e^{-x}$, $du = \left(e^x - e^{-x}\right)dx$.

$$\int \frac{2e^x - 2e^{-x}}{\left(e^x + e^{-x}\right)^2}\,dx = 2\int\left(e^x + e^{-x}\right)^{-2}\left(e^x - e^{-x}\right)dx$$
$$= \frac{-2}{e^x + e^{-x}} + C$$

105. $$\int \frac{5 - e^x}{e^{2x}}\,dx = \int 5e^{-2x}\,dx - \int e^{-x}\,dx$$
$$= -\frac{5}{2}e^{-2x} + e^{-x} + C$$

106. $$\int \frac{e^{-3x} + 2e^{2x} + 3}{e^x}\,dx = \int\left(e^{-4x} + 2e^x + 3e^{-x}\right)dx$$
$$= -\frac{1}{4}e^{-4x} + 2e^x - 3e^{-x} + C$$

107. $$\int e^{-x}\tan\left(e^{-x}\right)dx = -\int\left[\tan\left(e^{-x}\right)\right]\left(-e^{-x}\right)dx$$
$$= \ln\left|\cos\left(e^{-x}\right)\right| + C$$

108. $$\int e^{2x}\csc\left(e^{2x}\right)dx = \frac{1}{2}\int\csc\left(e^{2x}\right)\left(2e^{2x}\right)dx$$
$$= -\frac{1}{2}\ln\left|\csc\left(e^{2x}\right) + \cot\left(e^{2x}\right)\right| + C$$

109. $$\int_0^1 e^{-2x}\,dx = -\frac{1}{2}\int_0^1 e^{-2x}(-2)\,dx = \left[-\frac{1}{2}e^{-2x}\right]_0^1$$
$$= \frac{1}{2}\left(1 - e^{-2}\right) = \frac{e^2 - 1}{2e^2}$$

110. $\int_{-1}^{1} e^{1+4x}\,dx = \left[\frac{1}{4}e^{1+4x}\right]_{-1}^{1}$

$= \frac{1}{4}e^{5} - \frac{1}{4}e^{-3}$

$= \frac{1}{4}\left(e^{5} - e^{-3}\right)$

111. $\int_{0}^{1} xe^{-x^2}\,dx = -\frac{1}{2}\int_{0}^{1} e^{-x^2}(-2x)\,dx$

$= -\frac{1}{2}\left[e^{-x^2}\right]_{0}^{1}$

$= -\frac{1}{2}\left[e^{-1} - 1\right]$

$= \frac{1-(1/e)}{2} = \frac{e-1}{2e}$

112. $\int_{-2}^{0} x^2 e^{x^3/2}\,dx = \frac{2}{3}\int_{-2}^{0} e^{x^3/2}\left(\frac{3}{2}x^2\right)dx$

$= \frac{2}{3}\left[e^{x^3/2}\right]_{-2}^{0}$

$= \frac{2}{3}\left[1 - e^{-4}\right]$

$= \frac{2}{3}\left[1 - \frac{1}{e^4}\right] = \frac{2\left(e^4 - 1\right)}{3e^4}$

113. Let $u = \frac{3}{x}$, $du = -\frac{3}{x^2}\,dx$.

$\int_{1}^{3} \frac{e^{3/x}}{x^2}\,dx = -\frac{1}{3}\int_{1}^{3} e^{3/x}\left(-\frac{3}{x^2}\right)dx$

$= \left[-\frac{1}{3}e^{3/x}\right]_{1}^{3} = \frac{e}{3}\left(e^2 - 1\right)$

114. Let $u = \frac{-x^2}{2}$, $du = -x\,dx$.

$\int_{0}^{\sqrt{2}} xe^{-x^2/2}\,dx = -\int_{0}^{\sqrt{2}} e^{-x^2/2}(-x)\,dx$

$= \left[-e^{-x^2/2}\right]_{0}^{\sqrt{2}} = 1 - e^{-1} = \frac{e-1}{e}$

115. Let $u = 1 + e^{4x}$, $du = 4e^{4x}\,dx$.

$\int_{0}^{2} \frac{e^{4x}}{1+e^{4x}}\,dx = \left[\frac{1}{4}\ln\left(1 + e^{4x}\right)\right]_{0}^{2}$

$= \frac{1}{4}\ln\left(1 + e^{8}\right) - \frac{1}{4}\ln 2$

$= \frac{1}{4}\ln\left(\frac{1+e^8}{2}\right)$

116. Let $u = 7 - e^{x+1}$, $du = -e^{x+1}\,dx$.

$\int_{-2}^{0} \frac{e^{x+1}}{7 - e^{x+1}}\,dx = \left[-\ln\left|7 - e^{x+1}\right|\right]_{-2}^{0}$

$= -\ln(7 - e) + \ln\left(7 - e^{-1}\right)$

117. Let $u = \sin \pi x$, $du = \pi \cos \pi x\,dx$.

$\int_{0}^{\pi/2} e^{\sin \pi x} \cos \pi x\,dx = \frac{1}{\pi}\int_{0}^{\pi/2} e^{\sin \pi x}(\pi \cos \pi x)\,dx$

$= \frac{1}{\pi}\left[e^{\sin \pi x}\right]_{0}^{\pi/2}$

$= \frac{1}{\pi}\left[e^{\sin\left(\pi^2/2\right)} - 1\right]$

118. Let $u = \sec 2x$, $du = 2 \sec 2x \tan 2x\,dx$.

$\int_{\pi/3}^{\pi/2} e^{\sec 2x} \sec 2x \tan 2x\,dx = \frac{1}{2}\int_{\pi/3}^{\pi/2} e^{\sec 2x}(2 \sec 2x \tan 2x)dx$

$= \frac{1}{2}\left[e^{\sec 2x}\right]_{\pi/3}^{\pi/2}$

$= \frac{1}{2}\left[e^{-1} - e^{-2}\right]$

$= \frac{1}{2}\left[\frac{1}{e} - \frac{1}{e^2}\right] = \frac{e-1}{2e^2}$

119. Let $u = 9x^2$, $du = 18x\,dx$.

$y = \int xe^{9x^2}\,dx = \frac{1}{18}\int e^{9x^2}(18x\,dx)$

$= \frac{1}{18}e^{9x^2} + C$

120. $y = \int \left(e^{x} - e^{-x}\right)^2 dx$

$= \int \left(e^{2x} - 2 + e^{-2x}\right)dx$

$= \frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x} + C$

121. $f'(x) = \int \frac{1}{2}\left(e^{x} + e^{-x}\right)dx = \frac{1}{2}\left(e^{x} - e^{-x}\right) + C_1$

$f'(0) = C_1 = 0$

$f(x) = \int \frac{1}{2}\left(e^{x} - e^{-x}\right)dx = \frac{1}{2}\left(e^{x} + e^{-x}\right) + C_2$

$f(0) = 1 + C_2 = 1 \Rightarrow C_2 = 0$

$f(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right)$

122. $f'(x) = \int\left(\sin x + e^{2x}\right) dx = -\cos x + \frac{1}{2}e^{2x} + C_1$

$f'(0) = -1 + \frac{1}{2} + C_1 = \frac{1}{2} \Rightarrow C_1 = 1$

$f'(x) = -\cos x + \frac{1}{2}e^{2x} + 1$

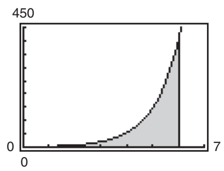
$f(x) = \int\left(-\cos x + \frac{1}{2}e^{2x} + 1\right) dx$

$= -\sin x + \frac{1}{4}e^{2x} + x + C_2$

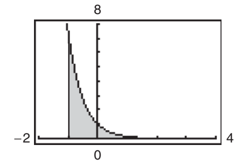
$f(0) = \frac{1}{4} + C_2 = \frac{1}{4} \Rightarrow C_2 = 0$

$f(x) = x - \sin x + \frac{1}{4}e^{2x}$

123. Area $= \int_0^6 e^x\, dx = \left[e^x\right]_0^6 = e^6 - 1 \approx 402.4$



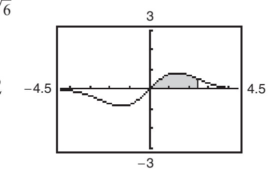
124. $A = \int_{-1}^{3} e^{-2x} dx = \left[-\frac{1}{2}e^{-2x}\right]_{-1}^{3} = -\frac{1}{2}\left(e^{-6} - e^{2}\right) \approx 3.693$



125. $\int_0^{\sqrt{6}} xe^{-x^2/4} dx = \left[-2e^{-x^2/4}\right]_0^{\sqrt{6}}$

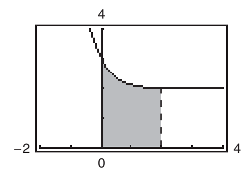
$= -2e^{-3/2} + 2$

≈ 1.554



126. $\int_0^2 \left(e^{-2x} + 2\right) dx = \left[-\frac{1}{2}e^{-2x} + 2x\right]_0^2$

$= -\frac{1}{2}e^{-4} + 4 + \frac{1}{2} \approx 4.491$



127. $\int_0^4 \sqrt{x}\, e^x\, dx,\ n = 12,\ \Delta x = \frac{4 - 0}{12} = \frac{1}{3}$

Midpoint approximation:

$$\int_0^4 \sqrt{x}\, e^x\, dx \approx \frac{1}{3}\left[f\left(\frac{1}{6}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{5}{6}\right) + f\left(\frac{7}{6}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{11}{6}\right) + f\left(\frac{13}{6}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{17}{6}\right) + f\left(\frac{19}{6}\right) + f\left(\frac{7}{2}\right) + f\left(\frac{23}{6}\right)\right] \approx 92.1898$$

128. $\int_0^2 2xe^{-x}\, dx,\ n = 12,\ \Delta x = \frac{2 - 0}{12} = \frac{1}{6}$

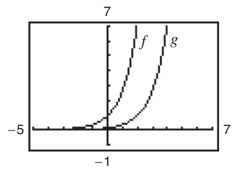
Midpoint approximation:

$$\int_0^2 2xe^{-x}\, dx \approx \frac{1}{6}\left[f\left(\frac{1}{12}\right) + f\left(\frac{1}{4}\right) + f\left(\frac{5}{12}\right) + f\left(\frac{7}{12}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{11}{12}\right) + f\left(\frac{13}{12}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{17}{12}\right) + f\left(\frac{19}{12}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{23}{12}\right)\right] \approx 1.1906$$

129. The natural exponential function has a horizontal asymptote $y = 0$ to the left.

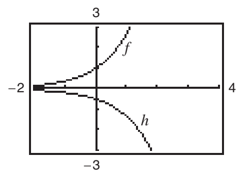
The natural logarithmic function has a vertical asymptote $x = 0$ from the right.

130. (a)



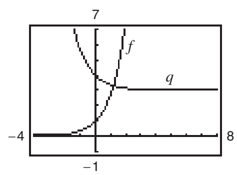
Horizontal shift 2 units to the right

(b)



A reflection in the x-axis and a vertical shrink

(c)



Vertical shift 3 units upward and a reflection in the y-axis

131. False.

$$\frac{d}{dx}\left[g(x)e^x\right] = e^x g'(x) + g(x)e^x = e^x\left(g(x) + g'(x)\right)$$

132. True

$$f\left(e^{n+1}\right) - f\left(e^n\right) = \ln e^{n+1} - \ln e^n = n + 1 - n = 1$$

133. True

$$f\left(f(x)\right) = 2 + e^{\ln(x-2)} = 2 + x - 2 = x$$

$$g\left(f(x)\right) = \ln\left(2 + e^x - 2\right) = \ln e^x = x$$

134. True

$$\frac{d^n y}{dx^n} = Ce^x = y \text{ for } n = 1, 2, 3, \ldots$$

135. $0.0665\int_{48}^{60} e^{-0.0139(t-48)^2}\,dt$

Graphing utility: $0.4772 = 47.72\%$

136. $\int_0^x 0.3e^{-0.3t}dt = \frac{1}{2}$

$$\left[-e^{-0.3t}\right]_0^x = \frac{1}{2}$$

$$-e^{-0.3x} + 1 = \frac{1}{2}$$

$$e^{-0.3x} = \frac{1}{2}$$

$$-0.3x = \ln\frac{1}{2} = -\ln 2$$

$$x = \frac{\ln 2}{0.3} \approx 2.31 \text{ min}$$

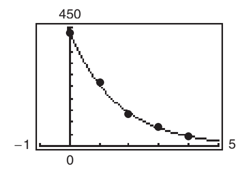
137.

t	0	1	2	3	4
R	425	240	118	71	36
$\ln R$	6.052	5.481	4.771	4.263	3.584

(a) $\ln R = -0.6155t + 6.0609$

$$R = e^{-0.6155t+6.0609} = 428.78e^{-0.6155t}$$

(b)



(c) $\int_0^4 R(t)\,dt = \int_0^4 428.78e^{-0.6155t}\,dt$

$$\approx 637.2 \text{ L}$$

138. Area $= \frac{8}{3} = \int_{-a}^{a} e^{-x}\,dx = -e^{-x}\Big]_{-a}^{a} = -e^{-a} + e^a$

Let $z = e^a$:

$$\frac{8}{3} = \frac{-1}{z} + z$$

$$\frac{8}{3}z = -1 + z^2$$

$$3z^2 - 8z - 3 = 0$$

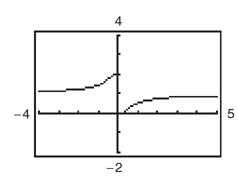
$$(3z + 1)(z - 3) = 0$$

$$z = 3 \Rightarrow e^a = 3 \Rightarrow a = \ln 3$$

$$\left(z = -\frac{1}{3} \Rightarrow e^a = -\frac{1}{3} \text{ impossible}\right)$$

So, $a = \ln 3$.

139. (a)



(b) When x increases without bound, $1/x$ approaches zero, and $e^{1/x}$ approaches 1. Therefore, $f(x)$ approaches $2/(1 + 1) = 1$. So, $f(x)$ has a horizontal asymptote at $y = 1$. As x approaches zero from the right, $1/x$ approaches ∞, $e^{1/x}$ approaches ∞ and $f(x)$ approaches zero. As x approaches zero from the left, $1/x$ approaches $-\infty$, $e^{1/x}$ approaches zero, and $f(x)$ approaches 2. The limit does not exist because the left limit does not equal the right limit. Therefore, $x = 0$ is a nonremovable discontinuity.

140. $f(x) = \dfrac{\ln x}{x}$

(a) $f'(x) = \dfrac{1 - \ln x}{x^2} = 0$ when $x = e$.

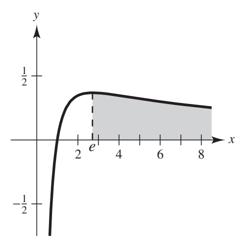
On $(0, e), f'(x) > 0 \Rightarrow f$ is increasing.

On $(e, \infty), f'(x) < 0 \Rightarrow f$ is decreasing.

(b) For $e \le A < B$, you have

$$\frac{\ln A}{A} > \frac{\ln B}{B}$$
$$B \ln A > A \ln B$$
$$\ln A^B > \ln B^A$$
$$A^B > B^A.$$

(c) Because $e < \pi$, from part (b) you have $e^\pi > \pi^e$.



141. $\int_0^x e^t\,dt \ge \int_0^x 1\,dt$

$$\left[e^t\right]_0^x \ge \left[t\right]_0^x$$

$e^x - 1 \ge x \Rightarrow e^x \ge 1 + x$ for $x \ge 0$

142. $e^{-x} = x \Rightarrow f(x) = x - e^{-x}$

$$f'(x) = 1 + e^{-x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}}$$

$$x_1 = 1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 0.5379$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 0.5670$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx 0.5671$$

Approximate the root of f to be $x \approx 0.567$.

143. $f(x) = xe^{-kx}, k > 0$

$$f'(x) = e^{-kx} - kxe^{-kx} = e^{-kx}(1 - kx)$$

$f'(x) = 0$ when $x = \dfrac{1}{k}$.

$$f''(x) = -ke^{-kx}(1 - kx) - ke^{-kx} = e^{-kx}\left(k^2x - 2k\right)$$

$f''(x) = 0$ when $x = \dfrac{2}{k}$.

Relative maximum: $\left(\dfrac{1}{k}, \dfrac{1}{ke}\right)$

Point of inflection: $\left(\dfrac{2}{k}, \dfrac{2}{ke^2}\right)$

144. $y = \dfrac{L}{1 + ae^{-x/b}}, \quad a > 0, b > 0, L > 0$

$$y' = \frac{-L\left(-\dfrac{a}{b}e^{-x/b}\right)}{\left(1 + ae^{-x/b}\right)^2} = \frac{\dfrac{aL}{b}e^{-x/b}}{\left(1 + ae^{-x/b}\right)^2}$$

$$y'' = \frac{\left(1 + ae^{-x/b}\right)^2\left(\dfrac{-aL}{b^2}e^{-x/b}\right) - \left(\dfrac{aL}{b}e^{-x/b}\right)2\left(1 + ae^{-x/b}\right)\left(\dfrac{-a}{b}e^{-x/b}\right)}{\left(1 + ae^{-x/b}\right)^4}$$

$$= \frac{\left(1 + ae^{-x/b}\right)\left(\dfrac{-aL}{b^2}e^{-x/b}\right) + 2\left(\dfrac{aL}{b}e^{-x/b}\right)\left(\dfrac{a}{b}e^{-x/b}\right)}{\left(1 + ae^{-x/b}\right)^3} = \frac{Lae^{-x/b}\left[ae^{-x/b} - 1\right]}{\left(1 + ae^{-x/b}\right)^3 b^2}$$

$y'' = 0$ if $ae^{-x/b} = 1 \Rightarrow \dfrac{-x}{b} = \ln\left(\dfrac{1}{a}\right) \Rightarrow x = b \ln a$

$$y(b \ln a) = \frac{L}{1 + ae^{-(b \ln a)/b}} = \frac{L}{1 + a(1/a)} = \frac{L}{2}$$

Therefore, the y-coordinate of the inflection point is $L/2$.

145. Let $f(x)$ and $g(x)$ be two functions in S. Since $\ln(x + 1) \in S$, $\ln(f(x) + 1)$ and $\ln(g(x) + 1)$ are in S by the composition property (ii). From the addition property (ii), and facts about logarithms,
$\ln(f(x) + 1) + \ln(g(x) + 1) = \ln[(f(x) + 1)(g(x) + 1)] \in S$.

Since $e^x - 1 \in S$, $e^{\ln[(f(x)+1)(g(x)+1)]} - 1 \in S$. Simplifying, $(f(x) + 1)(g(x) + 1) - 1 \in S$. Multiplying this expression out, $f(x)g(x) + f(x) + g(x) \in S$. You also have $f(x) + g(x) \in S$. Since $f(x)g(x) \ge 0$ and $f(x)g(x) + f(x) + g(x) \ge f(x) + g(x)$, property (iii) says that $f(x)g(x) \in S$, which is what you wanted to prove.

Section 5.5 Bases Other than *e* and Applications

1. Using Theorem 5.13,
$\frac{d}{dx}[6^{4x}] = (\ln 6)6^{4x}(4)$.
Therefore, $a = 4$ and $b = 6$.

2. You can use the formula on page 355,
$\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C$.

Or, you can convert $5t$ to $e^{(\ln 5)t}$ and integrate $\int e^{(\ln 5)t}\,dt$.

3. Logarithmic differentiation is necessary when you need to differentiate a function of the form $f(x) = u(x)^{v(x)}$.

4. Use the formula $A = Pe^{rt}$ if the interest is compounded continuously. Otherwise, use the formula
$A = P\left(1 + \frac{r}{n}\right)^{nt}$.

5. $\log_2 \frac{1}{8} = \log_2 2^{-3} = -3$

6. $\log_3 81 = \log_3(3^4) = 4 \log_3 3 = 4(1) = 4$

7. $\log_7 1 = 0$

8. $\log_a \frac{1}{a} = \log_a 1 - \log_a a = -1$

9. $\log_{64} 32 = \log_{64}(64)^{5/6} = \frac{5}{6}\log_{64} 64 = \frac{5}{6}$

10. $\log_{27}\left(\frac{1}{9}\right) = \log_{27}(27)^{-2/3} = -\frac{2}{3}\log_{27} 27 = -\frac{2}{3}$

11. (a) $2^3 = 8$
$\log_2 8 = 3$

(b) $3^{-1} = \frac{1}{3}$
$\log_3 \frac{1}{3} = -1$

12. (a) $27^{2/3} = 9$
$\log_{27} 9 = \frac{2}{3}$

(b) $16^{3/4} = 8$
$\log_{16} 8 = \frac{3}{4}$

13. (a) $\log_{10} 0.01 = -2$
$10^{-2} = 0.01$

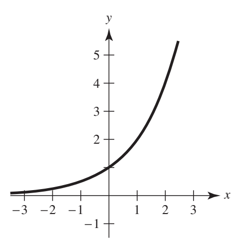
(b) $\log_{0.5} 8 = -3$
$0.5^{-3} = 8$
$\left(\frac{1}{2}\right)^{-3} = 8$

14. (a) $\log_3 \frac{1}{9} = -2$
$3^{-2} = \frac{1}{9}$

(b) $49^{1/2} = 7$
$\log_{49} 7 = \frac{1}{2}$

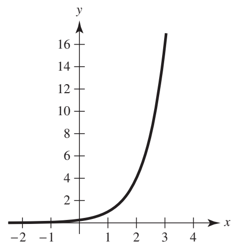
15. $y = 2^x$

x	-2	-1	0	1	2
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4



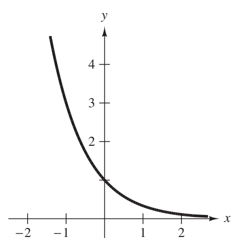
16. $y = 4^{x-1}$

x	-1	0	1	2	3
y	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16



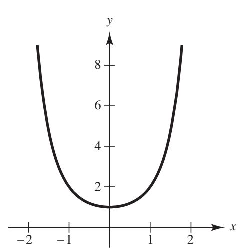
17. $y = \left(\frac{1}{3}\right)^x = 3^{-x}$

x	-2	-1	0	1	2
y	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$



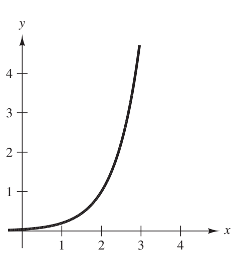
18. $y = 2^{x^2}$

x	-2	-1	0	1	2
y	16	2	1	2	16



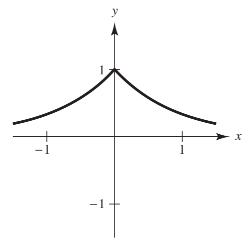
19. $h(x) = 5^{x-2}$

x	-1	0	1	2	3
y	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5



20. $y = 3^{-|x|}$

x	0	± 1	± 2
y	1	$\frac{1}{3}$	$\frac{1}{9}$



21. (a) $\log_{10} 1000 = x$

$10^x = 1000$

$x = 3$

(b) $\log_{10} 0.1 = x$

$10^x = 0.1$

$x = -1$

22. (a) $\log_3 \frac{1}{81} = x$

$3^x = \frac{1}{81}$

$x = -4$

(b) $\log_6 36 = x$

$6^x = 36$

$x = 2$

23. (a) $\log_3 x = -1$

$3^{-1} = x$

$x = \frac{1}{3}$

(b) $\log_2 x = -4$

$2^{-4} = x$

$x = \frac{1}{16}$

24. (a) $\log_4 x = -2$

$4^{-2} = x$

$x = \dfrac{1}{16}$

(b) $\log_5 x = 3$

$5^3 = x$

$x = 125$

25. (a)
$$x^2 - x = \log_5 25$$
$$x^2 - x = \log_5 5^2 = 2$$
$$x^2 - x - 2 = 0$$
$$(x+1)(x-2) = 0$$
$$x = -1 \quad \text{OR} \quad x = 2$$

(b)
$$3x + 5 = \log_2 64$$
$$3x + 5 = \log_2 2^6 = 6$$
$$3x = 1$$
$$x = \tfrac{1}{3}$$

26. (a)
$$\log_3 x + \log_3(x-2) = 1$$
$$\log_3[x(x-2)] = 1$$
$$x(x-2) = 3^1$$
$$x^2 - 2x - 3 = 0$$
$$(x+1)(x-3) = 0$$
$$x = -1 \quad \text{OR} \quad x = 3$$

$x = 3$ is the only solution because the domain of the logarithmic function is the set of all *positive* real numbers.

(b)
$$\log_{10}(x+3) - \log_{10} x = 1$$
$$\log_{10}\frac{x+3}{x} = 1$$
$$\frac{x+3}{x} = 10^1$$
$$x + 3 = 10x$$
$$3 = 9x$$
$$x = \frac{1}{3}$$

27.
$$3^{2x} = 75$$
$$2x \ln 3 = \ln 75$$
$$x = \left(\frac{1}{2}\right)\frac{\ln 75}{\ln 3} \approx 1.965$$

28.
$$6^{-2x} = 74$$
$$-2x \ln 6 = \ln 74$$
$$x = \frac{\ln 74}{-2 \ln 6} \approx -1.201$$

29.
$$2^{3-z} = 625$$
$$(3-z)\ln 2 = \ln 625$$
$$3 - z = \frac{\ln 625}{\ln 2}$$
$$z = 3 - \frac{\ln 625}{\ln 2} \approx -6.288$$

30.
$$3(5^{x-1}) = 86$$
$$5^{x-1} = \frac{86}{3}$$
$$(x-1)\ln 5 = \ln\left(\frac{86}{3}\right)$$
$$x - 1 = \frac{\ln(86/3)}{\ln 5}$$
$$x = 1 + \frac{\ln(86/3)}{\ln 5} \approx 3.085$$

31.
$$\left(1 + \frac{0.09}{12}\right)^{12t} = 3$$
$$12t \ln\left(1 + \frac{0.09}{12}\right) = \ln 3$$
$$t = \left(\frac{1}{12}\right)\frac{\ln 3}{\ln\left(1 + \frac{0.09}{12}\right)} \approx 12.253$$

32.
$$\left(1 + \frac{0.10}{365}\right)^{365t} = 2$$
$$365t \ln\left(1 + \frac{0.10}{365}\right) = \ln 2$$
$$t = \left(\frac{1}{365}\right)\frac{\ln 2}{\ln\left(1 + \frac{0.10}{365}\right)} \approx 6.932$$

33.
$$\log_2(x-1) = 5$$
$$x - 1 = 2^5 = 32$$
$$x = 33$$

34.
$$\log_{10}(t-3) = 2.6$$
$$t - 3 = 10^{2.6}$$
$$t = 3 + 10^{2.6} \approx 401.107$$

35.
$$\log_7 x^3 = 1.9$$
$$x^3 = 7^{1.9}$$
$$x = \left(7^{1.9}\right)^{1/3} \approx 3.429$$

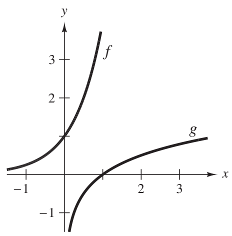
36.
$$\log_5 \sqrt{x-4} = 3.2$$
$$\sqrt{x-4} = 5^{3.2}$$
$$x - 4 = \left(5^{3.2}\right)^2 = 5^{6.4}$$
$$x = 4 + 5^{6.4}$$
$$\approx 29{,}748.593$$

37. $f(x) = 4^x$

$g(x) = \log_4 x$

x	-2	-1	0	$\frac{1}{2}$	1
$f(x)$	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4

x	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4
$g(x)$	-2	-1	0	$\frac{1}{2}$	1

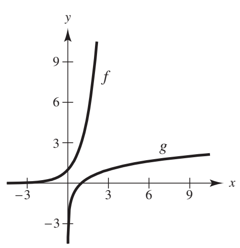


38. $f(x) = 3^x$

$g(x) = \log_3 x$

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$g(x)$	-2	-1	0	1	2



39. $f(x) = 4^x$

$f'(x) = (\ln 4)4^x$

40. $f(x) = 3^{4x}$

$f'(x) = 4(\ln 3)3^{4x} = 4(\ln 3)81^x$

41. $y = 5^{-4x}$

$y' = -4(\ln 5)5^{-4x}$

$= \dfrac{-4 \ln 5}{625^x}$

42. $y = 6^{3x-4}$

$y' = 3(\ln 6)6^{3x-4}$

43. $f(x) = x9^x$

$f'(x) = x(\ln 9)9^x + 9^x$

$= 9^x(1 + x \ln 9)$

44. $y = -7x(8^{-2x})$

$y' = -7(8^{-2x}) - 7x(\ln 8)8^{(-2x)}(-2)$

$= 8^{-2x}(14x \ln 8 - 7)$

45. $f(t) = \dfrac{-2t^2}{8^t}$

$f'(t) = \dfrac{8^t(-4t) + 2t^2(\ln 8)8^t}{8^{2t}}$

$= \dfrac{-4t + 2t^2 \ln 8}{8^t}$

46. $f(t) = \dfrac{3^{2t}}{t}$

$f'(t) = \dfrac{t(2 \ln 3)3^{2t} - 3^{2t}}{t^2}$

$= \dfrac{3^{2t}(2t \ln 3 - 1)}{t^2}$

47. $h(\theta) = 2^{-\theta} \cos \pi\theta$

$h'(\theta) = 2^{-\theta}(-\pi \sin \pi\theta) - (\ln 2)2^{-\theta} \cos \pi\theta$

$= -2^{-\theta}\left[(\ln 2) \cos \pi\theta + \pi \sin \pi\theta\right]$

48. $g(\alpha) = 5^{-\alpha/2} \sin 2\alpha$

$g'(\alpha) = 5^{-\alpha/2} 2 \cos 2\alpha - \frac{1}{2}(\ln 5)5^{-\alpha/2} \sin 2\alpha$

49. $y = \log_4 (6x + 1)$

$y' = \dfrac{6}{(\ln 4)(6x + 1)}$

50. $y = \log_3(x^2 - 3x)$

$y' = \dfrac{1}{(x^2 - 3x) \ln 3}(2x - 3)$

$= \dfrac{2x - 3}{x(x - 3) \ln 3}$

51. $h(t) = \log_5(4 - t)^2 = 2 \log_5(4 - t)$

$h'(t) = 2\dfrac{-1}{\ln(5)(4 - t)} = \dfrac{2}{(t - 4) \ln 5}$

52. $g(t) = \log_2(t^2+7)^3 = 3\log_2(t^2+7)$

$g'(t) = 3\dfrac{2t}{\ln 2(t^2+7)} = \dfrac{6t}{(t^2+7)\ln 2}$

53. $y = \log_5\sqrt{x^2-1} = \dfrac{1}{2}\log_5(x^2-1)$

$\dfrac{dy}{dx} = \dfrac{1}{2}\cdot\dfrac{2x}{(x^2-1)\ln 5} = \dfrac{x}{(x^2-1)\ln 5}$

54. $f(x) = \log_2\sqrt[3]{2x+1} = \dfrac{1}{3}\log_2(2x+1)$

$f'(x) = \dfrac{1}{3}\dfrac{1}{(2x+1)\ln 2}(2) = \dfrac{2}{3(2x+1)\ln 2}$

55. $f(x) = \log_2\dfrac{x^2}{x-1} = 2\log_2 x - \log_2(x-1)$

$f'(x) = \dfrac{2}{x\ln 2} - \dfrac{1}{(x-1)\ln 2} = \dfrac{x-2}{(\ln 2)x(x-1)}$

56. $y = \log_{10}\dfrac{x^2-1}{x} = \log_{10}(x^2-1) - \log_{10} x$

$\dfrac{dy}{dx} = \dfrac{2x}{(x^2-1)\ln 10} - \dfrac{1}{x\ln 10}$

$= \dfrac{1}{\ln 10}\left[\dfrac{2x}{x^2-1} - \dfrac{1}{x}\right]$

$= \dfrac{1}{\ln 10}\left[\dfrac{x^2+1}{x(x^2-1)}\right]$

57. $h(x) = \log_3\dfrac{x\sqrt{x-1}}{2}$

$= \log_3 x + \dfrac{1}{2}\log_3(x-1) - \log_3 2$

$h'(x) = \dfrac{1}{x\ln 3} + \dfrac{1}{2}\cdot\dfrac{1}{(x-1)\ln 3} - 0$

$= \dfrac{1}{\ln 3}\left[\dfrac{1}{x} + \dfrac{1}{2(x-1)}\right]$

$= \dfrac{1}{\ln 3}\left[\dfrac{3x-2}{2x(x-1)}\right]$

58. $g(x) = \log_5\left(\dfrac{4}{x^2\sqrt{1-x}}\right)$

$= \log_5 4 - \log_5 x^2 - \log_5\sqrt{1-x}$

$= \log_5 4 - 2\log_5 x - \dfrac{1}{2}\log_5(1-x)$

$g'(x) = -2\dfrac{1}{x\ln 5} - \dfrac{1}{2}\dfrac{1}{(1-x)\ln 5}(-1)$

$= \dfrac{-2}{x\ln 5} + \dfrac{1}{2(1-x)\ln 5}$

59. $g(t) = \dfrac{10\log_4 t}{t} = \dfrac{10}{\ln 4}\left(\dfrac{\ln t}{t}\right)$

$g'(t) = \dfrac{10}{\ln 4}\left[\dfrac{t(1/t) - \ln t}{t^2}\right]$

$= \dfrac{10}{t^2\ln 4}[1 - \ln t]$

$= \dfrac{5}{t^2\ln 2}(1 - \ln t)$

60. $f(t) = t^{3/2}\log_2\sqrt{t+1} = t^{3/2}\dfrac{1}{2}\dfrac{\ln(t+1)}{\ln 2}$

$f'(t) = \dfrac{1}{2\ln 2}\left[t^{3/2}\dfrac{1}{t+1} + \dfrac{3}{2}t^{1/2}\ln(t+1)\right]$

61. $y = 2^{-x}, \quad (-1, 2)$

$y' = -2^{-x}\ln(2)$

At $(-1, 2)$, $y' = -2\ln(2)$.

Tangent line: $y - 2 = -2\ln(2)(x+1)$

$y = -2x\ln 2 + 2 - 2\ln 2$

62. $y = 5^{x-2}, \quad (2, 1)$

$y' = 5^{x-2}\ln 5$

At $(2, 1)$, $y' = \ln 5$.

Tangent line: $y - 1 = \ln(5)(x-2)$

$y = x\ln 5 + 1 - 2\ln 5$

63. $y = \log_3 x, \quad (27, 3)$

$y' = \dfrac{1}{x\ln 3}$

At $(27, 3)$, $y' = \dfrac{1}{27\ln 3}$.

Tangent line: $y - 3 = \dfrac{1}{27\ln 3}(x - 27)$

$y = \dfrac{1}{27\ln 3}x + 3 - \dfrac{1}{\ln 3}$

64. $y = \log_{10}(2x), \quad (5, 1)$

$y' = \dfrac{1}{x\ln 10}$

At $(5, 1)$, $y' = \dfrac{1}{5\ln 10}$.

Tangent line: $y - 1 = \dfrac{1}{5\ln 10}(x - 5)$

$y = \dfrac{1}{5\ln 10}x + 1 - \dfrac{1}{\ln 10}$

65.
$$y = x^{2/x}$$
$$\ln y = \frac{2}{x}\ln x$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{2}{x}\left(\frac{1}{x}\right) + \ln x\left(-\frac{2}{x^2}\right) = \frac{2}{x^2}(1 - \ln x)$$
$$\frac{dy}{dx} = \frac{2y}{x^2}(1 - \ln x) = 2x^{(2/x)-2}(1 - \ln x)$$

66.
$$y = x^{x-1}$$
$$\ln y = (x - 1)(\ln x)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = (x - 1)\left(\frac{1}{x}\right) + \ln x$$
$$\frac{dy}{dx} = y\left[\frac{x - 1}{x} + \ln x\right]$$
$$= x^{x-2}(x - 1 + x\ln x)$$

67.
$$y = (x - 2)^{x+1}$$
$$\ln y = (x + 1)\ln(x - 2)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = (x + 1)\left(\frac{1}{x - 2}\right) + \ln(x - 2)$$
$$\frac{dy}{dx} = y\left[\frac{x + 1}{x - 2} + \ln(x - 2)\right]$$
$$= (x - 2)^{x+1}\left[\frac{x + 1}{x - 2} + \ln(x - 2)\right]$$

68.
$$y = (1 + x)^{1/x}$$
$$\ln y = \frac{1}{x}\ln(1 + x)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x}\left(\frac{1}{1 + x}\right) + \ln(1 + x)\left(-\frac{1}{x^2}\right)$$
$$\frac{dy}{dx} = \frac{y}{x}\left[\frac{1}{x + 1} - \frac{\ln(x + 1)}{x}\right]$$
$$= \frac{(1 + x)^{1/x}}{x}\left[\frac{1}{x + 1} - \frac{\ln(x + 1)}{x}\right]$$

69. $\int 3^x\,dx = \frac{3^x}{\ln 3} + C$

70.
$$\int 2^{-x}\,dx = -\int 2^{-x}(-dx)$$
$$= -\frac{2^{-x}}{\ln 2} + C$$

71.
$$\int\left(x^2 + 2^{-x}\right)dx = \frac{x^3}{3} + \frac{-1}{\ln 2}2^{-x} + C$$
$$= \frac{1}{3}x^3 - \frac{2^{-x}}{\ln 2} + C$$

72. $\int\left(x^4 + 5^x\right)dx = \frac{x^5}{5} + \frac{5^x}{\ln 5} + C$

73.
$$\int x\left(5^{-x^2}\right)dx = -\frac{1}{2}\int 5^{-x^2}(-2x)\,dx$$
$$= -\left(\frac{1}{2}\right)\frac{5^{-x^2}}{\ln 5} + C$$
$$= \frac{-1}{2\ln 5}\left(5^{-x^2}\right) + C$$

74. Let $u = (4 - x)^2$, $du = -2(4 - x)\,dx$.
$$\int (4 - x)6^{(4-x)^2}\,dx = -\frac{1}{2}\int 6^{(4-x)^2}\left[-2(4 - x)\,dx\right]$$
$$= -\frac{6^{(4-x)^2}}{2\ln 6} + C$$

75. $\int\frac{3^{2x}}{1 + 3^{2x}}\,dx$, $u = 1 + 3^{2x}$, $du = 2(\ln 3)3^{2x}\,dx$
$$\frac{1}{2\ln 3}\int\frac{(2\ln 3)3^{2x}}{1 + 3^{2x}}\,dx = \frac{1}{2\ln 3}\ln\left(1 + 3^{2x}\right) + C$$

76. $\int 2^{\sin x}\cos x\,dx$, $u = \sin x$, $du = \cos x\,dx$
$$\frac{1}{\ln 2}2^{\sin x} + C$$

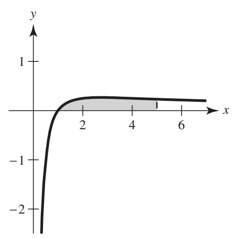
77. $\int_{-1}^{2} 2^x\,dx = \left[\frac{2^x}{\ln 2}\right]_{-1}^{2} = \frac{1}{\ln 2}\left[4 - \frac{1}{2}\right] = \frac{7}{2\ln 2} = \frac{7}{\ln 4}$

78.
$$\int_{-4}^{4} 3^{x/4}\,dx = 4\int_{-4}^{4} 3^{x/4}\left(\frac{1}{4}\,dx\right)$$
$$= \left[4\frac{1}{\ln 3}3^{x/4}\right]_{-4}^{4}$$
$$= \frac{4}{\ln 3}\left(3 - 3^{-1}\right)$$
$$= \frac{32}{3\ln 3}$$

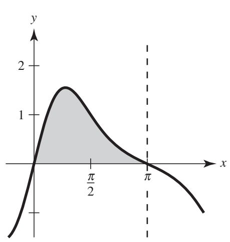
79.
$$\int_0^1\left(5^x - 3^x\right)dx = \left[\frac{5^x}{\ln 5} - \frac{3^x}{\ln 3}\right]_0^1$$
$$= \left(\frac{5}{\ln 5} - \frac{3}{\ln 3}\right) - \left(\frac{1}{\ln 5} - \frac{1}{\ln 3}\right)$$
$$= \frac{4}{\ln 5} - \frac{2}{\ln 3}$$

80. $$\int_1^3 \left(4^{x+1} + 2^x\right) dx = \left[\left(\frac{1}{\ln 4}\right)4^{x+1} + \frac{1}{\ln 2}2^x\right]_1^3$$
$$= \frac{1}{\ln 4}\left(4^4\right) + \frac{1}{\ln 2}\left(2^3\right) - \frac{1}{\ln 4}\left(4^2\right) - \frac{1}{\ln 2}(2)$$
$$= \frac{1}{\ln 4}(256 - 16) + \frac{1}{\ln 2}(8 - 2)$$
$$= \frac{240}{\ln 4} + \frac{6}{\ln 2}$$
$$= \frac{240}{2 \ln 2} + \frac{6}{\ln 2}$$
$$= \frac{126}{\ln 2}$$

81. $$\text{Area} = \int_1^5 \frac{\log_4 x}{x}\, dx = \int_1^5 \frac{\ln x}{x \ln 4}\, dx$$
$$= \frac{1}{\ln 4}\left[\frac{(\ln x)^2}{2}\right]_1^5 = \frac{1}{\ln 4}\frac{(\ln 5)^2}{2} = \frac{(\ln 5)^2}{2 \ln 4}$$
$$\approx 0.934$$



82. $$\text{Area} = \int_0^{\pi} 3^{\cos x} \sin x\, dx$$
$$= \left[\frac{-3^{\cos x}}{\ln 3}\right]_0^{\pi} = \frac{-1}{\ln 3}\left[3^{-1} - 3\right] = \frac{8}{3 \ln 3} \approx 2.4273$$



83. The rate of change grows more rapidly as a becomes larger:

$y = a^x \Rightarrow y' = (\ln a)a^x$.

84. The rate of change grows more slowly as a becomes larger:

$y = \log_a x \Rightarrow y' = \dfrac{1}{(\ln a)x}$.

85. $f(x) = \log_{10} x$

(a) Domain: $x > 0$

(b) $$y = \log_{10} x$$
$$10^y = x$$
$$f^{-1}(x) = 10^x$$

(c) $\log_{10} 1000 = \log_{10} 10^3 = 3$

$\log_{10} 10{,}000 = \log_{10} 10^4 = 4$

If $1000 \le x \le 10{,}000$, then $3 \le f(x) \le 4$.

(d) If $f(x) < 0$, then $0 < x < 1$.

(e) $f(x) + 1 = \log_{10} x + \log_{10} 10 = \log_{10}(10x)$

x must have been increased by a factor of 10.

(f) $$\log_{10}\left(\frac{x_1}{x_2}\right) = \log_{10} x_1 - \log_{10} x_2$$
$$= 3n - n = 2n$$

So, $x_1/x_2 = 10^{2n} = 100^n$.

86. $f(x) = \log_2 x \Rightarrow f'(x) = \dfrac{1}{x \ln 2}$

$g(x) = x^x \Rightarrow g'(x) = x^x(1 + \ln x)$

Note: Let $y = g(x)$. Then:

$$\ln y = \ln x^x = x \ln x$$
$$\frac{1}{y}y' = x \cdot \frac{1}{x} + \ln x$$
$$y' = y(1 + \ln x)$$
$$y' = x^x(1 + \ln x) = g'(x)$$

$h(x) = x^2 \Rightarrow h'(x) = 2x$

$k(x) = 2^x \Rightarrow k'(x) = (\ln 2)2^x$

From greatest to least rate of growth:

$g(x), k(x), h(x), f(x)$

87. $C(t) = P(1.05)^t$

(a) $C(10) = 24.95(1.05)^{10}$

$\approx \$40.64$

(b) $\dfrac{dC}{dt} = P(\ln 1.05)(1.05)^t$

When $t = 1$, $\dfrac{dC}{dt} \approx 0.051P$.

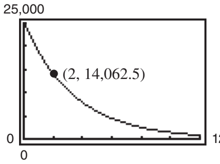
When $t = 8$, $\dfrac{dC}{dt} \approx 0.072P$.

(c) $\dfrac{dC}{dt} = (\ln 1.05)\left[P(1.05)^t\right] = (\ln 1.05)C(t)$

The constant of proportionality is ln 1.05.

88. $V(t) = 25{,}000\left(\dfrac{3}{4}\right)^t$

(a)



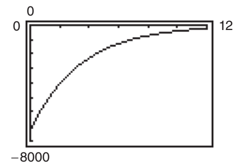
$V(2) = 25{,}000\left(\dfrac{3}{4}\right)^2 = \$14{,}062.50$

(b) $\dfrac{dV}{dt} = 25{,}000\left(\ln\dfrac{3}{4}\right)\left(\dfrac{3}{4}\right)^t$

When $t = 1$, $\dfrac{dV}{dt} \approx -5394.04$.

When $t = 4$, $\dfrac{dV}{dt} \approx -2275.61$.

(c)



Horizontal asymptote: $V' = 0$

As the car ages, it is worth less each year and depreciates less each year, but the value of the car will never reach \$0.

89. $P = \$1000$, $r = 3\frac{1}{2}\% = 0.035$, $t = 10$

$A = 1000\left(1 + \dfrac{0.035}{n}\right)^{10n}$

$A = 1000e^{(0.035)(10)}$

n	1	2	4	12	365	Continuous
A	\$1410.60	\$1414.78	\$1416.91	\$1418.34	\$1419.04	\$1419.07

90. $P = \$2500$, $r = 6\% = 0.06$, $t = 20$

$A = 2500\left(1 + \dfrac{0.06}{n}\right)^{20n}$

$A = 2500e^{(0.06)(20)}$

n	1	2	4	12	365	Continuous
A	\$8017.84	\$8155.09	\$8226.66	\$8275.51	\$8299.47	\$8300.29

91. $P = \$7500,\ r = 4.8\% = 0.048,\ t = 30$

$A = 7500\left(1 + \dfrac{0.048}{n}\right)^{30n}$

$A = 7500e^{(0.048)(30)}$

n	1	2	4	12	365	Continuous Compounded
A	\$30,612.57	\$31,121.37	\$31,385.05	\$31,564.42	\$31,652.22	\$31,655.22

92. $P = \$4000,\ r = 4\% = 0.04,\ t = 15$

$A = 4000\left(1 + \dfrac{0.04}{n}\right)^{15n}$

$A = 4000e^{0.04(15)}$

n	1	2	4	12	365	Continuous
A	\$7203.77	\$7245.45	\$7266.79	\$7281.21	\$7288.24	\$7288.48

93. $100{,}000 = Pe^{0.04t} \Rightarrow P = 100{,}000e^{-0.04t}$

t	1	10	20	30	40	50
P	\$96,078.94	\$67,032.00	\$44,932.90	\$30,119.42	\$20,189.65	\$13,533.53

94. $100{,}000 = Pe^{0.006t} \Rightarrow P = 100{,}000e^{-0.006t}$

t	1	10	20	30	40	50
P	99,401.80	94,176.45	88,692.04	83,527.02	78,662.79	74,081.82

95. $100{,}000 = P\left(1 + \frac{0.05}{12}\right)^{12t} \Rightarrow P = 100{,}000\left(1 + \frac{0.05}{12}\right)^{-12t}$

t	1	10	20	30	40	50
P	\$95,132.82	\$60,716.10	\$36,864.45	\$22,382.66	\$13,589.88	\$8251.24

96. $100{,}000 = P\left(1 + \frac{0.02}{365}\right)^{365t} \Rightarrow P = 100{,}000\left(1 + \frac{0.02}{365}\right)^{-365t}$

t	1	10	20	30	40	50
P	\$98,019.92	\$81,873.52	\$67,032.74	\$54,882.07	\$44,933.88	\$36,788.95

97. (a) $A = 20{,}000\left(1 + \frac{0.06}{365}\right)^{(365)(8)} \approx \$32{,}320.21$

(b) $A = \$30{,}000$

(c) $A = 8000\left(1 + \frac{0.06}{365}\right)^{(365)(8)} + 20{,}000\left(1 + \frac{0.06}{365}\right)^{(365)(4)} \approx \$12{,}928.09 + 25{,}424.48 = \$38{,}352.57$

(d) $A = 9000\left[\left(1 + \frac{0.06}{365}\right)^{(365)(8)} + \left(1 + \frac{0.06}{365}\right)^{(365)(4)} + 1\right] \approx \$34{,}985.11$

Take option (c).

98. Let $P = \$100,\ 0 \le t \le 20.$

(a) $A = 100e^{0.03t}$

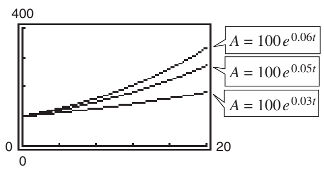
$A(20) \approx 182.21$

(b) $A = 100e^{0.05t}$

$A(20) \approx 271.83$

(c) $A = 100e^{0.06t}$

$A(20) \approx 332.01$



99. (a) $\lim_{t\to\infty} 6.7e^{(-48.1)/t} = 6.7e^0 = 6.7$ million ft^3/acre

(b) $V' = \dfrac{322.27}{t^2}e^{-(48.1)/t}$

$V'(20) \approx 0.073$ million ft^3/acre/yr

$V'(60) \approx 0.040$ million ft^3/acre/yr

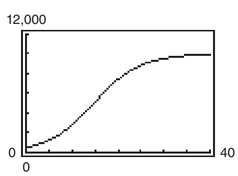
100. $P = \dfrac{0.86}{1 + e^{-0.25n}}$

(a) $\lim_{n\to\infty} \dfrac{0.86}{1 + e^{-0.25n}} = \dfrac{0.86}{1} = 0.86$, or 86%

(b) In the long run, $\dfrac{dP}{dn} \to 0$. The graph gets flatter.

101. $p(t) = \dfrac{10{,}000}{1 + 19e^{-t/5}}$

(a)



(b) $p(6) = 1487$ fish

$p(12) = 3672$ fish

$p(24) = 8648$ fish

$p(36) = 9860$ fish

$p(48) = 9987$ fish

The limiting size is 10,000 fish because $\lim_{t\to\infty} p(t) = 10{,}000.$

(c) $p'(t) = \dfrac{38{,}000e^{-t/5}}{\left(1 + 19^{-t/5}\right)^2}$

$p'(1) \approx 114$ fish/mo

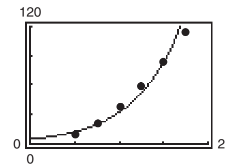
$p'(10) \approx 403$ fish/mo

(d) $p''(t) = 0$ when

$19e^{-t/5} = 1 \Rightarrow t = 5 \ln 19 \approx 14.72$, or 15 months.

102. (a) $B = 4.7539(6.7744)^d = 4.7539e^{1.9132d}$

(b)



(c) $B'(d) = 9.0952e^{1.9132d}$

$B'(0.8) \approx 42.03$ tons/in.

$B'(1.5) \approx 160.38$ tons/in.

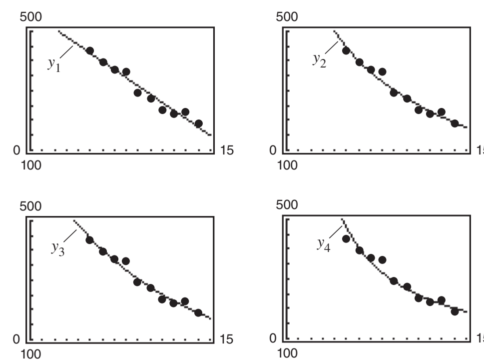
103. (a) $y_1 = -27.7x + 565$

$y_2 = 843 - 246.3 \ln x$

$y_3 = 706.995(0.9106)^x$

$y_4 = 1765.4563\, x$

(b)



Answers will vary. All the models seem like good fits for the data.

(c) For 2012, $t = 12$ and

$y_1'(12) = -27.7$ cases/yr

$y_2'(12) \approx -20.5$ cases/yr

$y_3'(12) \approx -21.5$ cases/yr

$y_4'(12) \approx -15.7$ cases/yr.

The model y_1 is decreasing at the greatest rate.

104.

x	1	10^{-1}	10^{-2}	10^{-4}	10^{-6}
$(1 + x)^{1/x}$	2	2.594	2.705	2.718	2.718

105.

t	0	1	2	3	4
y	1200	720	432	259.20	155.52

$y = C(k^t)$

When $t = 0, y = 1200 \Rightarrow C = 1200.$

$y = 1200(k^t)$

$\frac{720}{1200} = 0.6, \frac{432}{720} = 0.6, \frac{259.20}{432} = 0.6, \frac{155.52}{259.20} = 0.6$

Let $k = 0.6.$

$y = 1200(0.6)^t$

106.

t	0	1	2	3	4
y	600	630	661.50	694.58	729.30

$y = C(k^t)$

When $t = 0, y = 600 \Rightarrow C = 600.$

$y = 600(k^t)$

$\frac{630}{600} = 1.05, \frac{661.50}{630} = 1.05, \frac{694.58}{661.50} \approx 1.05,$

$\frac{729.30}{694.58} \approx 1.05$

Let $k = 1.05.$

$y = 600(1.05)^t$

107.

$$\begin{aligned} 5^{1/\ln 5} &= x \\ \ln\left[5^{1/\ln 5}\right] &= \ln x \\ \frac{1}{\ln 5}[\ln 5] &= \ln x \\ 1 &= \ln x \\ x &= e \end{aligned}$$

108.

$$\begin{aligned} 6^{\ln 10/\ln 6} &= x \\ \ln\left[6^{\ln 10/\ln 6}\right] &= \ln x \\ \frac{\ln 10}{\ln 6} \ln 6 &= \ln x \\ \ln 10 &= \ln x \\ x &= 10 \end{aligned}$$

109.

$$\begin{aligned} 9^{1/\ln 3} &= x \\ \ln\left[9^{1/\ln 3}\right] &= \ln x \\ \frac{1}{\ln 3} \ln(3^2) &= \ln x \\ \frac{1}{\ln 3} 2 \ln 3 &= \ln x \\ \ln x &= 2 \\ x &= e^2 \end{aligned}$$

110.

$$\begin{aligned} 32^{1/\ln 2} &= x \\ \ln\left[32^{1/\ln 2}\right] &= \ln x \\ \frac{1}{\ln 2}\left[\ln 2^5\right] &= \ln x \\ 5 &= \ln x \\ x &= e^5 \end{aligned}$$

111. (a) $(2^3)^2 = 8^2 = 64$

$2^{(3^2)} = 2^9 = 512$

(b) In general, $f(x) = (x^x)^x = x^{(x^2)}$ and $g(x) = x^{(x^x)}$ are not the same.

For example, $f(3) = 3^9 = 19683$, whereas $g(3) = 3^{27}$. Note that when $x = 2, f(2) = g(2) = 16.$

(c) (i)

$$\begin{aligned} y &= f(x) = (x^x)^x = x^{(x^2)} \\ \ln y &= x^2 \ln x \\ \frac{y'}{y} &= 2x \ln x + x^2\left(\frac{1}{x}\right) \\ y' &= x^{(x^2)}(x)(2 \ln x + 1) \\ &= x^{x^2+1}(2 \ln x + 1) \end{aligned}$$

(ii)

$$\begin{aligned} y &= g(x) = x^{(x^x)} \\ \ln y &= x^x \ln x \end{aligned}$$

(**Note:** $\frac{d}{dx}\left[x^x\right] = x^x(1 + \ln x)$ from Example 5)

$$\begin{aligned} \frac{y'}{y} &= x^x(1 + \ln x)\ln x + x^x\frac{1}{x} \\ y' &= x^{(x^x)} \cdot x^x\left[(1 + \ln x) \ln x + \frac{1}{x}\right] \\ &= x^{(x^x)+x-1}\left[x(\ln x)^2 + x \ln x + 1\right] \end{aligned}$$

112. $y = f(x) = \dfrac{a^x - 1}{a^x + 1}$

$$y' = \frac{(a^x + 1)(a^x \ln a) - (a^x - 1)a^x \ln a}{(a^x + 1)^2} = \frac{2a^x \ln a}{(a^x + 1)^2}$$

For $0 < a < 1, y' < 0 \Rightarrow$ one-to-one and has an inverse

For $a > 1, y' > 0 \Rightarrow$ one-to-one and has an inverse

$$y(a^x + 1) = a^x - 1$$
$$a^x(y - 1) = -1 - y$$
$$a^x = \frac{y + 1}{1 - y}$$
$$x \ln a = \ln\left(\frac{y + 1}{1 - y}\right)$$
$$x = \frac{1}{\ln a} \ln\left(\frac{y + 1}{1 - y}\right)$$
$$f^{-1}(x) = \frac{1}{\ln a} \ln\left(\frac{x + 1}{1 - x}\right)$$

113. $\dfrac{dy}{dt} = \dfrac{8}{25}y\left(\dfrac{5}{4} - y\right), \quad y(0) = 1$

$$\frac{dy}{y[(5/4) - y]} = \frac{8}{25}\,dt$$
$$\frac{4}{5}\int\left(\frac{1}{y} + \frac{1}{(5/4) - y}\right)dy = \int\frac{8}{25}\,dt$$
$$\ln y - \ln\left(\frac{5}{4} - y\right) = \frac{2}{5}t + C$$
$$\ln\left(\frac{y}{(5/4) - y}\right) = \frac{2}{5}t + C$$
$$\frac{y}{(5/4) - y} = e^{(2/5)t + C} = C_1 e^{(2/5)t}$$

When

$t = 0, y = 1 \Rightarrow C_1 = 4 \Rightarrow 4e^{(2/5)t} = \dfrac{y}{(5/4) - y}$

$$4e^{(2/5)t}\left(\frac{5}{4} - y\right) = y$$
$$5e^{(2/5)t} = 4e^{(2/5)t}y + y$$
$$= \left(4e^{(2/5)t} + 1\right)y$$
$$y = \frac{5e^{(2/5)t}}{4e^{(2/5)t} + 1}$$
$$= \frac{5}{4 + e^{-0.4t}}$$
$$= \frac{1.25}{1 + 0.25e^{-0.4t}}$$

114. $f(x) = a^x$

(a) $f(u + v) = a^{u+v} = a^u a^v = f(u)f(v)$

(b) $f(2x) = a^{2x} = (a^x)^2 = [f(x)]^2$

115. (a)

$$y^x = x^y$$
$$x \ln y = y \ln x$$
$$x\frac{y'}{y} + \ln y = \frac{y}{x} + y' \ln x$$
$$y'\left[\frac{x}{y} - \ln x\right] = \frac{y}{x} - \ln y$$
$$y' = \frac{(y/x) - \ln y}{(x/y) - \ln x}$$
$$y' = \frac{y^2 - xy \ln y}{x^2 - xy \ln x}$$

(b) (i) At (c, c): $y' = \dfrac{c^2 - c^2 \ln c}{c^2 - c^2 \ln c} = 1, (c \neq 0, e)$

(ii) At $(2, 4)$:

$$y' = \frac{16 - 8 \ln 4}{4 - 8 \ln 2} = \frac{4 - 4 \ln 2}{1 - 2 \ln 2} \approx -3.1774$$

(iii) At $(4, 2)$:

$$y' = \frac{4 - 8 \ln 2}{16 - 8 \ln 4} = \frac{1 - 2 \ln 2}{4 - 4 \ln 2} \approx -0.3147$$

(c) y' is undefined for

$x^2 = xy \ln x$

$x = y \ln x = \ln x^y$

$e^x = x^y.$

At (e, e), y' is undefined.

116. Let $f(x) = \dfrac{\ln x}{x}, \quad x > 0.$

$f'(x) = \dfrac{1 - \ln x}{x^2} < 0$ for $x > e \Rightarrow f$ is decreasing for $x \geq e$. So, for $e \leq x < y$:

$$f(x) > f(y)$$
$$\frac{\ln x}{x} > \frac{\ln y}{y}$$
$$(xy)\frac{\ln x}{x} > (xy)\frac{\ln y}{y}$$
$$\ln x^y > \ln y^x$$
$$x^y > y^x$$

For $n \geq 8, e < \sqrt{n} < \sqrt{n+1}, \left(\sqrt{8} \approx 2.828\right)$ and so letting $x = \sqrt{n}, y = \sqrt{n+1}$, you have

$$\left(\sqrt{n}\right)^{\sqrt{n+1}} > \left(\sqrt{n+1}\right)^{\sqrt{n}}.$$

Note: $\sqrt{8}^{\sqrt{9}} \approx 22.6$ and $\sqrt{9}^{\sqrt{8}} \approx 22.4.$

Note: This same argument shows $e^\pi > \pi^e$.

117. $\log_e\left(1 + \dfrac{1}{x}\right) = \ln\left(1 + \dfrac{1}{x}\right) = \displaystyle\int_x^{1+x} \frac{dt}{t}$

$$> \int_x^{1+x} \frac{dt}{1+x} \quad \begin{pmatrix}\text{because } 1 + x \geq t \\ \text{on } x \leq t \leq 1 + x\end{pmatrix}$$

$$= \left[\frac{t}{1+x}\right]_x^{1+x} = \frac{1+x}{1+x} - \frac{x}{1+x} = \frac{1}{1+x}$$

Note: You can confirm this result by graphing $y_1 = \ln\left(1 + \dfrac{1}{x}\right)$ and $y_2 = \dfrac{1}{1+x}$.

Section 5.6 Indeterminate Forms and L'Hôpital's Rule

1. L'Hôpital's Rule allows you to address limits of the form $0/0$ and ∞/∞.

2. (a) $\displaystyle\lim_{x\to 0} \frac{x^2}{\sin 2x}$ is of the form $\dfrac{0}{0}$.

Indeterminate

(b) $\displaystyle\lim_{x\to\infty} \left(e^x + x^2\right)$ is of form $\infty + \infty$.

Not indeterminate

(c) $\displaystyle\lim_{x\to\infty} \left(\ln x - e^x\right)$ is of the form $\infty - \infty$.

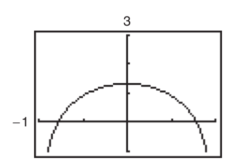
Indeterminate

(d) $\displaystyle\lim_{x\to 0^+} \left(\ln x^2 - \frac{1}{x}\right)$ is of the form $-\infty - \infty$.

Not indeterminate

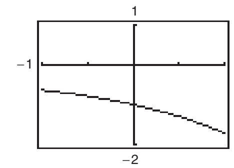
3. $\displaystyle\lim_{x\to 0} \frac{\sin 4x}{\sin 3x} \approx 1.3333 \quad \left(\text{exact: } \frac{4}{3}\right)$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	1.3177	1.3332	1.3333	1.3333	1.3332	1.3177



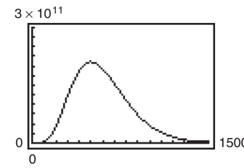
4. $\lim_{x\to 0} \dfrac{1 - e^x}{x} \approx -1$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	-0.9516	-0.9950	-0.9995	-1.0005	-1.0050	-1.0517



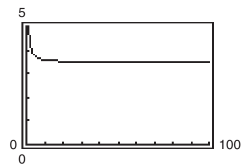
5. $\lim_{x\to \infty} x^5 e^{-x/100} \approx 0$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$	0.9900	$90{,}484$	3.7×10^9	4.5×10^{10}	0	0



6. $\lim_{x\to \infty} \dfrac{6x}{\sqrt{3x^2 - 2x}} \approx 3.4641 \quad \left(\text{exact: } \dfrac{6}{\sqrt{3}}\right)$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$	6	3.5857	3.4757	3.4653	3.4642	3.4641



7. (a) $\lim_{x\to 4} \dfrac{3(x - 4)}{x^2 - 16} = \lim_{x\to 4} \dfrac{3(x - 4)}{(x - 4)(x + 4)} = \lim_{x\to 4} \dfrac{3}{x + 4} = \dfrac{3}{8}$

(b) $\lim_{x\to 4} \dfrac{3(x - 4)}{x^2 - 16} = \lim_{x\to 4} \dfrac{d/dx[3(x - 4)]}{d/dx[x^2 - 16]} = \lim_{x\to 4} \dfrac{3}{2x} = \dfrac{3}{8}$

8. (a) $\lim_{x\to -4} \dfrac{2x^2 + 13x + 20}{x + 4} = \lim_{x\to -4} \dfrac{(x + 4)(2x + 5)}{x + 4} = \lim_{x\to -4}(2x + 5) = -8 + 5 = -3$

(b) $\lim_{x\to -4} \dfrac{2x^2 + 13x + 20}{x + 4} = \lim_{x\to -4} \dfrac{d/dx[2x^2 + 13x + 20]}{d/dx[x + 4]} = \lim_{x\to -4} \dfrac{4x + 13}{1} = -3$

9. (a) $\lim_{x\to 6} \dfrac{\sqrt{x + 10} - 4}{x - 6} = \lim_{x\to 6} \dfrac{\sqrt{x + 10} - 4}{x - 6} \cdot \dfrac{\sqrt{x + 10} + 4}{\sqrt{x + 10} + 4} = \lim_{x\to 6} \dfrac{(x + 10) - 16}{(x - 6)\left(\sqrt{x + 10} + 4\right)} = \lim_{x\to 6} \dfrac{1}{\sqrt{x + 10} + 4} = \dfrac{1}{8}$

(b) $\lim_{x\to 6} \dfrac{\sqrt{x + 10} - 4}{x - 6} = \lim_{x\to 6} \dfrac{d/dx\left[\sqrt{x + 10} - 4\right]}{d/dx[x - 6]} = \lim_{x\to 6} \dfrac{\frac{1}{2}(x + 10)^{-1/2}}{1} = \dfrac{1}{8}$

10. (a)
$$\lim_{x\to -1}\left(\frac{1 - \sqrt{x + 2}}{x + 1} \cdot \frac{1 + \sqrt{x + 2}}{1 + \sqrt{x + 2}}\right) = \lim_{x\to -1} \frac{1 - (x + 2)}{(x + 1)\left(1 + \sqrt{x + 2}\right)}$$
$$= \lim_{x\to -1} \frac{-1 - x}{(x + 1)\left(1 + \sqrt{x + 2}\right)}$$
$$= \lim_{x\to -1} \frac{-1}{1 + \sqrt{-1 + 2}} = \frac{-1}{1 + 1} = -\frac{1}{2}$$

(b)
$$\lim_{x\to -1} \frac{1 - \sqrt{x + 2}}{x + 1} = \lim_{x\to -1} \frac{-\frac{1}{2}(x + 2)^{-1/2}}{1}$$
$$= \lim_{x\to -1} \frac{-1}{2\sqrt{x + 2}} = -\frac{1}{2}$$

11. (a) $\lim_{x\to 0}\left(\frac{2-2\cos x}{6\,x}\right) = \lim_{x\to 0}\frac{(1-\cos x)}{3x}\cdot\frac{(1+\cos x)}{(1+\cos x)}$

$= \lim_{x\to 0}\frac{1-\cos^2 x}{3x(1+\cos x)}$

$= \lim_{x\to 0}\frac{\sin^2 x}{3x(1+\cos x)}$

$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{3(1+\cos x)} = (1)(0) = 0$

(b) $\lim_{x\to 0}\frac{2-2\cos x}{6x} = \lim_{x\to 0}\frac{2\sin x}{6} = 0$

12. (a) $\lim_{x\to 0}\frac{\sin 6x}{4x} = \lim_{x\to 0}\left(\frac{3}{2}\cdot\frac{\sin 6x}{6x}\right) = \frac{3}{2}(1) = \frac{3}{2}$

(b) $\lim_{x\to 0}\frac{\sin 6x}{4x} = \lim_{x\to 0}\frac{d/dx[\sin 6x]}{d/dx[4x]} = \lim_{x\to 0}\frac{6\cos 6x}{4} = \frac{3}{2}$

13. (a) $\lim_{x\to\infty}\frac{5x^2-3x+1}{3x^2-5} = \lim_{x\to\infty}\frac{5-(3/x)+(1/x^2)}{3-(5/x^2)} = \frac{5}{3}$

(b) $\lim_{x\to\infty}\frac{5x^2-3x+1}{3x^2-5} = \lim_{x\to\infty}\frac{(d/dx)[5x^2-3x+1]}{(d/dx)[3x^2-5]} = \lim_{x\to\infty}\frac{10x-3}{6x} = \lim_{x\to\infty}\frac{(d/dx)[10x-3]}{(d/dx)[6x]} = \lim_{x\to\infty}\frac{10}{6} = \frac{5}{3}$

14. (a) $\lim_{x\to\infty}\frac{x^3+2x}{4-x} = \lim_{x\to\infty}\frac{x^2+2}{(4/x)-1} = -\infty$

(b) $\lim_{x\to\infty}\frac{x^3+2x}{4-x} = \lim_{x\to\infty}\frac{3x^2+2}{-1} = -\infty$

15. $\lim_{x\to 3}\frac{x^2-2x-3}{x-3} = \lim_{x\to 3}\frac{2x-2}{1} = 4$

16. $\lim_{x\to -2}\frac{x^2-3x-10}{x+2} = \lim_{x\to -2}\frac{2x-3}{1} = -7$

17. $\lim_{x\to 0}\frac{\sqrt{25-x^2}-5}{x} = \lim_{x\to 0}\frac{\frac{1}{2}(25-x^2)^{-1/2}(-2x)}{1}$

$= \lim_{x\to 0}\frac{-x}{\sqrt{25-x^2}} = 0$

18. $\lim_{x\to 5^-}\frac{\sqrt{25-x^2}}{x-5} = \lim_{x\to 5^-}\frac{\frac{1}{2}(25-x^2)^{-1/2}(-2x)}{1}$

$= \lim_{x\to 5^-}\frac{-x}{\sqrt{25-x^2}} = -\infty$

19. $\lim_{x\to 0^+}\frac{e^x-(1+x)}{x^3} = \lim_{x\to 0^+}\frac{e^x-1}{3x^2} = \lim_{x\to 0^+}\frac{e^x}{6x} = \infty$

20. $\lim_{x\to 1}\frac{\ln x^3}{x^2-1} = \lim_{x\to 1}\frac{3\ln x}{x^2-1} = \lim_{x\to 1}\frac{3/x}{2x} = \frac{3}{2}$

21. $\lim_{x\to 1}\frac{x^{11}-1}{x^4-1} = \lim_{x\to 1}\frac{11x^{10}}{4x^3} = \frac{11}{4}$

22. $\lim_{x\to 1}\frac{x^a-1}{x^b-1} = \lim_{x\to 1}\frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}$

23. $\lim_{x\to 0}\frac{\sin 3x}{\sin 5x} = \lim_{x\to 0}\frac{3\cos 3x}{5\cos 5x} = \frac{3}{5}$

24. $\lim_{x\to 0}\frac{\sin ax}{\sin bx} = \lim_{x\to 0}\frac{a\cos ax}{b\cos bx} = \frac{a}{b}$

25. $\lim_{x\to\infty}\frac{7x^3-2x+1}{6x^3+1} = \lim_{x\to\infty}\frac{21x^2-2}{18x^2}$

$= \lim_{x\to\infty}\frac{42x}{36x} = \frac{42}{36} = \frac{7}{6}$

26. $\lim_{x\to\infty}\frac{8-x}{x^3} = \lim_{x\to\infty}\frac{-1}{3x^2} = 0$

27. $\lim_{x\to\infty}\frac{x^2+4x+7}{x-6} = \lim_{x\to\infty}\frac{2x+4}{1} = \infty$

28. $\lim_{x\to\infty}\frac{x^3}{x+1} = \lim_{x\to\infty}\frac{3x^2}{1} = \infty$

29. $\lim_{x\to\infty} \frac{x^3}{e^{x/2}} = \lim_{x\to\infty} \frac{3x^2}{(1/2)e^{x/2}}$

$= \lim_{x\to\infty} \frac{6x}{(1/4)e^{x/2}} = \lim_{x\to\infty} \frac{6}{(1/8)e^{x/2}} = 0$

30. $\lim_{x\to\infty} \frac{e^{x^2}}{1 - x^3} = \lim_{x\to\infty} \frac{2xe^{x^2}}{-3x^2}$

$= \lim_{x\to\infty} \frac{2e^{x^2}}{-3x}$

$= \lim_{x\to\infty} \frac{4xe^{x^2}}{-3} = -\infty$

31. $\lim_{x\to\infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1 + (1/x^2)}} = 1$

Note: L'Hôpital's Rule does not work on this limit. See Exercise 83.

32. $\lim_{x\to\infty} \frac{x^2}{\sqrt{x^2 + 1}} = \lim_{x\to\infty} \frac{x}{\sqrt{1 + (1/x)^2}} = \infty$

33. $\lim_{x\to\infty} \frac{\cos x}{x} = 0$ by Squeeze Theorem

$\left(\frac{\cos x}{x} \le \frac{1}{x}, \text{ for } x > 0\right)$

34. $\lim_{x\to\infty} \frac{\sin x}{x - \pi} = 0$

Note: Use the Squeeze Theorem for $x > \pi$.

$-\frac{1}{x - \pi} \le \frac{\sin x}{x - \pi} \le \frac{1}{x - \pi}$

35. $\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$

36. $\lim_{x\to\infty} \frac{\ln x^4}{x^3} = \lim_{x\to\infty} \frac{4 \ln x}{x^3} = \lim_{x\to\infty} \frac{4/x}{3x^2} = \lim_{x\to\infty} \frac{4}{3x^3} = 0$

37. $\lim_{x\to\infty} \frac{e^x}{x^4} = \lim_{x\to\infty} \frac{e^x}{4x^3}$

$= \lim_{x\to\infty} \frac{e^x}{12x^2}$

$= \lim_{x\to\infty} \frac{e^x}{24x}$

$= \lim_{x\to\infty} \frac{e^x}{24} = \infty$

38. $\lim_{x\to\infty} \frac{e^{2x-9}}{3x} = \lim_{x\to\infty} \frac{2e^{2x-9}}{3} = \infty$

39. $\lim_{x\to 0} \frac{\sin 5x}{\tan 9x} = \lim_{x\to 0} \frac{5 \cos 5x}{9 \sec^2 9x} = \frac{5}{9}$

40. $\lim_{x\to 1} \frac{\ln x}{\sin \pi x} = \lim_{x\to 1} \frac{1/x}{\pi \cos \pi x} = -\frac{1}{\pi}$

41. $\lim_{x\to\infty} \frac{\int_1^x \ln\left(e^{4t-1}\right) dt}{x}$

$= \lim_{x\to\infty} \frac{\int_1^x (4t - 1)\, dt}{x}$

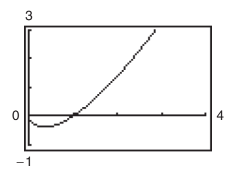
$= \lim_{x\to\infty} \frac{4x - 1}{1} = \infty$

42. $\lim_{x\to 1^+} \frac{\int_1^x \cos\theta\, d\theta}{x - 1} = \lim_{x\to 1^+} \frac{\cos x}{1} = \cos(1)$

43. (a) $\lim_{x\to\infty} x \ln x$, not indeterminate

(b) $\lim_{x\to\infty} x \ln x = (\infty)(\infty) = \infty$

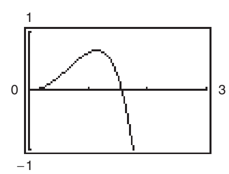
(c)



44. (a) $\lim_{x\to 0^+} x^3 \cot x = (0)(\infty)$

(b) $\lim_{x\to 0^+} x^3 \cot x = \lim_{x\to 0^+} \frac{x^3}{\tan x} = \lim_{x\to 0^+} \frac{3x^2}{\sec^2 x} = 0$

(c)



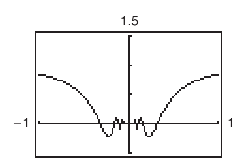
45. (a) $\lim_{x\to\infty} \left(x \sin \frac{1}{x}\right) = (\infty)(0)$

(b) $\lim_{x\to\infty} x \sin \frac{1}{x} = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x}$

$= \lim_{x\to\infty} \frac{\left(-1/x^2\right) \cos(1/x)}{-1/x^2}$

$= \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = 1$

(c)



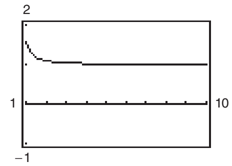
46. (a) $\lim_{x\to\infty}\left(x\tan\frac{1}{x}\right) = (\infty)(0)$

(b) $\lim_{x\to\infty} x\tan\frac{1}{x} = \lim_{x\to\infty}\frac{\tan(1/x)}{1/x}$

$= \lim_{x\to\infty}\frac{-(1/x^2)\sec^2(1/x)}{-(1/x^2)}$

$= \lim_{x\to\infty}\sec^2\left(\frac{1}{x}\right) = 1$

(c)



47. (a) $\lim_{x\to 0^+}\left(e^x + x\right)^{2/x} = 1^\infty$

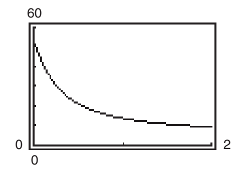
(b) Let $y = \lim_{x\to 0^+}\left(e^x + x\right)^{2/x}$.

$\ln y = \lim_{x\to 0^+}\frac{2\ln\left(e^x + x\right)}{x} = \lim_{x\to 0^+}\frac{2\left(e^x + 1\right)/\left(e^x + x\right)}{1} = 4$

So, $\ln y = 4 \Rightarrow y = e^4 \approx 54.598$.

Therefore, $\lim_{x\to 0^+}\left(e^x + x\right)^{2/x} = e^4$.

(c)



48. (a) $\lim_{x\to 0^+}\left(1 + \frac{1}{x}\right)^x = \infty^\circ$

(b) Let $y = \lim_{x\to 0^+}\left(1 + \frac{1}{x}\right)^x$

$\ln y = \lim_{x\to 0^+}\ln\left[\left(1 + \frac{1}{x}\right)^x\right]$

$= \lim_{x\to 0^+} x\ln\left(1 + \frac{1}{x}\right)$

$= \lim_{x\to 0^+}\frac{\ln(1 + 1/x)}{1/x}$

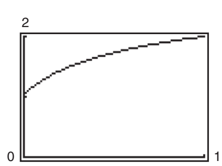
$= \lim_{x\to 0^+}\frac{\left[\frac{-1/x^2}{1+1/x}\right]}{\left(-1/x^2\right)}$

$= \lim_{x\to 0^+}\frac{1}{1 + 1/x} = 0$

So, $\ln y \to 0 \Rightarrow y = e^0 = 1$.

Therefore, $\lim_{x\to 0^+}\left(1 + \frac{1}{x}\right)^x = 1$.

(c)



49. (a) $\lim_{x\to\infty} x^{1/x} = \infty^0$

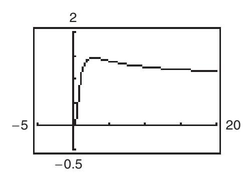
(b) Let $y = \lim_{x\to\infty} x^{1/x}$.

$\ln y = \lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\left(\frac{1/x}{1}\right) = 0$

So, $\ln y = 0 \Rightarrow y = e^0 = 1$.

Therefore, $\lim_{x\to\infty} x^{1/x} = 1$.

(c)



50. (a) $\lim_{x\to 0^+} x^{1/x} = 0^\infty = 0$, not indeterminate

(See Exercise 110).

(b) Let $y = x^{1/x}$

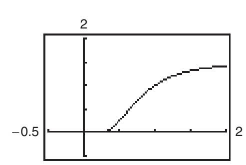
$\ln y = \ln x^{1/x} = \frac{1}{x}\ln x$.

Because $x \to 0^+$, $\frac{1}{x}\ln x \to (\infty)(-\infty) = -\infty$. So,

$\ln y \to -\infty \Rightarrow y \to 0^+$.

Therefore, $\lim_{x\to 0^+} x^{1/x} = 0$.

(c)



51. (a) $\lim_{x\to 0^+}(1+x)^{1/x} = 1^\infty$

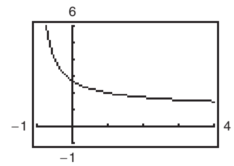
(b) Let $y = \lim_{x\to 0^+}(1+x)^{1/x}$.

$$\ln y = \lim_{x\to 0^+}\frac{\ln(1+x)}{x}$$
$$= \lim_{x\to 0^+}\left(\frac{1/(1+x)}{1}\right) = 1$$

So, $\ln y = 1 \Rightarrow y = e^1 = e$.

Therefore, $\lim_{x\to 0^+}(1+x)^{1/x} = e$.

(c)



52. (a) $\lim_{x\to\infty}(1+x)^{1/x} = \infty^0$

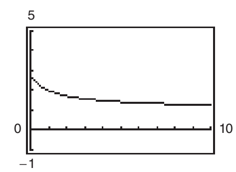
(b) Let $y = \lim_{x\to\infty}(1+x)^{1/x}$.

$$\ln y = \lim_{x\to\infty}\frac{\ln(1+x)}{x} = \lim_{x\to\infty}\left(\frac{1/(1+x)}{1}\right) = 0$$

So, $\ln y = 0 \Rightarrow y = e^0 = 1$.

Therefore, $\lim_{x\to\infty}(1+x)^{1/x} = 1$.

(c)



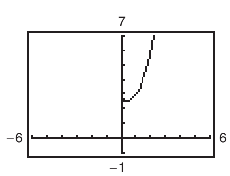
53. (a) $\lim_{x\to 0^+}\left[3(x)^{x/2}\right] = 0^0$

(b) Let $y = \lim_{x\to 0^+} 3(x)^{x/2}$.

$$\ln y = \lim_{x\to 0^+}\left[\ln 3 + \frac{x}{2}\ln x\right]$$
$$= \lim_{x\to 0^+}\left[\ln 3 + \frac{\ln x}{2/x}\right]$$
$$= \lim_{x\to 0^+}\ln 3 + \lim_{x\to 0^+}\frac{1/x}{-2/x^2}$$
$$= \lim_{x\to 0^+}\ln 3 - \lim_{x\to 0^+}\frac{x}{2}$$
$$= \ln 3$$

So, $\lim_{x\to 0^+} 3(x)^{x/2} = 3$.

(c)



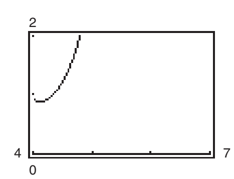
54. (a) $\lim_{x\to 4^+}\left[3(x-4)\right]^{x-4} = 0^0$

(b) Let $y = \lim_{x\to 4^+}\left[3(x-4)\right]^{x-4}$.

$$\ln y = \lim_{x\to 4^+}(x-4)\ln\left[3(x-4)\right]$$
$$= \lim_{x\to 4^+}\frac{\ln\left[3(x-4)\right]}{1/(x-4)}$$
$$= \lim_{x\to 4^+}\frac{1/(x-4)}{-1/(x-4)^2}$$
$$= \lim_{x\to 4^+}\left[-(x-4)\right] = 0$$

So, $\lim_{x\to 4^+}\left[3(x-4)\right]^{x-4} = 1$.

(c)



55. (a) $\lim_{x \to 1^+} (\ln x)^{x-1} = 0^0$

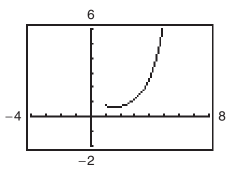
(b) Let $y = (\ln x)^{x-1}$.

$$\ln y = \ln\left[(\ln x)^{x-1}\right] = (x - 1)\ln(\ln x)$$
$$= \frac{\ln(\ln x)}{(x - 1)^{-1}}$$

$$\lim_{x \to 1^+} \ln y = \lim_{x \to 1^+} \frac{\ln(\ln x)}{(x - 1)^{-1}}$$
$$= \lim_{x \to 1^+} \frac{1/(x \ln x)}{-(x - 1)^{-2}}$$
$$= \lim_{x \to 1^+} \frac{-(x - 1)^2}{x \ln x}$$
$$= \lim_{x \to 1^+} \frac{-2(x - 1)}{1 + \ln x} = 0$$

Because $\lim_{x \to 1^+} \ln y = 0$, $\lim_{x \to 1^+} y = 1$.

(c)



56. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

(a) $\lim_{x \to 0^+} \left[\cos\left(\frac{\pi}{2} - x\right)\right]^x = \lim_{x \to 0^+} [\sin x]^x = 0^0$

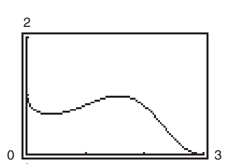
(b) Let $y = (\sin x)^x$

$$\ln y = x \ln(\sin x) = \frac{\ln(\sin x)}{1/x}$$

$$\lim_{x \to 0^+} \frac{\ln(\sin x)}{1/x} = \lim_{x \to 0^+} \frac{\cos x/\sin x}{-1/x^2}$$
$$= \lim_{x \to 0^+} \frac{-x^2 \cos x}{\sin x}$$
$$= \lim_{x \to 0^+} \frac{x}{\sin x}\left(\frac{-x \cos x}{1}\right)$$
$$= 0$$

So, $\lim_{x \to 0^+} \left[\cos\left(\frac{\pi}{2} - x\right)\right]^x = 1$.

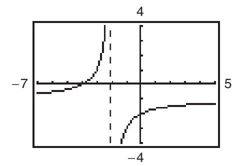
(c)



57. (a) $\lim_{x \to 2^+} \left(\frac{8}{x^2 - 4} - \frac{x}{x - 2}\right) = \infty - \infty$

(b)
$$\lim_{x \to 2^+} \left(\frac{8}{x^2 - 4} - \frac{x}{x - 2}\right) = \lim_{x \to 2^+} \frac{8 - x(x + 2)}{x^2 - 4}$$
$$= \lim_{x \to 2^+} \frac{(2 - x)(4 + x)}{(x + 2)(x - 2)}$$
$$= \lim_{x \to 2^+} \frac{-(x + 4)}{x + 2} = \frac{-3}{2}$$

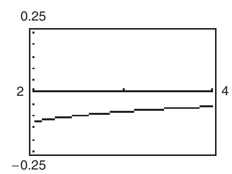
(c)



58. (a) $\lim_{x \to 2^+} \left(\frac{1}{x^2 - 4} - \frac{\sqrt{x - 1}}{x^2 - 4}\right) = \infty - \infty$

(b)
$$\lim_{x \to 2^+} \left(\frac{1}{x^2 - 4} - \frac{\sqrt{x - 1}}{x^2 - 4}\right) = \lim_{x \to 2^+} \frac{1 - \sqrt{x - 1}}{x^2 - 4}$$
$$= \lim_{x \to 2^+} \frac{-1/\left(2\sqrt{x - 1}\right)}{2x}$$
$$= \lim_{x \to 2^+} \frac{-1}{4x\sqrt{x - 1}} = \frac{-1}{8}$$

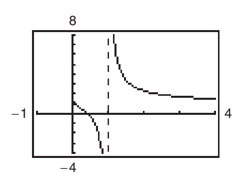
(c)



59. (a) $\lim_{x \to 1^+} \left(\frac{3}{\ln x} - \frac{2}{x - 1}\right) = \infty - \infty$

(b)
$$\lim_{x \to 1^+} \left(\frac{3}{\ln x} - \frac{2}{x - 1}\right) = \lim_{x \to 1^+} \frac{3x - 3 - 2\ln x}{(x - 1)\ln x}$$
$$= \lim_{x \to 1^+} \frac{3 - (2/x)}{\left[(x - 1)/x\right] + \ln x} = \infty$$

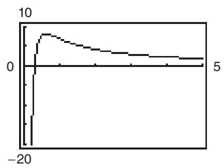
(c)



60. (a) $\lim_{x \to 0^+} \left(\frac{10}{x} - \frac{3}{x^2} \right) = \infty - \infty$

(b) $\lim_{x \to 0^+} \left(\frac{10}{x} - \frac{3}{x^2} \right) = \lim_{x \to 0^+} \left(\frac{10x - 3}{x^2} \right) = -\infty$

(c)



61. (a) $\lim_{x \to \infty} \left(e^x - x \right) = \infty - \infty$

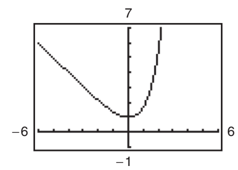
(b) $\lim_{x \to \infty} \left(e^x - x \right) = \lim_{x \to \infty} x \left(\frac{e^x}{x} - 1 \right)$

Now, $\lim_{x \to \infty} x = \infty$ and

$\lim_{x \to \infty} \left(\frac{e^x}{x} - 1 \right) = \lim_{x \to \infty} \frac{e^x}{1} = \infty.$

So, $\lim_{x \to \infty} x \left(\frac{e^x}{x} - 1 \right) = \infty$ and $\lim_{x \to \infty} \left(e^x - x \right) = \infty.$

(c)

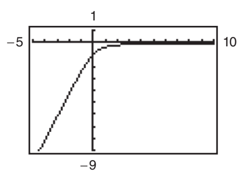


62. (a) $\lim_{x \to \infty} \left(x - \sqrt{x^2 + 1} \right) = \infty - \infty$

(b) $\lim_{x \to \infty} \frac{\left(x - \sqrt{x^2 + 1} \right)}{1} \cdot \frac{x + \sqrt{x^2 + 1}}{x + \sqrt{x^2 + 1}} = \lim_{x \to \infty} \frac{x^2 - \left(x^2 + 1 \right)}{x + \sqrt{x^2 + 1}}$

$= \lim_{x \to \infty} \frac{-1}{x + \sqrt{x^2 + 1}} = 0$

(c)



63. (a) Let $f(x) = x^2 - 25$ and $g(x) = x - 5$.

(b) Let $f(x) = (x - 5)^2$ and $g(x) = x^2 - 25$.

(c) Let $f(x) = x^2 - 25$ and $g(x) = (x - 5)^3$.

(Answers will vary.)

64. Let $f(x) = x + 25$ and $g(x) = x$.

(Answers will vary.)

65. (a) Yes: $\frac{0}{0}$ (b) No: $\frac{0}{-1}$ (c) Yes: $\frac{\infty}{\infty}$

(d) Yes: $\frac{0}{0}$ (e) No: $\frac{-1}{0}$ (f) Yes: $\frac{0}{0}$

66. (a) From the graph, $\lim_{x \to 1^-} f(x) = \infty$.

(b) From the graph, $\lim_{x \to 1^+} f(x) = -\infty$.

(c) From the graph, $\lim_{x \to 1} f(x)$ does not exist.

67.

x	10	10^2	10^4	10^6	10^8	10^{10}
$\frac{(\ln x)^4}{x}$	2.811	4.498	0.720	0.036	0.001	0.000

68.

x	1	5	10	20	30	40	50	100
$\frac{e^x}{x^5}$	2.718	0.047	0.220	151.614	4.40×10^5	2.30×10^9	1.66×10^{13}	2.69×10^{33}

69. $\lim_{x\to\infty} \frac{x^2}{e^{5x}} = \lim_{x\to\infty} \frac{2x}{5e^{5x}} = \lim_{x\to\infty} \frac{2}{25e^{5x}} = 0$

70. $\lim_{x\to\infty} \frac{x^3}{e^{2x}} = \lim_{x\to\infty} \frac{3x^2}{2e^{2x}} = \lim_{x\to\infty} \frac{6x}{4e^{2x}} = \lim_{x\to\infty} \frac{6}{8e^{2x}} = 0$

71.
$$\lim_{x\to\infty} \frac{(\ln x)^3}{x} = \lim_{x\to\infty} \frac{3(\ln x)^2(1/x)}{1} = \lim_{x\to\infty} \frac{3(\ln x)^2}{x} = \lim_{x\to\infty} \frac{6(\ln x)(1/x)}{1} = \lim_{x\to\infty} \frac{6(\ln x)}{x} = \lim_{x\to\infty} \frac{6}{x} = 0$$

72.
$$\lim_{x\to\infty} \frac{(\ln x)^2}{x^3} = \lim_{x\to\infty} \frac{(2\ln x)/x}{3x^2} = \lim_{x\to\infty} \frac{2\ln x}{3x^3} = \lim_{x\to\infty} \frac{2/x}{9x^2} = \lim_{x\to\infty} \frac{2}{9x^3} = 0$$

73.
$$\lim_{x\to\infty} \frac{(\ln x)^n}{x^m} = \lim_{x\to\infty} \frac{n(\ln x)^{n-1}/x}{mx^{m-1}} = \lim_{x\to\infty} \frac{n(\ln x)^{n-1}}{mx^m} = \lim_{x\to\infty} \frac{n(n-1)(\ln x)^{n-2}}{m^2x^m} = \cdots = \lim_{x\to\infty} \frac{n!}{m^nx^m} = 0$$

74.
$$\lim_{x\to\infty} \frac{x^m}{e^{nx}} = \lim_{x\to\infty} \frac{mx^{m-1}}{ne^{nx}} = \lim_{x\to\infty} \frac{m(m-1)x^{m-2}}{n^2e^{nx}} = \cdots = \lim_{x\to\infty} \frac{m!}{n^me^{nx}} = 0$$

75. $y = x^{1/x}, x > 0$

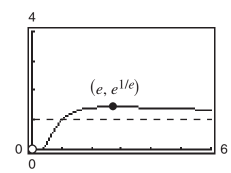
Horizontal asymptote: $y = 1$ (See Exercise 49.)

$$\ln y = \frac{1}{x}\ln x$$
$$\left(\frac{1}{y}\right)\frac{dy}{dx} = \frac{1}{x}\left(\frac{1}{x}\right) + (\ln x)\left(-\frac{1}{x^2}\right)$$
$$\frac{dy}{dx} = x^{1/x}\left(\frac{1}{x^2}\right)(1 - \ln x) = x^{(1/x)-2}(1 - \ln x) = 0$$

Critical number: $x = e$

Intervals:	$(0, e)$	(e, ∞)
Sign of dy/dx:	$+$	$-$
$y = f(x)$:	Increasing	Decreasing

Relative maximum: $(e, e^{1/e})$



76. $y = x^x, x > 0$

$\lim_{x\to\infty} x^x = \infty$ and $\lim_{x\to 0^+} x^x = 1$

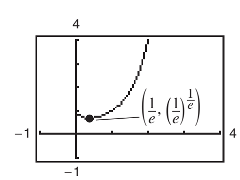
No horizontal asymptotes

$$\ln y = x\ln x$$
$$\left(\frac{1}{y}\right)\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x$$
$$\frac{dy}{dx} = x^x(1 + \ln x) = 0$$

Critical number: $x = e^{-1}$

Intervals:	$(0, e^{-1})$	$(e^{-1}, 0)$
Sign of dy/dx:	$-$	$+$
$y = f(x)$:	Decreasing	Increasing

Relative maximum: $\left(e^{-1}, (e^{-1})^{e^{-1}}\right) = \left(\frac{1}{e}, \left(\frac{1}{e}\right)^{1/e}\right)$



77. $y = 2xe^{-x}$

$$\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$$

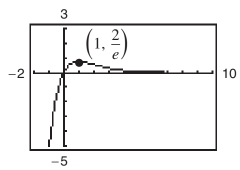
Horizontal asymptote: $y = 0$

$$\frac{dy}{dx} = 2x\left(-e^{-x}\right) + 2e^{-x}$$
$$= 2e^{-x}(1 - x) = 0$$

Critical number: $x = 1$

Intervals:	$(-\infty, 1)$	$(1, \infty)$
Sign of dy/dx:	$+$	$-$
$y = f(x)$:	Increasing	Decreasing

Relative maximum: $\left(1, \dfrac{2}{e}\right)$



78. $y = \dfrac{\ln x}{x}$

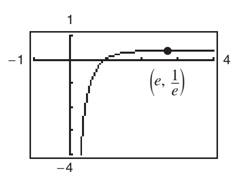
Horizontal asymptote: $y = 0$ (See Example 2.)

$$\frac{dy}{dx} = \frac{x(1/x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2} = 0$$

Critical number: $x = e$

Intervals:	$(0, e)$	(e, ∞)
Sign of dy/dx:	$+$	$-$
$y = f(x)$:	Increasing	Decreasing

Relative maximum: $\left(e, \dfrac{1}{e}\right)$



79. $\displaystyle\lim_{x\to 2} \frac{3x^2 + 4x + 1}{x^2 - x - 2} = \frac{21}{0}$

Limit is not of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

L'Hôpital's Rule does not apply.

80. $\displaystyle\lim_{x\to 0} \frac{e^{2x} - 1}{e^x} = \frac{0}{1} = 0$

Limit is not of the form $0/0$ or ∞/∞.

L'Hôpital's Rule does not apply.

81. $\displaystyle\lim_{x\to\infty} \frac{e^{-x}}{1 + e^{-x}} = \frac{0}{1 + 0} = 0$

Limit is not of the form $0/0$ or ∞/∞.

L'Hôpital's Rule does not apply.

82. $\displaystyle\lim_{x\to\infty} x \cos\frac{1}{x} = \infty(1) = \infty$

Limit is not of the form $0/0$ or ∞/∞.

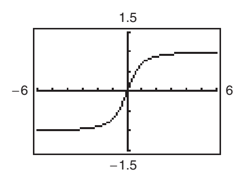
L'Hôpital's Rule does not apply.

83. (a) Applying L'Hôpital's Rule twice results in the original limit, so L'Hôpital's Rule fails:

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x\to\infty} \frac{1}{x/\sqrt{x^2 + 1}}$$
$$= \lim_{x\to\infty} \frac{\sqrt{x^2 + 1}}{x}$$
$$= \lim_{x\to\infty} \frac{x/\sqrt{x^2 + 1}}{1}$$
$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2 + 1}}$$

(b) $$\lim_{x\to\infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x\to\infty} \frac{x/x}{\sqrt{x^2 + 1}/x}$$
$$= \lim_{x\to\infty} \frac{1}{\sqrt{1 + 1/x^2}} = \frac{1}{\sqrt{1 + 0}} = 1$$

(c)



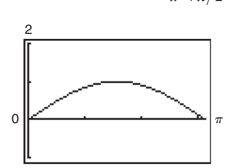
84. (a) Applying L'Hôpital's Rule twice results in the original limit, so L'Hôpital's Rule fails:

$\displaystyle\lim_{x\to\pi/2^-} \frac{\tan x}{\sec x}$ is indeterminant: $\dfrac{\infty}{\infty}$

$$\lim_{x\to\pi/2^-} \frac{\tan x}{\sec x} = \lim_{x\to\pi/2^-} \frac{\sec^2 x}{\sec x \tan x}$$
$$= \lim_{x\to\pi/2^-} \frac{\sec x}{\tan x} \quad \left(\frac{\infty}{\infty}\right)$$
$$= \lim_{x\to\pi/2^-} \frac{\sec x \tan x}{\sec^2 x}$$
$$= \lim_{x\to\pi/2^-} \frac{\tan x}{\sec x}$$

(b) $$\lim_{x\to\pi/2^-} \frac{\tan x}{\sec x} = \lim_{x\to\pi/2^-} \frac{\sin x}{\cos x}(\cos x)$$
$$= \lim_{x\to\pi/2^-} \sin x = 1$$

(c)



85. $f(x) = \sin(3x),\ g(x) = \sin(4x)$

$f'(x) = 3\cos(3x),\ g'(x) = 4\cos(4x)$

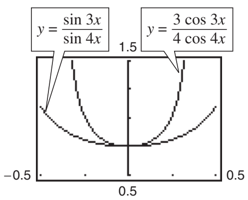
$$y_1 = \frac{f(x)}{g(x)} = \frac{\sin 3x}{\sin 4x},$$

$$y_2 = \frac{f'(x)}{g'(x)} = \frac{3\cos 3x}{4\cos 4x}$$

As $x \to 0,\ y_1 \to 0.75$ and $y_2 \to 0.75$

By L'Hôpital's Rule,

$$\lim_{x\to 0} \frac{\sin 3x}{\sin 4x} = \lim_{x\to 0} \frac{3\cos 3x}{4\cos 4x} = \frac{3}{4}$$



86. $f(x) = e^{3x} - 1,\ g(x) = x$

$f'(x) = 3e^{3x},\ g'(x) = 1$

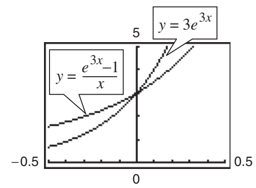
$$y_1 = \frac{f(x)}{g(x)} = \frac{e^{3x} - 1}{x},$$

$$y_2 = \frac{f'(x)}{g'(x)} = 3e^{3x}$$

As $x \to 0,\ y_1 \to 3$ and $y_2 \to 3$

By L'Hôpital's Rule,

$$\lim_{x\to 0} \frac{e^{3x} - 1}{x} = \lim_{x\to 0} \frac{3e^{3x}}{1} = 3$$



87. $$I = \frac{V\left(1 - e^{-Rt/L}\right)}{R}$$

$$\lim_{R\to 0^+} I = \frac{0}{0}$$

$$\lim_{R\to\infty} V\left(\frac{1 - e^{-Rt/L}}{R}\right) = \lim_{R\to\infty} V\left[\frac{(t/L)e^{-Rt/L}}{1}\right] = V\left(\frac{t}{L}\right) = \frac{Vt}{L}$$

88. $$\lim_{k\to 0} \frac{32\left(1 - e^{-kt} + \dfrac{v_0 k e^{-kt}}{32}\right)}{k} = \lim_{k\to 0} \frac{32\left(1 - e^{-kt}\right)}{k} + \lim_{k\to 0}\left(v_0 e^{-kt}\right) = \lim_{k\to 0} \frac{32\left(0 + te^{-kt}\right)}{1} + \lim_{k\to 0}\left(\frac{v_0}{e^{kt}}\right) = 32t + v_0$$

89. Let N be a fixed value for n. Then

$$\lim_{x\to\infty} \frac{x^{N-1}}{e^x} = \lim_{x\to\infty} \frac{(N-1)x^{N-2}}{e^x} = \lim_{x\to\infty} \frac{(N-1)(N-2)x^{N-3}}{e^x} = \cdots = \lim_{x\to\infty}\left[\frac{(N-1)!}{e^x}\right] = 0. \quad \text{(See Exercise 74.)}$$

90. $$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$\ln A = \ln P + nt \ln\left(1 + \frac{r}{n}\right) = \ln P + \frac{\ln\left(1 + \dfrac{r}{n}\right)}{\dfrac{1}{nt}}$$

$$\lim_{n\to\infty}\left[\frac{\ln\left(1 + \dfrac{r}{n}\right)}{\dfrac{1}{nt}}\right] = \lim_{n\to\infty}\left[\frac{-\dfrac{r}{n^2}\left(\dfrac{1}{1 + (r/n)}\right)}{-\left(\dfrac{1}{n^2 t}\right)}\right] = \lim_{n\to\infty}\left[rt\left(\frac{1}{1 + \dfrac{r}{n}}\right)\right] = rt$$

Because $\lim_{n\to\infty} \ln A = \ln P + rt$, you have $\lim_{n\to\infty} A = e^{(\ln P + rt)} = e^{\ln P}e^{rt} = Pe^{rt}$. Alternatively,

$$\lim_{n\to\infty} A = \lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{nt} = \lim_{n\to\infty} P\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} = Pe^{rt}.$$

91. $f(x) = x^3, g(x) = x^2 + 1, [0, 1]$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{f(1) - f(0)}{g(1) - g(0)} = \frac{3c^2}{2c}$$

$$\frac{1}{1} = \frac{3c}{2}$$

$$c = \frac{2}{3}$$

92. $f(x) = \frac{1}{x}, g(x) = x^2 - 4, [1, 2]$

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)}$$

$$\frac{-1/2}{3} = \frac{-1/c^2}{2c}$$

$$-\frac{1}{6} = -\frac{1}{2c^3}$$

$$2c^3 = 6$$

$$c = \sqrt[3]{3}$$

93. $f(x) = \sin x, g(x) = \cos x, \left[0, \frac{\pi}{2}\right]$

$$\frac{f(\pi/2) - f(0)}{g(\pi/2) - g(0)} = \frac{f'(c)}{g'(c)}$$

$$\frac{1}{-1} = \frac{\cos c}{-\sin c}$$

$$-1 = -\cot c$$

$$c = \frac{\pi}{4}$$

94. $f(x) = \ln x, g(x) = x^3, [1, 4]$

$$\frac{f(4) - f(1)}{g(4) - g(1)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\ln 4}{63} = \frac{1/c}{3c^2} = \frac{1}{3c^3}$$

$$3c^3 \ln 4 = 63$$

$$c^3 = \frac{21}{\ln 4}$$

$$c = \sqrt[3]{\frac{21}{\ln 4}} \approx 2.474$$

95. False. A limit of the form $\frac{\infty}{0}$ is equal to ∞ (or $-\infty$).

96. False. A limit of the form $\infty \cdot \infty$ equals ∞.

97. True

98. False. L'Hôpital's Rule does not apply because

$\lim_{x\to 0} (x^2 + x + 1) \neq 0.$

$$\lim_{x\to 0^+} \frac{x^2 + x + 1}{x} = \lim_{x\to 0^+} \left(x + 1 + \frac{1}{x}\right) = 1 + \infty = \infty$$

99. True

100. False. Let $f(x) = x$ and $g(x) = x + 1$. Then

$$\lim_{x\to\infty} \frac{x}{x+1} = 1, \text{ but } \lim_{x\to\infty} [x - (x + 1)] = -1.$$

101. Area of triangle: $\frac{1}{2}(2x)(1 - \cos x) = x - x\cos x$

Shaded area: Area of rectangle − Area under curve

$$2x(1 - \cos x) - 2\int_0^x (1 - \cos t)\,dt = 2x(1 - \cos x) - 2[t - \sin t]_0^x$$

$$= 2x(1 - \cos x) - 2(x - \sin x)$$

$$= 2\sin x - 2x\cos x$$

Ratio: $$\lim_{x\to 0} \frac{x - x\cos x}{2\sin x - 2x\cos x} = \lim_{x\to 0} \frac{1 + x\sin x - \cos x}{2\cos x + 2x\sin x - 2\cos x}$$

$$= \lim_{x\to 0} \frac{1 + x\sin x - \cos x}{2x\sin x}$$

$$= \lim_{x\to 0} \frac{x\cos x + \sin x + \sin x}{2x\cos x + 2\sin x}$$

$$= \lim_{x\to 0} \frac{x\cos x + 2\sin x}{2x\cos x + 2\sin x} \cdot \frac{1/\cos x}{1/\cos x} = \lim_{x\to 0} \frac{x + 2\tan x}{2x + 2\tan x} = \lim_{x\to 0} \frac{1 + 2\sec^2 x}{2 + 2\sec^2 x} = \frac{3}{4}$$

102. (a) $\sin\theta = BD$

$\cos\theta = DO \Rightarrow AD = 1 - \cos\theta$

$$\text{Area } \Delta ABD = \frac{1}{2}bh = \frac{1}{2}(1-\cos\theta)\sin\theta = \frac{1}{2}\sin\theta - \frac{1}{2}\sin\theta\cos\theta$$

(b) Area of sector: $\frac{1}{2}\theta$

$$\text{Shaded area: } \frac{1}{2}\theta - \text{Area } \Delta OBD = \frac{1}{2}\theta - \frac{1}{2}(\cos\theta)(\sin\theta) = \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta$$

(c) $$R = \frac{(1/2)\sin\theta - (1/2)\sin\theta\cos\theta}{(1/2)\theta - (1/2)\sin\theta\cos\theta} = \frac{\sin\theta - \sin\theta\cos\theta}{\theta - \sin\theta\cos\theta}$$

(d) $$\lim_{\theta\to 0} R = \lim_{\theta\to 0}\frac{\sin\theta - (1/2)\sin 2\theta}{\theta - (1/2)\sin 2\theta} = \lim_{\theta\to 0}\frac{\cos\theta - \cos 2\theta}{1-\cos 2\theta} = \lim_{\theta\to 0}\frac{-\sin\theta + 2\sin 2\theta}{2\sin 2\theta} = \lim_{\theta\to 0}\frac{-\cos\theta + 4\cos 2\theta}{4\cos 2\theta} = \frac{3}{4}$$

103. $$\lim_{x\to 0}\frac{4x - 2\sin 2x}{2x^3} = \lim_{x\to 0}\frac{4 - 4\cos 2x}{6x^2} = \lim_{x\to 0}\frac{8\sin 2x}{12x} = \lim_{x\to 0}\frac{16\cos 2x}{12} = \frac{16}{12} = \frac{4}{3}$$

Let $c = \frac{4}{3}$.

104. Let $y = (e^x + x)^{1/x}$.

$$\ln y = \frac{1}{x}\ln(e^x + x) = \frac{\ln(e^x + x)}{x}$$

$$\lim_{x\to 0}\frac{\ln(e^x + x)}{x} = \lim_{x\to 0}\frac{e^x + 1}{e^x + x} = \frac{2}{1} = 2$$

So, $\lim_{x\to 0}(e^x + x)^{1/x} = e^2$.

Let $c = e^2 \approx 7.389$.

105. $$\lim_{x\to 0}\frac{a - \cos bx}{x^2} = 2$$

Near $x = 0$, $\cos bx \approx 1$ and $x^2 \approx 0 \Rightarrow a = 1$.

Using L'Hôpital's Rule,

$$\lim_{x\to 0}\frac{1-\cos bx}{x^2} = \lim_{x\to 0}\frac{b\sin bx}{2x} = \lim_{x\to 0}\frac{b^2\cos bx}{2} = 2.$$

So, $b^2 = 4$ and $b = \pm 2$.

Answer: $a = 1, b = \pm 2$

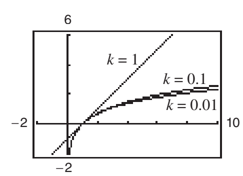
106. $$f(x) = \frac{x^k - 1}{k}$$

$k = 1,\quad f(x) = x - 1$

$$k = 0.1,\quad f(x) = \frac{x^{0.1} - 1}{0.1} = 10(x^{0.1} - 1)$$

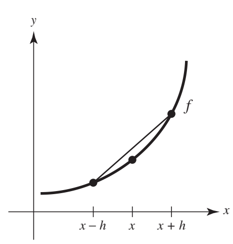
$$k = 0.01,\quad f(x) = \frac{x^{0.01} - 1}{0.01} = 100(x^{0.01} - 1)$$

$$\lim_{k\to 0^+}\frac{x^k - 1}{k} = \lim_{k\to 0^+}\frac{x^k(\ln x)}{1} = \ln x$$



107. (a) $$\lim_{h\to 0}\frac{f(x+h) - f(x-h)}{2h} = \lim_{h\to 0}\frac{f'(x+h)(1) - f'(x-h)(-1)}{2} = \lim_{h\to 0}\left[\frac{f'(x+h) + f'(x-h)}{2}\right] = \frac{f'(x) + f'(x)}{2} = f'(x)$$

(b)



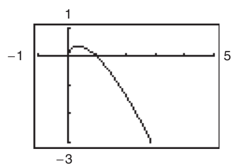
Graphically, the slope of the line joining $(x - h, f(x-h))$ and $(x + h, f(x+h))$ is approximately $f'(x)$.

So, $$\lim_{h\to 0}\frac{f(x+h) - f(x-h)}{2h} = f'(x).$$

108. $$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} = \lim_{h\to 0}\frac{f'(x+h)(1)+f'(x-h)(-1)}{2h}$$
$$= \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}$$
$$= \lim_{h\to 0}\frac{f''(x+h)(1)-f''(x-h)(-1)}{2}$$
$$= \lim_{h\to 0}\frac{f''(x+h)+f''(x-h)}{2}$$
$$= \frac{f''(x)+f''(x)}{2} = f''(x)$$

109. (a) $\lim_{x\to 0^+}(-x\ln x)$ is the form $0\cdot\infty$.

(b) $\lim_{x\to 0^+}\frac{-\ln x}{1/x} = \lim_{x\to 0^+}\frac{-1/x}{-1/x^2} = \lim_{x\to 0^+}(x) = 0$



110. $\lim_{x\to a} f(x)^{g(x)}$

$y = f(x)^{g(x)}$

$\ln y = g(x)\ln f(x)$

$\lim_{x\to a} g(x)\ln f(x) = (\infty)(-\infty) = -\infty$

As $x \to a$, $\ln y \Rightarrow -\infty$, and therefore $y = 0$. So,

$\lim_{x\to a} f(x)^{g(x)} = 0.$

111. $\lim_{x\to a} f(x)^{g(x)}$

$y = f(x)^{g(x)}$

$\ln y = g(x)\ln f(x)$

$\lim_{x\to a} g(x)\ln f(x) = (-\infty)(-\infty) = \infty$

As $x \to a$, $\ln y \Rightarrow \infty$, and therefore $y = \infty$. So,

$\lim_{x\to a} f(x)^{g(x)} = \infty.$

112. Method 1: Direct computation

$$\lim_{x\to\infty}\frac{\ln x^m}{\ln x^n} = \lim_{x\to\infty}\frac{m\ln x}{n\ln x} = \lim_{x\to\infty}\frac{m}{n} = \frac{m}{n}$$

Method 2: L' Hôpital's Rule for ∞/∞

$$\lim_{x\to\infty}\frac{\ln x^m}{\ln x^n} = \lim_{x\to\infty}\frac{(1/x^m)mx^{m-1}}{(1/x^n)nx^{n-1}}$$
$$= \lim_{x\to\infty}\frac{m/x}{n/x} = \lim_{x\to\infty}\frac{m}{n} = \frac{m}{n}$$

113. (a) $\lim_{x\to 0^+} x^{(\ln 2)/(1+\ln x)}$ is of form 0^0.

Let $y = x^{(\ln 2)/(1+\ln x)}$

$\ln y = \frac{\ln 2}{1+\ln x}\ln x$

$\lim_{x\to 0^+}\ln y = \frac{\ln 2(1/x)}{1/x} = \ln 2.$

So, $\lim_{x\to 0^+} x^{(\ln 2)/(1+\ln x)} = 2.$

(b) $\lim_{x\to\infty} x^{(\ln 2)/(1+\ln x)}$ is of form ∞^0.

Let $y = x^{(\ln 2)/(1+\ln x)}$

$\ln y = \frac{\ln 2}{1+\ln x}\ln x$

$\lim_{x\to\infty}\ln y = \frac{\ln 2(1/x)}{1/x} = \ln 2.$

So, $\lim_{x\to\infty} x^{(\ln 2)/(1+\ln x)} = 2.$

(c) $\lim_{x\to 0}(x+1)^{(\ln 2)/(x)}$ is of form 1^∞.

Let $y = (x+1)^{(\ln 2)/(x)}$

$\ln y = \frac{\ln 2}{x}\ln(x+1)$

$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{(\ln 2)1/(x+1)}{1} = \ln 2.$

So, $\lim_{x\to 0}(x+1)^{(\ln 2)/(x)} = 2.$

114. $$\lim_{x \to a} \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$$

$$= \lim_{x \to a} \frac{\frac{1}{2}\left(2a^3x - x^4\right)^{-1/2}\left(2a^3 - 4x^3\right) - \frac{a}{3}\left(a^2x\right)^{-2/3}a^2}{-\frac{1}{4}\left(ax^3\right)^{-3/4}}$$

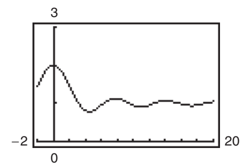
$$= \frac{\frac{1}{2}\left(a^4\right)^{-1/2}\left(-2a^3\right) - \frac{a^3}{3}\left(a^3\right)^{-2/3}}{-\frac{1}{4}\left(ax^3\right)^{-3/4}\left(3ax^2\right)}$$

$$= \frac{a + \frac{a}{3}}{\frac{1}{4}\left(a^{-3}\right)\left(3a^3\right)}$$

$$= \frac{\frac{4}{3}a}{\frac{3}{4}} = \frac{16}{9}a$$

115. (a) $h(x) = \dfrac{x + \sin x}{x}$

$\lim_{x \to \infty} h(x) = 1$



(b) $h(x) = \dfrac{x + \sin x}{x} = \dfrac{x}{x} + \dfrac{\sin x}{x} = 1 + \dfrac{\sin x}{x}, x > 0$

So, $\lim_{x \to \infty} h(x) = \lim_{x \to \infty}\left[1 + \dfrac{\sin x}{x}\right] = 1 + 0 = 1.$

(c) No. $h(x)$ is not an indeterminate form.

116. (a) $\lim_{x \to \infty} \dfrac{f(x)}{g(x)} = \lim_{x \to \infty} \dfrac{x + x\sin x}{x^2 - 4} = \lim_{x \to \infty} \dfrac{1 + \sin x}{x - 4/x} = 0$

(Because $|1 + \sin x| \le 1$ and $x \to \infty$.)

(b) $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} x(1 + \sin x) = \infty$

$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \left(x^2 - 4\right) = \infty$

(c) $\lim_{x \to \infty} \dfrac{f'(x)}{g'(x)} = \lim_{x \to \infty} \dfrac{1 + \sin x + x\cos x}{2x}$ undefined

(d) No. If $\lim_{x \to \infty} \dfrac{f'(x)}{g'(x)}$ does not exist, then you cannot assume anything about $\lim_{x \to \infty} \dfrac{f(x)}{g(x)}$.

117. Let $f(x) = \left[\dfrac{1}{x} \cdot \dfrac{a^x - 1}{a - 1}\right]^{1/x}$.

For $a > 1$ and $x > 0$,

$$\ln f(x) = \frac{1}{x}\left[\ln\frac{1}{x} + \ln\left(a^x - 1\right) - \ln(a - 1)\right] = -\frac{\ln x}{x} + \frac{\ln\left(a^x - 1\right)}{x} - \frac{\ln(a - 1)}{x}.$$

As $x \to \infty$, $\dfrac{\ln x}{x} \to 0$, $\dfrac{\ln(a - 1)}{x} \to 0$, and $\dfrac{\ln\left(a^x - 1\right)}{x} = \dfrac{\ln\left[\left(1 - a^{-x}\right)a^x\right]}{x} = \dfrac{\ln\left(1 - a^{-x}\right)}{x} + \ln a \to \ln a.$

So, $\ln f(x) \to \ln a$.

For $0 < a < 1$ and $x > 0$,

$$\ln f(x) = \frac{-\ln x}{x} + \frac{\ln\left(1 - a^x\right)}{x} - \frac{\ln(1 - a)}{x} \to 0 \text{ as } x \to \infty.$$

Combining these results, $\lim_{x \to \infty} f(x) = \begin{cases} a & \text{if } a > 1 \\ 1 & \text{if } 0 < a < 1 \end{cases}$.

Section 5.7 Inverse Trigonometric Functions: Differentiation

1. arccos x is the angle θ whose cosine is x, where $0 \le \theta \le \pi$.

2. A restricted domain is a subset of the entire domain of a function. One must restrict the domain of the six trigonometric functions in order for them to be one-to-one on that restricted domain.

3. The arccotangent function has a range of $0 < y < \pi$.

4. The missing value is $-u' = -3x^2$.

5. $y = \arccos x$

$\left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)$ because $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$\left(\frac{1}{2}, \frac{\pi}{3}\right)$ because $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$\left(\frac{\sqrt{3}}{2}, \frac{\pi}{6}\right)$ because $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

6. $y = \arctan x$

$\left(1, \frac{\pi}{4}\right)$ because $\tan\left(\frac{\pi}{4}\right) = 1$

$\left(-\frac{\sqrt{3}}{3}, -\frac{\pi}{6}\right)$ because $\tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$

$\left(-\sqrt{3}, -\frac{\pi}{3}\right)$ because $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$

7. $y = \arcsin \frac{1}{2} \Rightarrow \sin y = \frac{1}{2}$

In $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $y = \frac{\pi}{6}$. So, $\arcsin \frac{1}{2} = \frac{\pi}{6}$.

8. $y = \arcsin 0 \Rightarrow \sin y = 0$

In $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $y = 0$. So, $\arcsin 0 = 0$.

9. $y = \arccos \frac{1}{2} \Rightarrow \cos y = \frac{1}{2}$

In $[0, \pi]$, $y = \frac{\pi}{3}$. So, $\arccos \frac{1}{2} = \frac{\pi}{3}$.

10. $y = \arccos(-1) \Rightarrow \cos y = -1$

In $[0, \pi]$, $y = \pi$. So, $\arccos(-1) = \pi$.

11. $y = \arctan \frac{\sqrt{3}}{3} \Rightarrow \tan y = \frac{\sqrt{3}}{3}$

In $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $y = \frac{\pi}{6}$. So, $\arctan \frac{\sqrt{3}}{3} = \frac{\pi}{6}$.

12. $y = \operatorname{arccot}\left(-\sqrt{3}\right) \Rightarrow \cot y = -\sqrt{3}$

In $(0, \pi)$, $y = \frac{5\pi}{6}$. So, $\operatorname{arccot}\left(-\sqrt{3}\right) = \frac{5\pi}{6}$.

13. $y = \operatorname{arccsc}\left(-\sqrt{2}\right) \Rightarrow \csc y = -\sqrt{2}$

In $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $y = -\frac{\pi}{4}$. So, $\operatorname{arccsc}\left(-\sqrt{2}\right) = -\frac{\pi}{4}$.

14. $y = \operatorname{arcsec} 2 \Rightarrow \sec y = 2$

In $[0, \pi]$, $y = \frac{\pi}{3}$. So, $\operatorname{arcsec} 2 = \frac{\pi}{3}$.

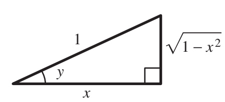
15. $\arccos(0.051) \approx 1.52$

16. $\arcsin(-0.39) \approx -0.40$

17. $\operatorname{arcsec}(1.269) = \arccos\left(\frac{1}{1.269}\right) \approx 0.66$

18. $\operatorname{arccsc}(-4.487) \approx -0.22$

In Exercises 19–24, use the triangle.



19. $y = \arccos x$

$\cos y = x$

20. $\sin y = \sqrt{1 - x^2}$

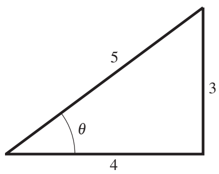
21. $\tan y = \frac{\sqrt{1 - x^2}}{x}$

22. $\cot y = \frac{x}{\sqrt{1 - x^2}}$

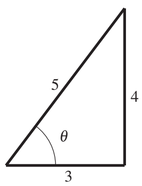
23. $\sec y = \frac{1}{x}$

24. $\csc y = \frac{1}{\sqrt{1 - x^2}}$

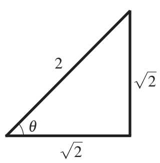
25. (a) $\sin\left(\arctan \frac{3}{4}\right) = \frac{3}{5}$



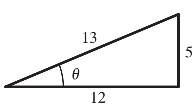
(b) $\sec\left(\arcsin \frac{4}{5}\right) = \frac{5}{3}$



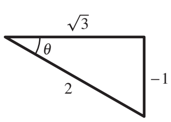
26. (a) $\tan\left(\arccos \dfrac{\sqrt{2}}{2}\right) = \tan\left(\dfrac{\pi}{4}\right) = 1$



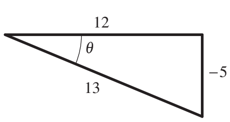
(b) $\cos\left(\arcsin \dfrac{5}{13}\right) = \dfrac{12}{13}$



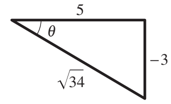
27. (a) $\cot\left[\arcsin\left(-\dfrac{1}{2}\right)\right] = \cot\left(-\dfrac{\pi}{6}\right) = -\sqrt{3}$



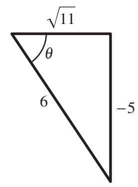
(b) $\csc\left[\arctan\left(-\dfrac{5}{12}\right)\right] = -\dfrac{13}{5}$



28. (a) $\sec\left[\arctan\left(-\dfrac{3}{5}\right)\right] = \dfrac{\sqrt{34}}{5}$



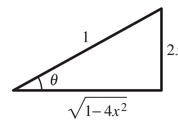
(b) $\tan\left[\arcsin\left(-\dfrac{5}{6}\right)\right] = -\dfrac{5\sqrt{11}}{11}$



29. $y = \cos(\arcsin 2x)$

$\theta = \arcsin 2x$

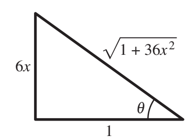
$y = \cos\theta = \sqrt{1 - 4x^2}$



30. $y = \sec(\arctan 6x)$

$\theta = \arctan 6x$

$y = \sec\theta = \sqrt{1 + 36x^2}$

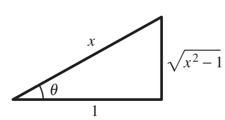


31. $y = \sin(\operatorname{arcsec} x)$

$\theta = \operatorname{arcsec} x,\ 0 \le \theta \le \pi,\ \theta \ne \dfrac{\pi}{2}$

$y = \sin\theta = \dfrac{\sqrt{x^2 - 1}}{|x|}$

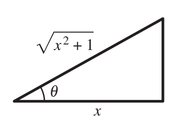
The absolute value bars on x are necessary because of the restriction $0 \le \theta \le \pi, \theta \ne \pi/2$, and $\sin\theta$ for this domain must always be nonnegative.



32. $y = \cos(\operatorname{arccot} x)$

$\theta = \operatorname{arccot} x$

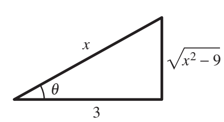
$y = \cos\theta = \dfrac{x}{\sqrt{x^2 + 1}}$



33. $y = \tan\left(\operatorname{arcsec} \dfrac{x}{3}\right)$

$\theta = \operatorname{arcsec} \dfrac{x}{3}$

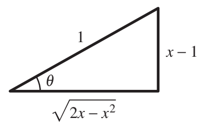
$y = \tan\theta = \dfrac{\sqrt{x^2 - 9}}{3}$



34. $y = \sec\left[\arcsin(x - 1)\right]$

$\theta = \arcsin(x - 1)$

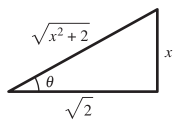
$y = \sec\theta = \dfrac{1}{\sqrt{2x - x^2}}$



35. $y = \csc\left(\arctan\dfrac{x}{\sqrt{2}}\right)$

$\theta = \arctan\dfrac{x}{\sqrt{2}}$

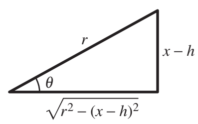
$y = \csc\theta = \dfrac{\sqrt{x^2 + 2}}{x}$



36. $y = \cos\left(\arcsin\dfrac{x - h}{r}\right)$

$\theta = \arcsin\dfrac{x - h}{r}$

$y = \cos\theta = \dfrac{\sqrt{r^2 - (x - h)^2}}{r}$



37. $\arcsin(3x - \pi) = \frac{1}{2}$

$3x - \pi = \sin\left(\frac{1}{2}\right)$

$x = \frac{1}{3}\left[\sin\left(\frac{1}{2}\right) + \pi\right] \approx 1.207$

38. $\arctan(2x - 5) = -1$

$2x - 5 = \tan(-1)$

$x = \frac{1}{2}\left(\tan(-1) + 5\right) \approx 1.721$

39. $\arcsin\sqrt{2x} = \arccos\sqrt{x}$

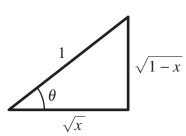
$\sqrt{2x} = \sin\left(\arccos\sqrt{x}\right)$

$\sqrt{2x} = \sqrt{1 - x}, \quad 0 \le x \le 1$

$2x = 1 - x$

$3x = 1$

$x = \frac{1}{3}$



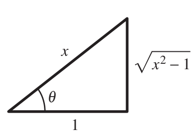
40. $\arccos x = \operatorname{arcsec} x$

$x = \cos(\operatorname{arcsec} x)$

$x = \dfrac{1}{x}$

$x^2 = 1$

$x = \pm 1$



41. $f(x) = \arcsin(x - 1)$

$f'(x) = \dfrac{1}{\sqrt{1 - (x - 1)^2}} = \dfrac{1}{\sqrt{2x - x^2}}$

42. $f(t) = \operatorname{arccsc}\left(t^2\right)$

$f'(t) = \dfrac{2t}{\left|-t^2\right|\sqrt{\left[(-t)^2\right]^2 - 1}}$

$= \dfrac{2}{t\sqrt{t^4 - 1}}$

43. $g(x) = 3\arccos\dfrac{x}{2}$

$g'(x) = \dfrac{-3(1/2)}{\sqrt{1 - \left(x^2/4\right)}} = \dfrac{-3}{\sqrt{4 - x^2}}$

44. $f(x) = \operatorname{arcsec} 2x$

$f'(x) = \dfrac{2}{|2x|\sqrt{4x^2 - 1}} = \dfrac{1}{|x|\sqrt{4x^2 - 1}}$

45. $f(x) = \arctan\left(e^x\right)$

$f'(x) = \dfrac{1}{1 + \left(e^x\right)^2}e^x = \dfrac{e^x}{1 + e^{2x}}$

46. $f(x) = \operatorname{arccot}\sqrt{x}$

$f'(x) = \dfrac{-1/2\sqrt{x}}{1 + x}$

$= -\dfrac{1}{2\sqrt{x}(1 + x)}$

47. $g(x) = \dfrac{\arcsin 3x}{x}$

$g'(x) = \dfrac{x\left(3/\sqrt{1 - 9x^2}\right) - \arcsin 3x}{x^2}$

$= \dfrac{3x - \sqrt{1 - 9x^2}\arcsin 3x}{x^2\sqrt{1 - 9x^2}}$

48. $h(x) = x^2\arctan 5x$

$h'(x) = 2x\arctan 5x + x^2\dfrac{1}{1 + (5x)^2}(5)$

$= 2x\arctan 5x + \dfrac{5x^2}{1 + 25x^2}$

49. $h(t) = \sin(\arccos t) = \sqrt{1 - t^2}$

$h'(t) = \dfrac{1}{2}\left(1 - t^2\right)^{-1/2}(-2t)$

$= \dfrac{-t}{\sqrt{1 - t^2}}$

50. $f(x) = \arcsin x + \arccos x = \dfrac{\pi}{2}$

$f'(x) = 0$

51. $y = 2x \arccos x - 2\sqrt{1 - x^2}$

$$y' = 2 \arccos x - 2x \frac{1}{\sqrt{1 - x^2}} - 2\left(\frac{1}{2}\right)(1 - x^2)^{-1/2}(-2x)$$

$$= 2 \arccos x - \frac{2x}{\sqrt{1 - x^2}} + \frac{2x}{\sqrt{1 - x^2}} = 2 \arccos x$$

52. $y = x \arctan 2x - \dfrac{1}{4} \ln(1 + 4x^2)$

$$\frac{dy}{dx} = \frac{2x}{1 + 4x^2} + \arctan 2x - \frac{1}{4}\left(\frac{8x}{1 + 4x^2}\right) = \arctan 2x$$

53. $y = \dfrac{1}{2}\left(\dfrac{1}{2} \ln \dfrac{x + 1}{x - 1} + \arctan x\right)$

$$= \frac{1}{4}\left[\ln(x + 1) - \ln(x - 1)\right] + \frac{1}{2} \arctan x$$

$$\frac{dy}{dx} = \frac{1}{4}\left(\frac{1}{x + 1} - \frac{1}{x - 1}\right) + \frac{1/2}{1 + x^2}$$

$$= \frac{1}{1 - x^4}$$

54. $y = \dfrac{1}{2}\left[x\sqrt{4 - x^2} + 4 \arcsin\left(\dfrac{x}{2}\right)\right]$

$$y' = \frac{1}{2}\left[x\frac{1}{2}(4 - x^2)^{-1/2}(-2x) + \sqrt{4 - x^2} + 2\frac{1}{\sqrt{1 - (x/2)^2}}\right] = \frac{1}{2}\left[\frac{-x^2}{\sqrt{4 - x^2}} + \sqrt{4 - x^2} + \frac{4}{\sqrt{4 - x^2}}\right] = \sqrt{4 - x^2}$$

55. $y = 8 \arcsin \dfrac{x}{4} - \dfrac{x\sqrt{16 - x^2}}{2}$

$$y' = 2\frac{1}{\sqrt{1 - (x/4)^2}} - \frac{\sqrt{16 - x^2}}{2} - \frac{x}{4}(16 - x^2)^{-1/2}(-2x)$$

$$= \frac{8}{\sqrt{16 - x^2}} - \frac{\sqrt{16 - x^2}}{2} + \frac{x^2}{2\sqrt{16 - x^2}} = \frac{16 - (16 - x^2) + x^2}{2\sqrt{16 - x^2}} = \frac{x^2}{\sqrt{16 - x^2}}$$

56. $y = \arctan x + \dfrac{x}{1 + x^2}$

$$y' = \frac{1}{1 + x^2} + \frac{(1 + x^2) - x(2x)}{(1 + x^2)^2} = \frac{(1 + x^2) + (1 - x^2)}{(1 + x^2)^2} = \frac{2}{(1 + x^2)^2}$$

57. $y = 2 \arcsin x, \quad \left(\dfrac{1}{2}, \dfrac{\pi}{3}\right)$

$$y' = \frac{2}{\sqrt{1 - x^2}}$$

At $\left(\dfrac{1}{2}, \dfrac{\pi}{3}\right)$, $y' = \dfrac{2}{\sqrt{1 - (1/4)}} = \dfrac{4}{\sqrt{3}}$.

Tangent line: $y - \dfrac{\pi}{3} = \dfrac{4}{\sqrt{3}}\left(x - \dfrac{1}{2}\right)$

$$y = \frac{4}{\sqrt{3}}x + \frac{\pi}{3} - \frac{2}{\sqrt{3}}$$

$$y = \frac{4\sqrt{3}}{3}x + \frac{\pi}{3} - \frac{2\sqrt{3}}{3}$$

58. $y = -\dfrac{1}{4} \arccos x, \ \left(-\dfrac{1}{2}, -\dfrac{\pi}{6}\right)$

$$y' = \frac{1}{4\sqrt{1 - x^2}}$$

At $\left(-\dfrac{1}{2}, -\dfrac{\pi}{6}\right)$, $y' = \dfrac{1}{4\sqrt{1 - 1/4}} = \dfrac{1}{2\sqrt{3}}$.

Tangent line: $y + \dfrac{\pi}{6} = \dfrac{1}{2\sqrt{3}}\left(x + \dfrac{1}{2}\right)$

$$y = \frac{1}{2\sqrt{3}}x + \frac{1}{4\sqrt{3}} - \frac{\pi}{6}$$

59. $y = \arctan\left(\frac{x}{2}\right), \quad \left(2, \frac{\pi}{4}\right)$

$$y' = \frac{1}{1 + (x^2/4)}\left(\frac{1}{2}\right) = \frac{2}{4 + x^2}$$

At $\left(2, \frac{\pi}{4}\right)$, $y' = \frac{2}{4 + 4} = \frac{1}{4}$.

Tangent line: $y - \frac{\pi}{4} = \frac{1}{4}(x - 2)$

$$y = \frac{1}{4}x + \frac{\pi}{4} - \frac{1}{2}$$

60. $y = \operatorname{arcsec}(4x), \quad \left(\frac{\sqrt{2}}{4}, \frac{\pi}{4}\right)$

$$y' = \frac{4}{|4x|\sqrt{16x^2 - 1}} = \frac{1}{x\sqrt{16x^2 - 1}} \text{ for } x > 0$$

At $\left(\frac{\sqrt{2}}{4}, \frac{\pi}{4}\right)$, $y' = \frac{1}{(\sqrt{2}/4)\sqrt{2 - 1}} = 2\sqrt{2}$.

Tangent line: $y - \frac{\pi}{4} = 2\sqrt{2}\left(x - \frac{\sqrt{2}}{4}\right)$

$$y = 2\sqrt{2}x + \frac{\pi}{4} - 1$$

61. $y = 4x\arccos(x - 1), \quad (1, 2\pi)$

$$y' = 4x\frac{-1}{\sqrt{1 - (x - 1)^2}} + 4\arccos(x - 1)$$

At $(1, 2\pi)$, $y' = -4 + 2\pi$.

Tangent line: $y - 2\pi = (2\pi - 4)(x - 1)$

$$y = (2\pi - 4)x + 4$$

62. $y = 3x\arcsin x, \quad \left(\frac{1}{2}, \frac{\pi}{4}\right)$

$$y' = 3x\frac{1}{\sqrt{1 - x^2}} + 3\arcsin x$$

At $\left(\frac{1}{2}, \frac{\pi}{4}\right)$, $y' = \frac{3}{2}\frac{1}{\sqrt{3/4}} + 3\left(\frac{\pi}{6}\right) = \sqrt{3} + \frac{\pi}{2}$.

Tangent line: $y - \frac{\pi}{4} = \left(\sqrt{3} + \frac{\pi}{2}\right)\left(x - \frac{1}{2}\right)$

$$y = \left(\sqrt{3} + \frac{\pi}{2}\right)x - \frac{\sqrt{3}}{2}$$

63. $f(x) = \operatorname{arcsec} x - x$

$$f'(x) = \frac{1}{|x|\sqrt{x^2 - 1}} - 1 = 0 \text{ when } |x|\sqrt{x^2 - 1} = 1$$

$$x^2(x^2 - 1) = 1$$

$$x^4 - x^2 - 1 = 0 \text{ when } x^2 = \frac{1 + \sqrt{5}}{2} \text{ or}$$

$$x = \pm\sqrt{\frac{1 + \sqrt{5}}{2}} = \pm 1.272$$

Relative maximum: $(1.272, -0.606)$

Relative minimum: $(-1.272, 3.747)$

64. $f(x) = \arcsin x - 2x$

$$f'(x) = \frac{1}{\sqrt{1 - x^2}} - 2$$

$$= 0 \text{ when } \sqrt{1 - x^2} = \frac{1}{2} \text{ or } x = \pm\frac{\sqrt{3}}{2}$$

$$f''(x) = \frac{x}{(1 - x^2)^{3/2}}$$

$$f''\left(\frac{\sqrt{3}}{2}\right) > 0$$

Relative minimum: $\left(\frac{\sqrt{3}}{2}, -0.68\right)$

$$f''\left(-\frac{\sqrt{3}}{2}\right) < 0$$

Relative maximum: $\left(-\frac{\sqrt{3}}{2}, 0.68\right)$

65. $f(x) = \arctan x - \arctan(x - 4)$

$$f'(x) = \frac{1}{1 + x^2} - \frac{1}{1 + (x - 4)^2} = 0$$

$$1 + x^2 = 1 + (x - 4)^2$$

$$0 = -8x + 16$$

$$x = 2$$

By the First Derivative Test, $(2, 2.214)$ is a relative maximum.

66. $f(x) = \arcsin x - 2\arctan x$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{2}{1+x^2} = 0$$

$$1 + x^2 = 2\sqrt{1-x^2}$$

$$1 + 2x^2 + x^4 = 4(1 - x^2)$$

$$x^4 + 6x^2 - 3 = 0$$

$$x = \pm 0.681$$

By the First Derivative Test, $(-0.681, 0.447)$ is a relative maximum and $(0.681, -0.447)$ is a relative minimum.

67. $f(x) = \arcsin(x - 1)$

$$f'(x) = \frac{1}{\sqrt{1-(x-1)^2}} = \frac{1}{\sqrt{2x - x^2}}$$

$$f''(x) = \frac{x-1}{(2x - x^2)^{3/2}}$$

Maximum: $\left(2, \frac{\pi}{2}\right)$

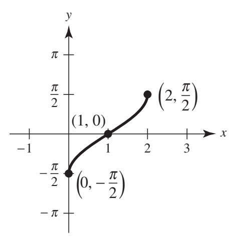
Minimum: $\left(0, -\frac{\pi}{2}\right)$

Point of inflection: $(1, 0)$

Domain: $[0, 2]$

Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

The graph of f is $y = \arcsin x$ shifted 1 unit to the right.



68. $f(x) = \arctan x + \frac{\pi}{2}$

$$f'(x) = \frac{1}{1+x^2}$$

$$f''(x) = \frac{-2x}{(1+x^2)^2}$$

Increasing on $(-\infty, \infty)$

No relative extrema

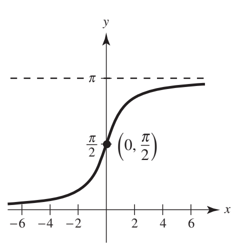
Point of inflection: $\left(0, \frac{\pi}{2}\right)$

Horizontal asymptotes:
$y = 0$ and $y = \pi$

Domain: $(-\infty, \infty)$

Range: $(0, \pi)$

f is $\arctan x$ shifted $\frac{\pi}{2}$ units upward.



69. $f(x) = \operatorname{arcsec} 2x$

$$f'(x) = \frac{1}{|x|\sqrt{4x^2 - 1}}$$

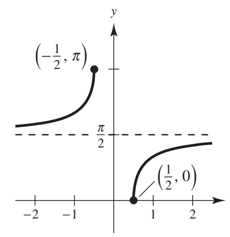
Domain: $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$

Range: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

Maximum: $\left(-\frac{1}{2}, \pi\right)$

Minimum: $\left(\frac{1}{2}, 0\right)$

Horizontal asymptote: $y = \frac{\pi}{2}$



70. $f(x) = \arccos \frac{x}{4}$

$$f'(x) = \frac{-1}{\sqrt{16 - x^2}} < 0$$

$$f''(x) = \frac{-x}{(16 - x^2)^{3/2}}$$

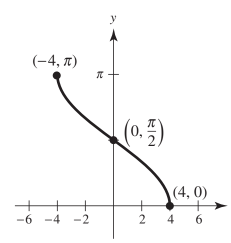
Domain: $[-4, 4]$

Range: $[0, \pi]$

Maximum: $(-4, \pi)$

Minimum: $(4, 0)$

Point of Inflection: $(0, \pi/2)$



71. $x^2 + x\arctan y = y - 1, \quad \left(-\frac{\pi}{4}, 1\right)$

$$2x + \arctan y + \frac{x}{1+y^2}y' = y'$$

$$\left(1 - \frac{x}{1+y^2}\right)y' = 2x + \arctan y$$

$$y' = \frac{2x + \arctan y}{1 - \frac{x}{1+y^2}}$$

At $\left(-\frac{\pi}{4}, 1\right)$: $y' = \dfrac{-\frac{\pi}{2} + \frac{\pi}{4}}{1 - \frac{-\pi/4}{2}} = \dfrac{-\frac{\pi}{2}}{2 + \frac{\pi}{4}} = \dfrac{-2\pi}{8 + \pi}$

Tangent line: $y - 1 = \frac{-2\pi}{8+\pi}\left(x + \frac{\pi}{4}\right)$

$$y = \frac{-2\pi}{8+\pi}x + 1 - \frac{\pi^2}{16 + 2\pi}$$

72. $\arctan(xy) = \arcsin(x + y), \quad (0, 0)$

$$\frac{1}{1 + (xy)^2}[y + xy'] = \frac{1}{\sqrt{1 - (x + y)^2}}[1 + y']$$

At $(0, 0)$: $0 = 1 + y' \Rightarrow y' = -1$

Tangent line: $y = -x$

73. $\arcsin x + \arcsin y = \frac{\pi}{2}, \quad \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$$\frac{1}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - y^2}}y' = 0$$

$$\frac{1}{\sqrt{1 - y^2}}y' = \frac{-1}{\sqrt{1 - x^2}}$$

At $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$: $y' = -1$

Tangent line: $y - \frac{\sqrt{2}}{2} = -1\left(x - \frac{\sqrt{2}}{2}\right)$

$$y = -x + \sqrt{2}$$

74. $\arctan(x + y) = y^2 + \frac{\pi}{4}, \quad (1, 0)$

$$\frac{1}{1 + (x + y)^2}[1 + y'] = 2yy'$$

At $(1, 0)$: $\frac{1}{2}[1 + y'] = 0 \Rightarrow y' = -1$

Tangent line: $y - 0 = -1(x - 1)$

$$y = -x + 1$$

75. (a) $\arcsin(\arcsin(0.5)) \approx 0.551$

$\arcsin(\arcsin(1.0))$ does not exist

(b) In order for $f(x) = \arcsin(\arcsin x)$ to be real, you must have $-1 \le \arcsin x \le 1$.

Because $\arcsin x = 1 \Rightarrow \sin(1) = x$ and $\arcsin x = -1 \Rightarrow \sin(-1) = -\sin 1 = x$, you have

$$-\sin(1) \le x \le \sin(1)$$
$$-0.84147 \le x \le 0.84147$$

76. $\left(-\frac{1}{2}, \frac{2\pi}{3}\right)$ and $\left(0, \frac{\pi}{2}\right)$ lie on the graph of $y = \arccos x$ because both $\frac{2\pi}{3}$ and $\frac{\pi}{2}$ lie in the interval $[0, \pi]$. $\left(\frac{1}{2}, -\frac{\pi}{3}\right)$ does not lie on the graph of $y = \arccos x$ because $-\frac{\pi}{3}$ is not in the interval $[0, \pi]$.

77. They are not equal. For example,

$$\arctan 1 = \frac{\pi}{4} \text{ but } \frac{\arcsin(1)}{\arccos(1)} = \frac{\pi/2}{0}, \text{ undefined.}$$

78. (a) No, the secant function is not one-to-one on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(b) No, the cosecant function is not one-to-one on $(0, \pi)$.

79. In order to have a true inverse function, the domain of sine must be restricted. As a result, 2π is not in the range of the arcsine function.

80. $\arctan 0 = 0$. π is not in the range of $y = \arctan x$.

81. (a) $\operatorname{arccsc} x = \arcsin \frac{1}{x}, \quad |x| \ge 1$

Let $y = \operatorname{arccsc} x$. Then for

$$-\frac{\pi}{2} \le y < 0 \text{ and } 0 < y \le \frac{\pi}{2},$$

$\csc y = x \Rightarrow \sin y = 1/x$. So, $y = \arcsin(1/x)$.

Therefore, $\operatorname{arccsc} x = \arcsin(1/x)$.

(b) $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}, \quad x > 0$

Let $y = \arctan x + \arctan(1/x)$. Then,

$$\tan y = \frac{\tan(\arctan x) + \tan[\arctan(1/x)]}{1 - \tan(\arctan x)\tan[\arctan(1/x)]}$$
$$= \frac{x + (1/x)}{1 - x(1/x)}$$
$$= \frac{x + (1/x)}{0} \text{ (which is undefined).}$$

So, $y = \pi/2$. Therefore,

$\arctan x + \arctan(1/x) = \pi/2$.

82. (a) $\arcsin(-x) = -\arcsin x, \quad |x| \le 1$

Let $y = \arcsin(-x)$. Then,

$-x = \sin y \Rightarrow x = -\sin y \Rightarrow x = \sin(-y)$.

So, $-y = \arcsin x \Rightarrow y = -\arcsin x$. Therefore,

$\arcsin(-x) = -\arcsin x$.

(b) $\arccos(-x) = \pi - \arccos x, \quad |x| \le 1$

Let $y = \arccos(-x)$. Then,

$-x = \cos y \Rightarrow x = -\cos y \Rightarrow x = \cos(\pi - y)$.

So, $\pi - y = \arccos x \Rightarrow y = \pi - \arccos x$.

Therefore, $\arccos(-x) = \pi - \arccos x$.

83. True

$$\frac{d}{dx}[\arctan x] = \frac{1}{1 + x^2} > 0 \text{ for all } x.$$

84. False

The range of $y = \arcsin x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

85. True

$$\frac{d}{dx}[\arctan(\tan x)] = \frac{\sec^2 x}{1 + \tan^2 x} = \frac{\sec^2 x}{\sec^2 x} = 1$$

86. False

$$\arcsin^2 0 + \arccos^2 0 = 0 + \left(\frac{\pi}{2}\right)^2 \neq 1$$

87. (a) $\cot \theta = \frac{x}{5}$

$$\theta = \operatorname{arccot}\left(\frac{x}{5}\right)$$

(b) $\frac{d\theta}{dt} = \frac{-1/5}{1 + (x/5)^2}\frac{dx}{dt} = \frac{-5}{x^2 + 25}\frac{dx}{dt}$

If $\frac{dx}{dt} = -400$ and $x = 10$, $\frac{d\theta}{dt} = 16$ rad/h.

If $\frac{dx}{dt} = -400$ and $x = 3$, $\frac{d\theta}{dt} \approx 58.824$ rad/h.

88. (a) $\cot \theta = \frac{x}{3}$

$$\theta = \operatorname{arccot}\left(\frac{x}{3}\right)$$

(b) $\frac{d\theta}{dt} = \frac{-3}{x^2 + 9}\frac{dx}{dt}$

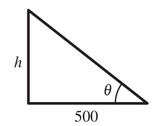
If $x = 10$, $\frac{d\theta}{dt} \approx 11.001$ rad/h.

If $x = 3$, $\frac{d\theta}{dt} \approx 66.667$ rad/h.

A lower altitude results in a greater rate of change of θ.

89. (a) $h(t) = -16t^2 + 256$

$-16t^2 + 256 = 0$ when $t = 4$ sec



(b) $\tan \theta = \frac{h}{500} = \frac{-16t^2 + 256}{500}$

$$\theta = \arctan\left[\frac{16}{500}(-t^2 + 16)\right]$$

$$\frac{d\theta}{dt} = \frac{-8t/125}{1 + [(4/125)(-t^2 + 16)]^2}$$

$$= \frac{-1000t}{15{,}625 + 16(16 - t^2)^2}$$

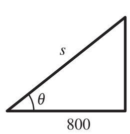
When $t = 1$, $d\theta/dt \approx -0.0520$ rad/sec.

When $t = 2$, $d\theta/dt \approx -0.1116$ rad/sec.

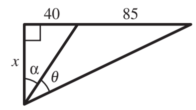
90. $\cos \theta = \frac{800}{s}$

$$\theta = \arccos\left(\frac{800}{s}\right)$$

$$\frac{d\theta}{dt} = \frac{d\theta}{ds} \cdot \frac{ds}{dt} = \frac{-1}{\sqrt{1 - (800/s)^2}}\left(\frac{-800}{s^2}\right)\frac{ds}{dt} = \frac{800}{s\sqrt{s^2 - 800^2}}\frac{ds}{dt}, \quad s > 800$$



91. $\tan \alpha = \dfrac{40}{x}$



$$\tan(\alpha + \theta) = \frac{40 + 85}{x} = \frac{125}{x}$$

$$\tan(\alpha + \theta) = \frac{\tan \alpha + \tan \theta}{1 - \tan \alpha \tan \theta}$$

$$\frac{125}{x} = \frac{40/x + \tan \theta}{1 - \dfrac{40}{x}\tan \theta} = \frac{40 + x \tan \theta}{x - 40 \tan \theta}$$

$$125(x - 40 \tan \theta) = x(40 + x \tan \theta)$$

$$85x = (x^2 + 5000)\tan \theta$$

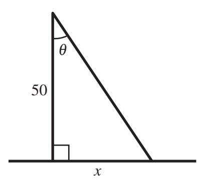
$$\theta = \arctan\left(\frac{85x}{x^2 + 5000}\right)$$

$$\frac{d\theta}{dx} = \frac{85(5000 - x^2)}{(x^2 + 1600)(x^2 + 15625)} = 0 \Rightarrow x = \sqrt{5000} = 50\sqrt{2}$$

By the First Derivative Test, this is a maximum $x = 50\sqrt{2} \approx 70.71$ ft.

92. $\dfrac{d\theta}{dt} = 30(2\pi) = 60\pi$ rad/min

$$\tan \theta = \frac{x}{50}$$



$$\theta = \arctan\left(\frac{x}{50}\right)$$

$$\frac{d\theta}{dt} = \frac{d\theta}{dx}\frac{dx}{dt} = \frac{50}{x^2 + 2500}\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x^2 + 2500}{50}\frac{d\theta}{dt}$$

When $\theta = 45° = \dfrac{\pi}{4}, x = 50$:

$$\frac{dx}{dt} = \frac{(50)^2 + 2500}{50}(60\pi) = 6000\pi \text{ ft/min}$$

93. (a) $\tan(\arctan x + \arctan y) = \dfrac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = \dfrac{x + y}{1 - xy}, \qquad xy \neq 1$

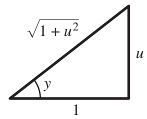
Therefore, $\arctan x + \arctan y = \arctan\left(\dfrac{x + y}{1 - xy}\right), xy \neq 1.$

(b) Let $x = \dfrac{1}{2}$ and $y = \dfrac{1}{3}$.

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan \frac{(1/2) + (1/3)}{1 - [(1/2)\cdot(1/3)]} = \arctan \frac{5/6}{1 - (1/6)} = \arctan \frac{5/6}{5/6} = \arctan 1 = \frac{\pi}{4}$$

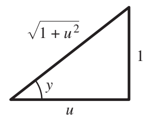
94. (a) Let $y = \arctan u$. Then

$$\tan y = u$$
$$\sec^2 y \frac{dy}{dx} = u'$$
$$\frac{dy}{dx} = \frac{u'}{\sec^2 y} = \frac{u'}{1 + u^2}.$$



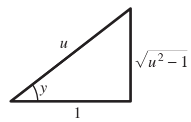
(b) Let $y = \operatorname{arccot} u$. Then

$$\cot y = u$$
$$-\csc^2 y \frac{dy}{dx} = u'$$
$$\frac{dy}{dx} = \frac{u'}{-\csc^2 y} = -\frac{u'}{1 + u^2}.$$



(c) Let $y = \operatorname{arcsec} u$. Then

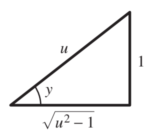
$$\sec y = u$$
$$\sec y \tan y \frac{dy}{dx} = u'$$
$$\frac{dy}{dx} = \frac{u'}{\sec y \tan y} = \frac{u'}{|u|\sqrt{u^2 - 1}}.$$



Note: The absolute value notation in the formula for the derivative of arcsec u is necessary because the inverse secant function has a positive slope at every value in its domain.

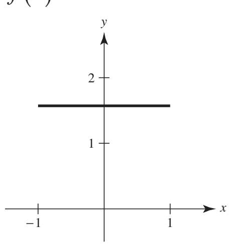
(d) Let $y = \operatorname{arccsc} u$. Then

$$\csc y = u$$
$$-\csc y \cot y \frac{dy}{dx} = u'$$
$$\frac{dy}{dx} = \frac{u'}{-\csc y \cot y} = -\frac{u'}{|u|\sqrt{u^2 - 1}}$$



Note: The absolute value notation in the formula for the derivative of arccsc u is necessary because the inverse cosecant function has a negative slope at every value in its domain.

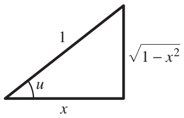
95. (a) $f(x) = \arccos x + \arcsin x$

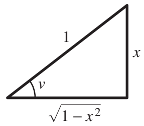


(b) The graph of f is the constant function $y = \pi/2$.

(c) Let $u = \arccos x$ and $v = \arcsin x$

$\cos u = x$ and $\sin v = x$.



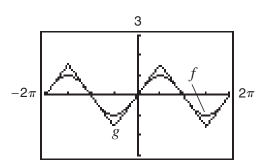


$$\sin(u + v) = \sin u \cos v + \sin v \cos u$$
$$= \sqrt{1 - x^2}\sqrt{1 - x^2} + x \cdot x$$
$$= 1 - x^2 + x^2 = 1$$

So, $u + v = \pi/2$. Therefore, $\arccos x + \arcsin x = \pi/2$.

96. $f(x) = \sin x$

$g(x) = \arcsin(\sin x)$

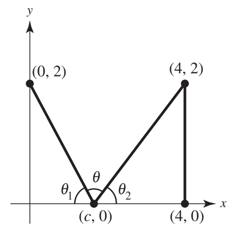


(a) The range of $y = \arcsin x$ is $-\pi/2 \le y \le \pi/2$.

(b) Maximum: $\pi/2$

Minimum: $-\pi/2$

97. $\tan\theta_1 = \dfrac{2}{c}, \tan\theta_2 = \dfrac{2}{4-c}, \quad 0 < c < 4$



To maximize θ, minimize $f(c) = \theta_1 + \theta_2$.

$$f(c) = \arctan\left(\frac{2}{c}\right) + \arctan\left(\frac{2}{4-c}\right)$$

$$f'(c) = \frac{-2}{c^2 + 4} + \frac{2}{(4-c)^2 + 4} = 0$$

$$\frac{1}{c^2+4} = \frac{1}{(4-c)^2 + 4}$$

$$c^2 + 4 = c^2 - 8c + 16 + 4$$

$$8c = 16$$

$$c = 2$$

By the First Derivative Test, $c = 2$ is a minimum. So, $(c, f(c)) = (2, \pi/2)$ is a relative maximum for the angle θ.

Checking the endpoints:

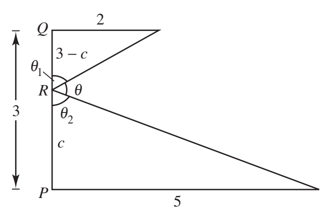
$c = 0: \tan\theta = \dfrac{4}{2} = 2 \Rightarrow \theta \approx 1.107$

$c = 4: \tan\theta = \dfrac{4}{2} = 2 \Rightarrow \theta \approx 1.107$

$c = 2: \theta = \pi - \theta_1 - \theta_2 = \dfrac{\pi}{2} \approx 1.5708$

So, $(2, \pi/2)$ is the absolute maximum.

98. $\tan\theta_1 = \dfrac{2}{3-c}, \tan\theta_2 = \dfrac{5}{c}, \quad 0 < c < 3$



To maximize θ, minimize $f(c) = \theta_1 + \theta_2$.

$$f(c) = \arctan\left(\frac{2}{3-c}\right) + \arctan\left(\frac{5}{c}\right)$$

$$f'(c) = \frac{2}{(3-c)^2 + 4} + \frac{-5}{c^2 + 25} = 0$$

$$2(c^2 + 25) = 5(c^2 - 6c + 9 + 4)$$

$$3c^2 - 30c + 15 = 0$$

$$c^2 - 10c + 5 = 0$$

$$c = 5 - 2\sqrt{5} \approx 0.5279 \text{ (because } c \in [0, 3])$$

$$\theta_1 + \theta_2 \approx 2.1458 \text{ and}$$

$$\theta \approx \pi - (\theta_1 + \theta_2) \approx 0.9958$$

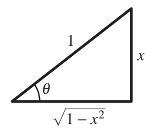
Checking the endpoints:

$c = 3: \tan\theta = \dfrac{3}{5} \Rightarrow \theta \approx 0.5404$

$c = 0: \tan\theta = \dfrac{3}{2} \Rightarrow \theta \approx 0.9828$

So, $c = 5 - 2\sqrt{5}$ yields the absolute maximum.

99. Let $\theta = \arctan\left(\dfrac{x}{\sqrt{1 - x^2}}\right), \quad -1 < x < 1$



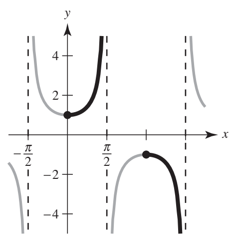
$$\tan\theta = \frac{x}{\sqrt{1 - x^2}}$$

$$\sin\theta = \frac{x}{1} = x$$

$$\arcsin x = \theta.$$

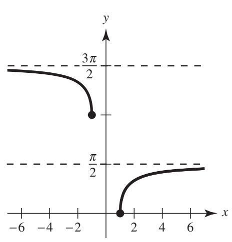
So, $\arcsin x = \arctan\left(\dfrac{x}{\sqrt{1 - x^2}}\right)$ for $-1 < x < 1$.

100. $f(x) = \sec x, \quad 0 \le x < \dfrac{\pi}{2}, \pi \le x < \dfrac{3\pi}{2}$



(a) $y = \operatorname{arcsec} x, \quad x \le -1 \quad \text{or} \quad x \ge 1$

$0 \le y < \dfrac{\pi}{2} \quad \text{or} \quad \pi \le y < \dfrac{3\pi}{2}$



(b)

$$y = \operatorname{arcsec} x$$

$$x = \sec y$$

$$1 = \sec y \tan y \cdot y'$$

$$y' = \frac{1}{\sec y \tan y} = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\tan^2 y + 1 = \sec^2 y$$

$$\tan y = \pm\sqrt{\sec^2 y - 1}$$

On $0 \le y < \pi/2$ and $\pi \le y < 3\pi/2$, $\tan y \ge 0$.

Section 5.8 Inverse Trigonometric Functions: Integration

1. (a) No

(b) Yes. Use the formula involving the arcsecant function.

2. To complete the square of a quadratic function $f(x) = x^2 + bx + c$, add and subtract the square of half the coefficient of x and then rewrite $x^2 + bx + \left(\dfrac{b}{2}\right)^2$ as a perfect square trinomial.

Completing the square helps when quadratic functions are involved in the integrand.

3. $\displaystyle\int \frac{dx}{\sqrt{9 - x^2}} = \arcsin\left(\frac{x}{3}\right) + C$

4. $\displaystyle\int \frac{dx}{\sqrt{1 - 4x^2}} = \frac{1}{2}\int \frac{2}{\sqrt{1 - 4x^2}}\,dx = \frac{1}{2}\arcsin(2x) + C$

5. $\displaystyle\int \frac{1}{x\sqrt{4x^2 - 1}}\,dx = \int \frac{2}{2x\sqrt{(2x)^2 - 1}}\,dx = \operatorname{arcsec}|2x| + C$

6. $\displaystyle\int \frac{12}{1 + 9x^2}\,dx = 4\int \frac{3}{1 + 9x^2}\,dx = 4\arctan(3x) + C$

7. $\displaystyle\int \frac{1}{\sqrt{1 - (x + 1)^2}}\,dx = \arcsin(x + 1) + C$

8. $\int \frac{7}{4 + (3 - x)^2}\,dx = -7\int \frac{1}{2^2 + (3 - x)^2}(-dx)$

$= -7\left(\frac{1}{2}\right)\arctan\frac{3 - x}{2} + C$

$= -\frac{7}{2}\arctan\frac{3 - x}{2} + C$

9. Let $u = t^2$, $du = 2t\,dt$.

$\int \frac{t}{\sqrt{1 - t^4}}\,dt = \frac{1}{2}\int \frac{1}{\sqrt{1 - (t^2)^2}}(2t)\,dt = \frac{1}{2}\arcsin t^2 + C$

10. Let $u = x^2$, $du = 2x\,dx$.

$\int \frac{1}{x\sqrt{x^4 - 4}}\,dx = \frac{1}{2}\int \frac{1}{x^2\sqrt{(x^2)^2 - 2^2}}(2x)\,dx$

$= \frac{1}{4}\operatorname{arcsec}\frac{x^2}{2} + C$

11. $\int \frac{t}{t^4 + 25}\,dt = \frac{1}{2}\int \frac{1}{(t^2)^2 + 5^2}(2)\,dt$

$= \frac{1}{2}\,\frac{1}{5}\arctan\left(\frac{t^2}{5}\right) + C$

$= \frac{1}{10}\arctan\left(\frac{t^2}{5}\right) + C$

12. $\int \frac{1}{x\sqrt{1 - (\ln x)^2}}\,dx = \int \frac{1}{\sqrt{1 - (\ln x)^2}}\cdot\frac{1}{x}\,dx$

$= \arcsin(\ln x) + C$

13. Let $u = e^{2x}$, $du = 2e^{2x}\,dx$.

$\int \frac{e^{2x}}{4 + e^{4x}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{4 + (e^{2x})^2}\,dx = \frac{1}{4}\arctan\frac{e^{2x}}{2} + C$

14. Let $u = 3x$, $du = 3\,dx$, $a = \sqrt{11}$.

$\int \frac{5}{x\sqrt{9x^2 - 11}}\,dx = \frac{5}{3}\int \frac{1}{x\sqrt{(3x)^2 - 11}}(3\,dx)$

$= \frac{5}{3\sqrt{11}}\operatorname{arcsec}\frac{|3x|}{\sqrt{11}} + C$

15. Let $u = \csc x$, $du = -\csc x\cot x\,dx$, $a = 5$.

$\int \frac{-\csc x\cot x}{\sqrt{25 - \csc^2 x}}\,dx = \arcsin\left(\frac{\csc x}{5}\right) + C$

16. $\int \frac{\sin x}{7 + \cos^2 x}\,dx = \int \frac{-1}{(\sqrt{7})^2 + \cos^2 x}(-\sin x)\,dx$

$= -\frac{1}{\sqrt{7}}\arctan\left(\frac{\cos x}{\sqrt{7}}\right) + C$

$= -\frac{\sqrt{7}}{7}\arctan\left(\frac{\sqrt{7}\cos x}{7}\right) + C$

17. $\int \frac{1}{\sqrt{x}\sqrt{1 - x}}\,dx$, $u = \sqrt{x}$, $x = u^2$, $dx = 2u\,du$

$\int \frac{1}{u\sqrt{1 - u^2}}(2u\,du) = 2\int \frac{du}{\sqrt{1 - u^2}} = 2\arcsin u + C = 2\arcsin\sqrt{x} + C$

18. $\int \frac{3}{2\sqrt{x}(1 + x)}\,dx$, $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$, $dx = 2u\,du$

$\frac{3}{2}\int \frac{2u\,du}{u(1 + u^2)} = 3\int \frac{du}{1 + u^2} = 3\arctan u + C = 3\arctan\sqrt{x} + C$

19. $\int \frac{x - 3}{x^2 + 1}\,dx = \frac{1}{2}\int \frac{2x}{x^2 + 1}\,dx - 3\int \frac{1}{x^2 + 1}\,dx = \frac{1}{2}\ln(x^2 + 1) - 3\arctan x + C$

20. $\int \frac{x^2 + 8}{x\sqrt{x^2 - 4}}\,dx = \int \frac{x}{\sqrt{x^2 - 4}}\,dx + \int \frac{8}{x\sqrt{x^2 - 4}}\,dx$

$= \frac{1}{2}\int (x^2 - 4)^{-1/2}(2x)\,dx + 8\int \frac{dx}{x\sqrt{x^2 - 4}}$

$= (x^2 - 4)^{1/2} + \frac{8}{2}\operatorname{arcsec}\frac{|x|}{2} + C$

$= \sqrt{x^2 + 4} + 4\operatorname{arcsec}\frac{|x|}{2} + C$

21. $\int \frac{x+5}{\sqrt{9-(x-3)^2}}\,dx = \int \frac{x-3+8}{\sqrt{9-(x-3)^2}}\,dx = \int \frac{x-3}{\sqrt{9-(x-3)^2}}\,dx + \int \frac{8}{\sqrt{9-(x-3)^2}}\,dx$

For the first integral on the right, use the substitution $u = 9-(x-3)^2$, which gives $du = -2(x-3)\,dx$. Then you have

$$\begin{aligned}\int \frac{x+5}{\sqrt{9-(x-3)^2}}\,dx &= \int \frac{x-3}{\sqrt{9-(x-3)^2}}\,dx + \int \frac{8}{\sqrt{9-(x-3)^2}}\,dx\\ &= -\frac{1}{2}\int \frac{-2(x-3)}{\left(9-(x-3)^2\right)^{1/2}}\,dx + 8\int \frac{1}{\sqrt{3^2-(x-3)^2}}\,dx + C\\ &= -\frac{1}{2}\frac{\left(9-(x-3)^2\right)^{1/2}}{(1/2)} + 8\arcsin\frac{x-3}{3} + C\\ &= -\sqrt{9-(x-3)^2} + 8\arcsin\frac{x-3}{3} + C\\ &= -\sqrt{6x-x^2} + 8\arcsin\left(\frac{x}{3}-1\right) + C.\end{aligned}$$

22. $$\begin{aligned}\int \frac{x-2}{(x+1)^2+4}\,dx &= \frac{1}{2}\int \frac{2x+2}{(x+1)^2+4}\,dx - \int \frac{3}{(x+1)^2+4}\,dx\\ &= \frac{1}{2}\ln\left(x^2+2x+5\right) - \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + C\end{aligned}$$

23. Let $u = 3x$, $du = 3\,dx$.

$$\int_0^{1/6} \frac{3}{\sqrt{1-9x^2}}\,dx = \int_0^{1/6} \frac{1}{\sqrt{1-(3x)^2}}(3)\,dx = \Big[\arcsin(3x)\Big]_0^{1/6} = \frac{\pi}{6}$$

24. $$\begin{aligned}\int_0^{\sqrt{2}} \frac{1}{\sqrt{4-x^2}}\,dx &= \left[\arcsin\frac{x}{2}\right]_0^{\sqrt{2}}\\ &= \arcsin\frac{\sqrt{2}}{2} - \arcsin 0\\ &= \frac{\pi}{4}\end{aligned}$$

25. Let $u = 2x$, $du = 2\,dx$.

$$\begin{aligned}\int_0^{\sqrt{3}/2} \frac{1}{1+4x^2}\,dx &= \frac{1}{2}\int_0^{\sqrt{3}/2} \frac{2}{1+(2x)^2}\,dx\\ &= \left[\frac{1}{2}\arctan(2x)\right]_0^{\sqrt{3}/2} = \frac{\pi}{6}\end{aligned}$$

26. $$\begin{aligned}\int_{\sqrt{3}}^{3} \frac{1}{x\sqrt{4x^2-9}}\,dx &= \left[\frac{1}{3}\operatorname{arcsec}\frac{|2x|}{3}\right]_{\sqrt{3}}^{3}\\ &= \frac{1}{3}\operatorname{arcsec}(2) - \frac{1}{3}\operatorname{arcsec}\frac{2\sqrt{3}}{3}\\ &= \frac{1}{3}\left(\frac{\pi}{3}\right) - \frac{1}{3}\left(\frac{\pi}{6}\right) = \frac{\pi}{18}\end{aligned}$$

27. Let $u = x+2$, $du = dx$, $a = 3$.

$$\begin{aligned}\int_1^7 \frac{1}{9+(x+2)^2}\,dx &= \frac{1}{3}\arctan\left(\frac{x+2}{3}\right)\Bigg]_1^7\\ &= \frac{1}{3}\arctan 3 - \frac{1}{3}\arctan 1\\ &= \frac{1}{3}\arctan 3 - \frac{\pi}{12} \approx 0.155\end{aligned}$$

28. $$\begin{aligned}\int_1^4 \frac{1}{x\sqrt{16x^2-5}}\,dx &= \int_1^4 \frac{4\,dx}{(4x)\sqrt{(4x)^2-\left(\sqrt{5}\right)^2}}\\ &= \left[\left(\frac{1}{\sqrt{5}}\right)\operatorname{arcsec}\frac{|4x|}{\sqrt{5}}\right]_1^4 = \frac{1}{\sqrt{5}}\operatorname{arcsec}\frac{16}{\sqrt{5}} - \frac{1}{\sqrt{5}}\operatorname{arcsec}\left(\frac{4}{\sqrt{5}}\right) \approx 0.091\end{aligned}$$

29. Let $u = e^x$, $du = e^x\,dx$.

$$\int_0^{\ln 5} \frac{e^x}{1 + e^{2x}}\,dx = \Big[\arctan\big(e^x\big)\Big]_0^{\ln 5} = \arctan 5 - \frac{\pi}{4} \approx 0.588$$

30. Let $u = e^{-x}$, $du = -e^{-x}\,dx$.

$$\int_{\ln 2}^{\ln 4} \frac{e^{-x}}{\sqrt{1 - e^{-2x}}}\,dx = \Big[-\arcsin\big(e^{-x}\big)\Big]_{\ln 2}^{\ln 4} = -\arcsin\left(\frac{1}{4}\right) + \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6} - \arcsin\left(\frac{1}{4}\right) \approx 0.271$$

31. Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int_{\pi/2}^{\pi} \frac{\sin x}{1 + \cos^2 x}\,dx = -\int_{\pi/2}^{\pi} \frac{-\sin x}{1 + \cos^2 x}\,dx = \Big[-\arctan(\cos x)\Big]_{\pi/2}^{\pi} = \frac{\pi}{4}$$

32. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\,dx = \Big[\arctan(\sin x)\Big]_0^{\pi/2} = \frac{\pi}{4}$

33. Let $u = \arcsin x$, $du = \dfrac{1}{\sqrt{1 - x^2}}\,dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1 - x^2}}\,dx = \left[\frac{1}{2}\arcsin^2 x\right]_0^{1/\sqrt{2}} = \frac{\pi^2}{32} \approx 0.308$$

34. Let $u = \arccos x$, $du = -\dfrac{1}{\sqrt{1 - x^2}}\,dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1 - x^2}}\,dx = -\int_0^{1/\sqrt{2}} \frac{-\arccos x}{\sqrt{1 - x^2}}\,dx = \left[-\frac{1}{2}\arccos^2 x\right]_0^{1/\sqrt{2}} = \frac{3\pi^2}{32} \approx 0.925$$

35. $\displaystyle\int_0^2 \frac{dx}{x^2 - 2x + 2} = \int_0^2 \frac{1}{1 + (x - 1)^2}\,dx = \Big[\arctan(x - 1)\Big]_0^2 = \frac{\pi}{2}$

36.
$$\begin{aligned}\int_{-2}^{3} \frac{dx}{x^2 + 4x + 8} &= \int_{-2}^{3} \frac{dx}{\left(x^2 + 4x + 4\right) + 4}\\ &= \int_{-2}^{3} \frac{dx}{(x + 2)^2 + 4}\\ &= \left[\frac{1}{2}\arctan\frac{x + 2}{2}\right]_{-2}^{3}\\ &= \frac{1}{2}\arctan\frac{5}{2}\end{aligned}$$

37.
$$\begin{aligned}\int \frac{dx}{\sqrt{-2x^2 + 8x + 4}} &= \int \frac{dx}{\sqrt{12 - 2\left(x^2 - 4x + 4\right)}}\\ &= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{6 - (x - 2)^2}}\\ &= \frac{1}{\sqrt{2}}\arcsin\left(\frac{x - 2}{\sqrt{6}}\right) + C\\ &= \frac{\sqrt{2}}{2}\arcsin\left[\frac{\sqrt{6}}{6}(x - 2)\right] + C\end{aligned}$$

38.
$$\begin{aligned}\int \frac{dx}{3x^2 - 6x + 12} &= \frac{1}{3}\int \frac{dx}{x^2 - 2x + 4}\\ &= \frac{1}{3}\int \frac{dx}{\left(x^2 - 2x + 1\right) + 3}\\ &= \frac{1}{3}\int \frac{dx}{(x - 1)^2 + 3}\\ &= \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)\arctan\frac{x - 1}{\sqrt{3}} + C\\ &= \frac{\sqrt{3}}{9}\arctan\left[\frac{\sqrt{3}}{3}(x - 1)\right] + C\end{aligned}$$

39.
$$\begin{aligned}\int \frac{1}{\sqrt{-x^2 - 4x}}\,dx &= \int \frac{1}{\sqrt{4 - (x + 2)^2}}\,dx\\ &= \arcsin\left(\frac{x + 2}{2}\right) + C\end{aligned}$$

40. $\displaystyle\int \frac{2}{\sqrt{-x^2 + 4x}}\,dx = \int \frac{2}{\sqrt{4 - (x^2 - 4x + 4)}}\,dx = \int \frac{2}{\sqrt{4 - (x-2)^2}}\,dx = 2\arcsin\left(\frac{x-2}{2}\right) + C$

41. $\displaystyle\int_2^3 \frac{2x - 3}{\sqrt{4x - x^2}}\,dx = \int_2^3 \frac{2x - 4}{\sqrt{4x - x^2}}\,dx + \int_2^3 \frac{1}{\sqrt{4x - x^2}}\,dx$

$$= -\int_2^3 (4x - x^2)^{-1/2}(4 - 2x)\,dx + \int_2^3 \frac{1}{\sqrt{4 - (x-2)^2}}\,dx$$

$$= \left[-2\sqrt{4x - x^2} + \arcsin\left(\frac{x-2}{2}\right)\right]_2^3 = 4 - 2\sqrt{3} + \frac{\pi}{6} \approx 1.059$$

42. $\displaystyle\int_3^4 \frac{1}{(x-1)\sqrt{x^2 - 2x}}\,dx = \int_3^4 \frac{1}{(x-1)\sqrt{x^2 - 2x + 1 - 1}}$

$$= \int_3^4 \frac{1}{(x-1)\sqrt{(x-1)^2 - 1}}$$

$$= \Big[\operatorname{arcsec}|x - 1|\Big]_3^4$$

$$= \operatorname{arcsec} 3 - \operatorname{arcsec} 2 \approx 0.1838$$

43. Let $u = \sqrt{e^t - 3}$. Then $u^2 + 3 = e^t$, $2u\,du = e^t\,dt$, and $\dfrac{2u\,du}{u^2 + 3} = dt$.

$$\int \sqrt{e^t - 3}\,dt = \int \frac{2u^2}{u^2 + 3}\,du = \int 2\,du - \int 6\frac{1}{u^2 + 3}\,du$$

$$= 2u - 2\sqrt{3}\arctan\frac{u}{\sqrt{3}} + C = 2\sqrt{e^t - 3} - 2\sqrt{3}\arctan\sqrt{\frac{e^t - 3}{3}} + C$$

44. Let $u = \sqrt{x - 2}$, $u^2 + 2 = x$, $2u\,du = dx$.

$$\int \frac{\sqrt{x - 2}}{x + 1}\,dx = \int \frac{2u^2}{u^2 + 3}\,du = \int \frac{2u^2 + 6 - 6}{u^2 + 3}\,du = 2\int du - 6\int \frac{1}{u^2 + 3}\,du$$

$$= 2u - \frac{6}{\sqrt{3}}\arctan\frac{u}{\sqrt{3}} + C = 2\sqrt{x - 2} - 2\sqrt{3}\arctan\sqrt{\frac{x - 2}{3}} + C$$

45. $\displaystyle\int_1^3 \frac{dx}{\sqrt{x}(1 + x)}$

Let $u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$, $1 + x = 1 + u^2$.

$$\int_1^{\sqrt{3}} \frac{2u\,du}{u(1 + u^2)} = \int_1^{\sqrt{3}} \frac{2}{1 + u^2}\,du$$

$$= \Big[2\arctan(u)\Big]_1^{\sqrt{3}}$$

$$= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{6}$$

46. $\displaystyle\int_0^1 \frac{dx}{2\sqrt{3 - x}\sqrt{x + 1}}$

Let $u = \sqrt{x + 1}$, $u^2 = x + 1$, $2u\,du = dx$,
$\sqrt{3 - x} = \sqrt{4 - u^2}$.

$$\int_1^{\sqrt{2}} \frac{2u\,du}{2\sqrt{4 - u^2}u} = \int_1^{\sqrt{2}} \frac{du}{\sqrt{4 - u^2}}$$

$$= \arcsin\left(\frac{u}{2}\right)\Bigg]_1^{\sqrt{2}}$$

$$= \arcsin\left(\frac{\sqrt{2}}{2}\right) - \arcsin\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

47. (a) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C, \quad u = x$

(b) $\int \frac{x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2} + C, \quad u = 1 - x^2$

(c) $\int \frac{1}{x\sqrt{1-x^2}}\,dx$ cannot be evaluated using the basic integration rules.

48. (a) $\int e^{x^2}\,dx$ cannot be evaluated using the basic integration rules.

(b) $\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C, \quad u = x^2$

(c) $\int \frac{1}{x^2}e^{1/x}\,dx = -e^{1/x} + C, \quad u = \frac{1}{x}$

49. (a) $\int \sqrt{x-1}\,dx = \frac{2}{3}(x-1)^{3/2} + C, \quad u = x - 1$

(b) Let $u = \sqrt{x-1}$. Then $x = u^2 + 1$ and $dx = 2u\,du$.

$$\begin{aligned}\int x\sqrt{x-1}\,dx &= \int (u^2+1)(u)(2u)\,du \\ &= 2\int (u^4 + u^2)\,du \\ &= 2\left(\frac{u^5}{5} + \frac{u^3}{3}\right) + C \\ &= \frac{2}{15}u^3(3u^2 + 5) + C \\ &= \frac{2}{15}(x-1)^{3/2}\left[3(x-1) + 5\right] + C \\ &= \frac{2}{15}(x-1)^{3/2}(3x+2) + C\end{aligned}$$

(c) Let $u = \sqrt{x-1}$. Then $x = u^2 + 1$ and $dx = 2u\,du$.

$$\begin{aligned}\int \frac{x}{\sqrt{x-1}}\,dx &= \int \frac{u^2+1}{u}(2u)\,du \\ &= 2\int (u^2+1)\,du \\ &= 2\left(\frac{u^3}{3} + u\right) + C \\ &= \frac{2}{3}u(u^2+3) + C \\ &= \frac{2}{3}\sqrt{x-1}(x+2) + C\end{aligned}$$

Note: In (b) and (c), substitution was necessary *before* the basic integration rules could be used.

50. (a) $\int \frac{1}{1+x^4}\,dx$ cannot be evaluated using the basic integration rules.

(b) $$\begin{aligned}\int \frac{x}{1+x^4}\,dx &= \frac{1}{2}\int \frac{2x}{1+(x^2)^2}\,dx \\ &= \frac{1}{2}\arctan(x^2) + C, \quad u = x^2\end{aligned}$$

(c) $$\begin{aligned}\int \frac{x^3}{1+x^4}\,dx &= \frac{1}{4}\int \frac{4x^3}{1+x^4}\,dx \\ &= \frac{1}{4}\ln(1+x^4) + C, \quad u = 1 + x^4\end{aligned}$$

51. $$\frac{d}{dx}\left[\arcsin x^3 + C\right] = \frac{3x^2}{\sqrt{1-x^6}}$$

$$\begin{aligned}\frac{d}{dx}\left[\arccos\sqrt{1-x^6} + C\right] &= \frac{-\frac{1}{2}(1-x^6)^{-1/2}(-6x^5)}{\sqrt{1-(1-x^6)}} \\ &= \frac{3x^5}{\sqrt{1-x^6}\cdot x^3} \\ &= \frac{3x^2}{\sqrt{1-x^6}}\end{aligned}$$

52. $$\frac{d}{dx}\left[\arctan\frac{3x}{2} + C\right] = \frac{3/2}{1 + (3x/2)^2} = \frac{6}{9x^2 + 4}$$

$$\frac{d}{dx}\left[\operatorname{arccsc}\left(\frac{\sqrt{4 + 9x^2}}{3x}\right) + C\right] = \frac{\dfrac{4}{3x^2\sqrt{4 + 9x^2}}}{\left|\dfrac{\sqrt{4 + 9x^2}}{3x}\right|\sqrt{\left(\dfrac{\sqrt{4 + 9x^2}}{3x}\right)^2 - 1}}$$

$$= \frac{\dfrac{4}{3x^2\sqrt{4 + 9x^2}}}{\left|\dfrac{\sqrt{4 + 9x^2}}{3x}\right|\dfrac{2}{3x}}$$

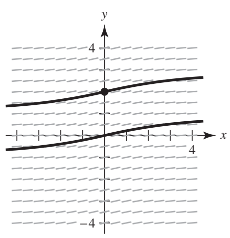
$$= \frac{4(9x^2)}{2(3x^2)(4 + 9x^2)}$$

$$= \frac{6}{9x^2 + 4}$$

53. No. Graphing $f(x) = \arcsin x$ and $g(x) = -\arccos x$, you can see that the graph of f is the graph of g shifted vertically.

54. The area is approximately the area of a square of side 1. So, (c) best approximates the area.

55. (a)

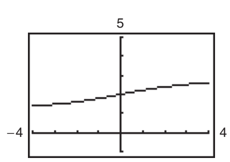


(b) $y' = \dfrac{2}{9 + x^2}, \quad (0, 2)$

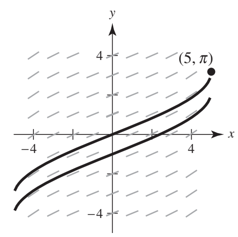
$$y = \int \frac{2}{9 + x^2}\,dx = \frac{2}{3}\arctan\left(\frac{x}{3}\right) + C$$

$$2 = C$$

$$y = \frac{2}{3}\arctan\left(\frac{x}{3}\right) + 2$$



56. (a)

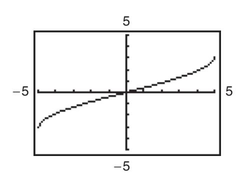


(b) $y' = \dfrac{2}{\sqrt{25 - x^2}}, \quad (5, \pi)$

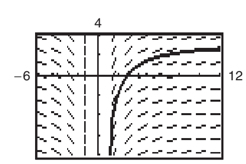
$$y = \int \frac{2}{\sqrt{25 - x^2}}\,dx = 2\arcsin\left(\frac{x}{5}\right) + C$$

$$\pi = 2\arcsin(1) + C \Rightarrow C = 0$$

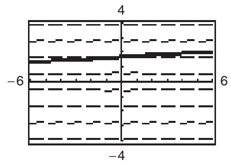
$$y = 2\arcsin\left(\frac{x}{5}\right)$$



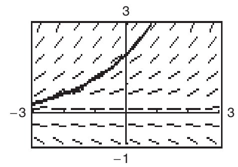
57. $\dfrac{dy}{dx} = \dfrac{10}{x\sqrt{x^2 - 1}}, \quad (3, 0)$



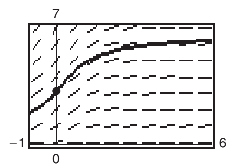
58. $\dfrac{dy}{dx} = \dfrac{1}{12 + x^2}, \quad (4, 2)$



59. $\dfrac{dy}{dx} = \dfrac{2y}{\sqrt{16 - x^2}}, \quad (0, 2)$



60. $\dfrac{dy}{dx} = \dfrac{\sqrt{y}}{1 + x^2}, \quad (0, 4)$



61. $y' = \dfrac{1}{\sqrt{4 - x^2}}, \quad (0, \pi)$

$$y = \int \frac{1}{\sqrt{4 - x^2}}\,dx = \arcsin\left(\frac{x}{2}\right) + C$$

When $x = 0, y = \pi \Rightarrow C = \pi$

$$y = \arcsin\left(\frac{x}{2}\right) + \pi$$

62. $y' = \dfrac{1}{4 + x^2}, \quad (2, \pi)$

$$y = \int \frac{1}{4 + x^2}\,dx = \frac{1}{2}\arctan\frac{x}{2} + C$$

When $x = 2, y = \pi$:

$$\pi = \frac{1}{2}\arctan\left(\frac{2}{2}\right) + C$$

$$\pi = \frac{\pi}{8} + C \Rightarrow C = \frac{7\pi}{8}$$

$$y = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + \frac{7\pi}{8}$$

63. $$\text{Area} = \int_0^1 \frac{2}{\sqrt{4 - x^2}}\,dx$$

$$= \left[2\arcsin\left(\frac{x}{2}\right)\right]_0^1$$

$$= 2\arcsin\left(\frac{1}{2}\right) - 2\arcsin(0)$$

$$= 2\left(\frac{\pi}{6}\right) = \frac{\pi}{3}$$

64. $$\text{Area} = \int_{2/\sqrt{2}}^{2} \frac{1}{x\sqrt{x^2 - 1}}\,dx$$

$$= \left[\operatorname{arcsec} x\right]_{2/\sqrt{2}}^{2}$$

$$= \operatorname{arcsec}(2) - \operatorname{arcsec}\left(\frac{2}{\sqrt{2}}\right)$$

$$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

65. $$\text{Area} = \int_{-\pi/2}^{\pi/2} \frac{3\cos x}{1 + \sin^2 x}\,dx = 3\int_{-\pi/2}^{\pi/2} \frac{1}{1 + \sin^2 x}(\cos x\,dx)$$

$$= \left[3\arctan(\sin x)\right]_{-\pi/2}^{\pi/2}$$

$$= 3\arctan(1) - 3\arctan(-1)$$

$$= \frac{3\pi}{4} + \frac{3\pi}{4} = \frac{3\pi}{2}$$

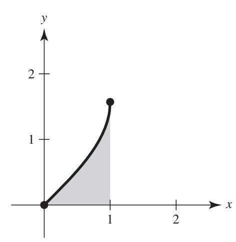
66. $$\text{Area} = \int_0^{\ln\sqrt{3}} \frac{4e^x}{1 + e^{2x}}\,dx, \quad (u = e^x)$$

$$= 4\left[\arctan(e^x)\right]_0^{\ln\sqrt{3}}$$

$$= 4\left[\arctan(\sqrt{3}) - \arctan(1)\right]$$

$$= 4\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{3}$$

67. (a)



Shaded area is given by $\int_0^1 \arcsin x\,dx$.

(b) $\int_0^1 \arcsin x\,dx \approx 0.5708$

(c) Divide the rectangle into two regions.

$$\text{Area rectangle} = (\text{base})(\text{height}) = 1\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\text{Area rectangle} = \int_0^1 \arcsin x\,dx + \int_0^{\pi/2} \sin y\,dy$$

$$\frac{\pi}{2} = \int_0^1 \arcsin x\,dx + \left(-\cos y\right)\Big]_0^{\pi/2}$$

$$= \int_0^1 \arcsin x\,dx + 1$$

So, $\int_0^1 \arcsin x\,dx = \dfrac{\pi}{2} - 1, \ (\approx 0.5708).$

68. (a) $\int_0^1 \frac{4}{1+x^2}\,dx = \left[4\arctan x\right]_0^1 = 4\arctan 1 - 4\arctan 0 = 4\left(\frac{\pi}{4}\right) - 4(0) = \pi$

(b) Let $n = 6$.

$$4\int_0^1 \frac{1}{1+x^2}\,dx \approx 4\left(\frac{1}{18}\right)\left[1 + \frac{4}{1+(1/36)} + \frac{2}{1+(1/9)} + \frac{4}{1+(1/4)} + \frac{2}{1+(4/9)} + \frac{4}{1+(25/36)} + \frac{1}{2}\right] \approx 3.1415918$$

(c) 3.1415927

69. $F(x) = \frac{1}{2}\int_x^{x+2} \frac{2}{t^2+1}\,dt$

(a) $F(x)$ represents the average value of $f(x)$ over the interval $[x, x+2]$. Maximum at $x = -1$, because the graph is greatest on $[-1, 1]$.

(b) $F(x) = \left[\arctan t\right]_x^{x+2} = \arctan(x+2) - \arctan x$

$$F'(x) = \frac{1}{1+(x+2)^2} - \frac{1}{1+x^2} = \frac{(1+x^2) - (x^2+4x+5)}{(x^2+1)(x^2+4x+5)} = \frac{-4(x+1)}{(x^2+1)(x^2+4x+5)} = 0 \text{ when } x = -1.$$

70. $\int \frac{1}{\sqrt{6x - x^2}}\,dx$

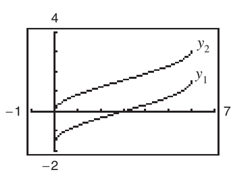
(a) $6x - x^2 = 9 - (x^2 - 6x + 9) = 9 - (x-3)^2$

$$\int \frac{1}{\sqrt{6x-x^2}}\,dx = \int \frac{dx}{\sqrt{9-(x-3)^2}} = \arcsin\left(\frac{x-3}{3}\right) + C$$

(b) $u = \sqrt{x}, u^2 = x, 2u\,du = dx$

$$\int \frac{1}{\sqrt{6u^2 - u^4}}(2u\,du) = \int \frac{2}{\sqrt{6-u^2}}\,du = 2\arcsin\left(\frac{u}{\sqrt{6}}\right) + C = 2\arcsin\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$

(c)



The antiderivatives differ by a constant, $\pi/2$.

Domain: $[0, 6]$

71. False, $\int \frac{dx}{3x\sqrt{9x^2-16}} = \frac{1}{12}\operatorname{arcsec}\frac{|3x|}{4} + C$

72. False, $\int \frac{dx}{25+x^2}\,dx = \frac{1}{5}\arctan\frac{x}{5} + C$

73. $\frac{d}{dx}\left[\arcsin\left(\frac{u}{a}\right) + C\right] = \frac{1}{\sqrt{1-(u^2/a^2)}}\left(\frac{u'}{a}\right) = \frac{u'}{\sqrt{a^2-u^2}}$

So, $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C.$

74. $\frac{d}{dx}\left[\frac{1}{a}\arctan\frac{u}{a} + C\right] = \frac{1}{a}\left[\frac{u'/a}{1+(u/a)^2}\right] = \frac{1}{a^2}\left[\frac{u'}{(a^2+u^2)/a^2}\right] = \frac{u'}{a^2+u^2}$

So, $\int \frac{du}{a^2+u^2} = \int \frac{u'}{a^2+u^2}\,dx = \frac{1}{a}\arctan\frac{u}{a} + C.$

75. Assume $u > 0$.

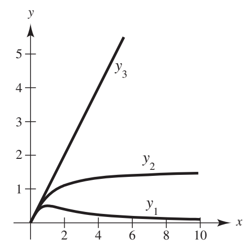
$$\frac{d}{dx}\left[\frac{1}{a}\operatorname{arcsec}\frac{u}{a} + C\right] = \frac{1}{a}\left[\frac{u'/a}{(u/a)\sqrt{(u/a)^2 - 1}}\right] = \frac{1}{a}\left[\frac{u'}{u\sqrt{(u^2 - a^2)/a^2}}\right] = \frac{u'}{u\sqrt{u^2 - a^2}}.$$

The case $u < 0$ is handled in a similar manner.

So, $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \int \frac{u'}{u\sqrt{u^2 - a^2}}\,dx = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C.$

76. Let $f(x) = \arctan x - \dfrac{x}{1 + x^2}$

$$f'(x) = \frac{1}{1 + x^2} - \frac{1 - x^2}{(1 + x^2)^2} = \frac{2x^2}{(1 + x^2)} > 0 \text{ for } x > 0.$$



Because $f(0) = 0$ and f is increasing for $x > 0$,

$\arctan x - \dfrac{x}{1 + x^2} > 0$ for $x > 0$. So, $\arctan x > \dfrac{x}{1 + x^2}$.

Let $g(x) = x - \arctan x$

$$g'(x) = 1 - \frac{1}{1 + x^2} = \frac{x^2}{1 + x^2} > 0 \text{ for } x > 0.$$

Because $g(0) = 0$ and g is increasing for $x > 0$, $x - \arctan x > 0$ for $x > 0$. So, $x > \arctan x$. Therefore,

$$\frac{x}{1 + x^2} < \arctan x < x.$$

77. (a) Area $= \displaystyle\int_0^1 \frac{1}{1 + x^2}\,dx$

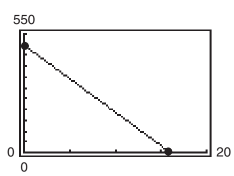
(b) Trapezoidal Rule: $n = 8, b - a = 1 - 0 = 1$

Area ≈ 0.7847

(c) Because $\displaystyle\int_0^1 \frac{1}{1 + x^2}\,dx = \left[\arctan x\right]_0^1 = \frac{\pi}{4}$, you can use the Trapezoidal Rule to approximate $\pi/4$, and therefore, π.

For example, using $n = 200$, you obtain $\pi \approx 4(0.785397) = 3.141588$.

78. (a) $v(t) = -32t + 500$



(b) $s(t) = \displaystyle\int v(t)\,dt = \int(-32t + 500)\,dt = -16t^2 + 500t + C$

$s(0) = -16(0) + 500(0) + C = 0 \Rightarrow C = 0$

$s(t) = -16t^2 + 500t$

When the object reaches its maximum height, $v(t) = 0$.

$$v(t) = -32t + 500 = 0$$
$$-32t = -500$$
$$t = 15.625$$

$s(15.625) = -16(15.625)^2 + 500(15.625) = 3906.25$ ft (Maximum height)

(c) $$\int \frac{1}{32 + kv^2}\,dv = -\int dt$$

$$\frac{1}{\sqrt{32k}}\arctan\left(\sqrt{\frac{k}{32}}v\right) = -t + C_1$$

$$\arctan\left(\sqrt{\frac{k}{32}}v\right) = -\sqrt{32k}t + C$$

$$\sqrt{\frac{k}{32}}v = \tan\left(C - \sqrt{32k}t\right)$$

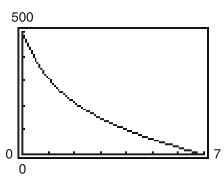
$$v = \sqrt{\frac{32}{k}}\tan\left(C - \sqrt{32k}t\right)$$

When $t = 0, v = 500, C = \arctan\left(500\sqrt{k/32}\right)$, and you have

$$v(t) = \sqrt{\frac{32}{k}}\tan\left[\arctan\left(500\sqrt{\frac{k}{32}}\right) - \sqrt{32k}t\right].$$

(d) When $k = 0.001$:

$$v(t) = \sqrt{32{,}000}\,\tan\left[\arctan\left(500\sqrt{0.00003125}\right) - \sqrt{0.032}t\right]$$



$v(t) = 0$ when $t_0 \approx 6.86$ sec.

(e) $h = \int_0^{6.86} \sqrt{32{,}000}\,\tan\left[\arctan\left(500\sqrt{0.00003125}\right) - \sqrt{0.032}\,t\right]dt \approx 1088$ ft

(f) Air resistance lowers the maximum height.

Section 5.9 Hyperbolic Functions

1. The name *hyperbolic function* came from comparing the area of a semicircular region with the area of a region bounded by a hyperbola.

2. The domains of $f(x) = \operatorname{csch} x$ and $f(x) = \coth x$ are restricted to $x \neq 0$.

3. $\sinh^2 x = \dfrac{-1 + \cosh 2x}{2}$

4. From Theorem 5.22, $\dfrac{d}{dx}\left[\operatorname{sech}^{-1}(3x)\right] = \dfrac{-3}{3x\sqrt{1 - 9x^2}}$.

5. (a) $\sinh 3 = \dfrac{e^3 - e^{-3}}{2} \approx 10.018$

(b) $\tanh(-2) = \dfrac{\sinh(-2)}{\cosh(-2)} = \dfrac{e^{-2} - e^2}{e^{-2} + e^2} \approx -0.964$

6. (a) $\cosh 0 = \dfrac{e^0 + e^0}{2} = 1$

(b) $\operatorname{sech} 1 = \dfrac{2}{e + e^{-1}} \approx 0.648$

7. (a) $\operatorname{csch}(\ln 2) = \dfrac{2}{e^{\ln 2} - e^{-\ln 2}} = \dfrac{2}{2 - (1/2)} = \dfrac{4}{3}$

(b) $$\coth(\ln 5) = \frac{\cosh(\ln 5)}{\sinh(\ln 5)} = \frac{e^{\ln 5} + e^{-\ln 5}}{e^{\ln 5} - e^{-\ln 5}} = \frac{5 + (1/5)}{5 - (1/5)} = \frac{13}{12}$$

8. (a) $\sinh^{-1} 0 = 0$

(b) $\tanh^{-1} 0 = 0$

9. (a) $\cosh^{-1} 2 = \ln\left(2 + \sqrt{3}\right) \approx 1.317$

(b) $\operatorname{sech}^{-1} \dfrac{2}{3} = \ln\left(\dfrac{1 + \sqrt{1 - (4/9)}}{2/3}\right) \approx 0.962$

10. (a) $\operatorname{csch}^{-1} 2 = \ln\left(\dfrac{1 + \sqrt{5}}{2}\right) \approx 0.481$

(b) $\coth^{-1} 3 = \dfrac{1}{2}\ln\left(\dfrac{4}{2}\right) \approx 0.347$

11. $\sinh x + \cosh x = \dfrac{e^x - e^{-x}}{2} + \dfrac{e^x + e^{-x}}{2} = e^x$

12. $\cosh x - \sinh x = \dfrac{e^x + e^{-x}}{2} - \dfrac{e^x - e^{-x}}{2} = e^{-x}$

13.
$$\begin{aligned}\tanh^2 x + \operatorname{sech}^2 x &= \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 + \left(\frac{2}{e^x + e^{-x}}\right)^2\\ &= \frac{e^{2x} - 2 + e^{-2x} + 4}{\left(e^x + e^{-x}\right)^2}\\ &= \frac{e^{2x} + 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} = 1\end{aligned}$$

14.
$$\begin{aligned}\coth^2 x - \operatorname{csch}^2 x &= \frac{\cosh^2 x}{\sinh^2 x} - \frac{1}{\sinh^2 x}\\ &= \frac{\cosh^2 x - 1}{\sinh^2 x}\\ &= \frac{\sinh^2 x}{\sinh^2 x} = 1\end{aligned}$$

15.
$$\begin{aligned}\frac{1 + \cosh 2x}{2} &= \frac{1 + \left(e^{2x} + e^{-2x}\right)/2}{2}\\ &= \frac{e^{2x} + 2 + e^{-2x}}{4}\\ &= \left(\frac{e^x + e^{-x}}{2}\right)^2 = \cosh^2 x\end{aligned}$$

16. $\sinh^2 x = \left(\dfrac{e^x - e^{-x}}{2}\right)^2 = \dfrac{e^{2x} - 2 + e^{-2x}}{4}$

$$\frac{-1 + \cosh 2x}{2} = \frac{-1 + \left(\dfrac{e^{2x} + e^{-2x}}{2}\right)}{2} = \frac{-2 + e^{2x} + e^{-2x}}{4}$$

So, $\sinh^2 x = \dfrac{-1 + \cosh 2x}{2}$.

17. $2 \sinh x \cosh x = 2\left(\dfrac{e^x - e^{-x}}{2}\right)\left(\dfrac{e^x + e^{-x}}{2}\right) = \dfrac{e^{2x} - e^{-2x}}{2} = \sinh 2x$

18.
$$\begin{aligned}\sinh x \cosh y + \cosh x \sinh y &= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)\\ &= \frac{1}{4}\left[e^{x+y} - e^{-x+y} + e^{x-y} - e^{-(x+y)} + e^{x+y} + e^{-x+y} - e^{x-y} - e^{-(x+y)}\right]\\ &= \frac{1}{4}\left[2\left(e^{x+y} - e^{-(x+y)}\right)\right] = \frac{e^{(x+y)} - e^{-(x+y)}}{2} = \sinh(x + y)\end{aligned}$$

19. $\sinh x = \frac{3}{2}$

$\cosh^2 x - \left(\frac{3}{2}\right)^2 = 1 \Rightarrow \cosh^2 x = \frac{13}{4} \Rightarrow \cosh x = \frac{\sqrt{13}}{2}$

$\tanh x = \frac{3/2}{\sqrt{13}/2} = \frac{3\sqrt{13}}{13}$

$\operatorname{csch} x = \frac{1}{3/2} = \frac{2}{3}$

$\operatorname{sech} x = \frac{1}{\sqrt{13}/2} = \frac{2\sqrt{13}}{13}$

$\coth x = \frac{1}{3/\sqrt{13}} = \frac{\sqrt{13}}{3}$

20. $\tanh x = \frac{1}{2}$

$\left(\frac{1}{2}\right)^2 + \operatorname{sech}^2 x = 1 \Rightarrow \operatorname{sech}^2 x = \frac{3}{4} \Rightarrow \operatorname{sech} x = \frac{\sqrt{3}}{2}$

$\cosh x = \frac{1}{\sqrt{3}/2} = \frac{2\sqrt{3}}{3}$

$\coth x = \frac{1}{1/2} = 2$

$\sinh x = \tanh x \cosh x = \left(\frac{1}{2}\right)\left(\frac{2\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{3}$

$\operatorname{csch} x = \frac{1}{\sqrt{3}/3} = \sqrt{3}$

Putting these in order:

$\sinh x = \frac{\sqrt{3}}{3}$ $\qquad \operatorname{csch} x = \sqrt{3}$

$\cosh x = \frac{2\sqrt{3}}{3}$ $\qquad \operatorname{sech} x = \frac{\sqrt{3}}{2}$

$\tanh x = \frac{1}{2}$ $\qquad \coth x = 2$

21. $\lim_{x\to\infty} \sinh x = \infty$

22. $\lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = -1$

23. $\lim_{x\to 0} \frac{\sinh x}{x} = \lim_{x\to 0} \frac{e^x - e^{-x}}{2x} = 1$

24. $\lim_{x\to 0^-} \coth x = \lim_{x\to 0^-} \frac{1}{\tanh x} = -\infty$

25. $f(x) = \sinh(9x)$

$f'(x) = 9\cosh(9x)$

26. $f(x) = \cosh(8x + 1)$

$f'(x) = 8\sinh(8x + 1)$

27. $y = \operatorname{sech}(5x^2)$

$y' = -\operatorname{sech}(5x^2)\tanh(5x^2)(10x)$

$= -10x\operatorname{sech}(5x^2)\tanh(5x^2)$

28. $f(x) = \tanh(4x^2 + 3x)$

$f'(x) = (8x + 3)\operatorname{sech}^2(4x^2 + 3x)$

29. $f(x) = \ln(\sinh x)$

$f'(x) = \frac{1}{\sinh x}(\cosh x) = \coth x$

30. $y = \ln\left(\tanh\frac{x}{2}\right)$

$y' = \frac{1/2}{\tanh(x/2)}\operatorname{sech}^2\left(\frac{x}{2}\right)$

$= \frac{1}{2\sinh(x/2)\cosh(x/2)}$

$= \frac{1}{\sinh x} = \operatorname{csch} x$

31. $h(t) = \frac{t}{6}\sinh(-3t)$

$h'(t) = \frac{t}{6}\cosh(-3t)(-3) + \frac{1}{6}\sinh(-3t)$

$= -\frac{t}{2}\cosh(-3t) + \frac{1}{6}\sinh(-3t)$

32. $y = (x^2 + 1)\coth\frac{x}{3}$

$y' = (x^2 + 1)\left[-\operatorname{csch}^2\left(\frac{x}{3}\right)\left(\frac{1}{3}\right)\right] + 2x\coth\frac{x}{3}$

$= -\frac{1}{3}(x^2 + 1)\operatorname{csch}^2\frac{x}{3} + 2x\coth\frac{x}{3}$

33. $f(t) = \arctan(\sinh t)$

$f'(t) = \dfrac{1}{1 + \sinh^2 t}(\cosh t) = \dfrac{\cosh t}{\cosh^2 t} = \operatorname{sech} t$

34. $g(x) = \operatorname{sech}^2 3x$

$g'(x) = -2 \operatorname{sech}(3x) \operatorname{sech}(3x) \tanh(3x)(3)$

$= -6 \operatorname{sech}^2 3x \tanh 3x$

35. $y = \sinh(1 - x^2), \quad (1, 0)$

$y' = \cosh(1 - x^2)(-2x)$

$y'(1) = -2$

Tangent line: $y - 0 = -2(x - 1)$

$y = -2x + 2$

36. $y = x^{\cosh x}, \quad (1, 1)$

$\ln y = \cosh x \ln x$

$\dfrac{y'}{y} = \dfrac{\cosh x}{x} + \sinh x \ln x$

At $(1, 1)$, $y' = \cosh(1)$.

Tangent line: $y - 1 = \cosh(1)(x - 1)$

$y = \cosh(1)x - \cosh(1) + 1$

Note: $\cosh(1) \approx 1.5431$

37. $y = (\cosh x - \sinh x)^2, \quad (0, 1)$

$y' = 2(\cosh x - \sinh x)(\sinh x - \cosh x)$

At $(0, 1)$, $y' = 2(1)(-1) = -2$.

Tangent line: $y - 1 = -2(x - 0)$

$y = -2x + 1$

38. $y = e^{\sinh x}, \quad (0, 1)$

$y' = e^{\sinh x} \cosh x$

$y'(0) = e^0(1) = 1$

Tangent line: $y - 1 = 1(x - 0)$

$y = x + 1$

39. $g(x) = x \operatorname{sech} x$

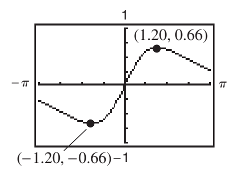
$g'(x) = \operatorname{sech} x - x \operatorname{sech} x \tanh x$

$= \operatorname{sech} x(1 - x \tanh x) = 0$

$x \tanh x = 1$

Using a graphing utility, $x \approx \pm 1.1997$.

By the First Derivative Test, $(1.1997, 0.6627)$ is a relative maximum and $(-1.1997, -0.6627)$ is a relative minimum.



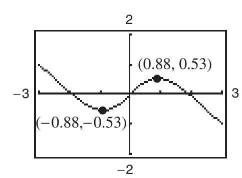
40. $h(x) = 2 \tanh x - x$

$h'(x) = 2 \operatorname{sech}^2 x - 1 = 0$

$\operatorname{sech}^2 x = \frac{1}{2}$

Using a graphing utility, $x \approx 0.8814$.

From the First Derivative Test, $(0.8814, 0.5328)$ is a relative maximum and $(-0.8814, -0.5328)$ is a relative minimum.



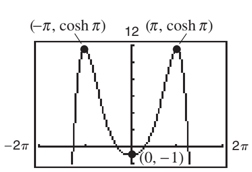
41. $f(x) = \sin x \sinh x - \cos x \cosh x, \quad -4 \le x \le 4$

$f'(x) = \sin x \cosh x + \cos x \sinh x - \cos x \sinh x + \sin x \cosh x$

$= 2 \sin x \cosh x = 0$ when $x = 0, \pm\pi$.

Relative maxima: $(\pm\pi, \cosh \pi)$

Relative minimum: $(0, -1)$

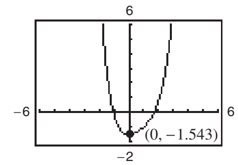


42. $f(x) = x\sinh(x-1) - \cosh(x-1)$

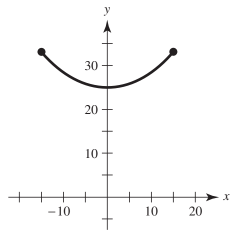
$f'(x) = x\cosh(x-1) + \sinh(x-1) - \sinh(x-1) = x\cosh(x-1)$

$f'(x) = 0$ for $x = 0$.

By the First Derivative Test, $(0, -\cosh(-1)) \approx (0, -1.543)$ is a relative minimum.



43. (a) $y = 10 + 15\cosh\dfrac{x}{15}, \quad -15 \le x \le 15$

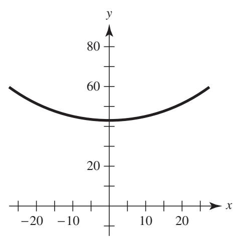


(b) At $x = \pm 15$, $y = 10 + 15\cosh(1) \approx 33.146$.

At $x = 0$, $y = 10 + 15\cosh(0) = 25$.

(c) $y' = \sinh\dfrac{x}{15}$. At $x = 15$, $y' = \sinh(1) \approx 1.175$.

44. (a) $y = 18 + 25\cosh\dfrac{x}{25}, \quad -25 \le x \le 25$



(b) At $x = \pm 25$, $y = 18 + 25\cosh(1) \approx 56.577$.

At $x = 0$, $y = 18 + 25 = 43$.

(c) $y' = \sinh\dfrac{x}{25}$. At $x = 25$, $y' = \sinh(1) \approx 1.175$.

45. $$\int \cosh 4x\,dx = \frac{1}{4}\int \cosh 4x(4\,dx) = \frac{1}{4}\sinh 4x + C$$

46. $$\int \operatorname{sech}^2(3x)\,dx = \tfrac{1}{3}\int \operatorname{sech}^2(3x)(3\,dx) = \tfrac{1}{3}\tanh(3x) + C$$

47. Let $u = 1 - 2x$, $du = -2\,dx$.

$$\int \sinh(1-2x)\,dx = -\tfrac{1}{2}\int \sinh(1-2x)(-2)\,dx = -\tfrac{1}{2}\cosh(1-2x) + C$$

48. Let $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$.

$$\int \frac{\cosh\sqrt{x}}{\sqrt{x}}\,dx = 2\int \cosh\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)dx = 2\sinh\sqrt{x} + C$$

49. Let $u = \cosh(x-1)$, $du = \sinh(x-1)\,dx$.

$$\int \cosh^2(x-1)\sinh(x-1)\,dx = \tfrac{1}{3}\cosh^3(x-1) + C$$

50. Let $u = \cosh x$, $du = \sinh x\,dx$.

$$\int \frac{\sinh}{1 + \sinh^2 x}\,dx = \int \frac{\sinh x}{\cosh^2 x}\,dx = \frac{-1}{\cosh x} + C = -\operatorname{sech} x + C$$

51. Let $u = \sinh x$, $du = \cosh x\,dx$.

$$\int \frac{\cosh x}{\sinh x}\,dx = \ln|\sinh x| + C$$

52. Let $u = \dfrac{1}{x}$, $du = -\dfrac{1}{x^2}\,dx$.

$$\int \frac{\operatorname{csch}(1/x)\coth(1/x)}{x^2}\,dx = -\int \operatorname{csch}\frac{1}{x}\coth\frac{1}{x}\left(-\frac{1}{x^2}\right)dx = \operatorname{csch}\frac{1}{x} + C$$

53. Let $u = \dfrac{x^2}{2}$, $du = x\,dx$.

$$\int x\operatorname{csch}^2\frac{x^2}{2}\,dx = \int\left(\operatorname{csch}^2\frac{x^2}{2}\right)x\,dx = -\coth\frac{x^2}{2} + C$$

54. Let $u = \operatorname{sech} x$, $du = -\operatorname{sech} x\tanh x\,dx$.

$$\int \operatorname{sech}^3 x\tanh x\,dx = -\int \operatorname{sech}^2 x(-\operatorname{sech} x\tanh x)\,dx = -\tfrac{1}{3}\operatorname{sech}^3 x + C$$

55. $$\int_0^{\ln 2} \tanh x\,dx = \int_0^{\ln 2} \frac{\sinh x}{\cosh x}\,dx, \quad (u = \cosh x)$$
$$= \Big[\ln(\cosh x)\Big]_0^{\ln 2} = \ln(\cosh(\ln 2)) - \ln(\cosh(0)) = \ln\left(\frac{5}{4}\right) - 0 = \ln\left(\frac{5}{4}\right)$$

Note: $\cosh(\ln 2) = \dfrac{e^{\ln 2} + e^{-\ln 2}}{2} = \dfrac{2 + (1/2)}{2} = \dfrac{5}{4}$

56. $\int_0^1 \cosh^2 x\,dx = \int_0^1 \frac{1+\cosh(2x)}{2}\,dx$
$= \frac{1}{2}\left[x + \frac{1}{2}\sinh(2x)\right]_0^1$
$= \frac{1}{2}\left[1 + \frac{1}{2}\sinh(2)\right]$
$= \frac{1}{2} + \frac{1}{2}\sinh(1)\cosh(1)$

57. $\int_3^4 \operatorname{csch}^2(x-2)\,dx = \left[-\coth(x-2)\right]_3^4$
$= -\coth 2 + \coth 1$

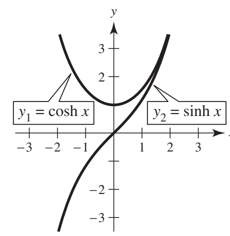
58. $\int_{1/2}^1 \operatorname{sech}^2(2x-1)\,dx = \frac{1}{2}\int_{1/2}^1 \operatorname{sech}^2(2x-1)2\,dx$
$= \left[\frac{1}{2}\tanh(2x-1)\right]_{1/2}^1$
$= \frac{1}{2}\tanh 1 - \frac{1}{2}\tanh 0$
$= \frac{1}{2}\tanh 1$

59. $\int_{5/3}^2 \operatorname{csch}(3x-4)\coth(3x-4)\,dx = \frac{1}{3}\int_{5/3}^2 \operatorname{csch}(3x-4)\coth(3x-4)(3\,dx)$
$= -\frac{1}{3}\left[\operatorname{csch}(3x-4)\right]_{5/3}^2$
$= -\frac{1}{3}\operatorname{csch} 2 + \frac{1}{3}\operatorname{csch} 1$

60. $2e^{-x}\cosh x = 2e^{-x}\left[\frac{e^x + e^{-x}}{2}\right] = 1 + e^{-2x}$

$\int_0^{\ln 2} 2e^{-x}\cosh x\,dx = \int_0^{\ln 2}\left(1 + e^{-2x}\right)dx$
$= \left[x - \frac{1}{2}e^{-2x}\right]_0^{\ln 2}$
$= \left[\ln 2 - \frac{1}{2}\left(\frac{1}{4}\right)\right] - \left[0 - \frac{1}{2}\right]$
$= \frac{3}{8} + \ln 2$

61. The graph of $y_1 = \cosh x$ lies above the graph of $y_2 = \sinh x$. They do not intersect.



62. The graphs of $y = \sinh x$, $y = \tanh x$, $y = \operatorname{csch} x$, and $y = \coth x$ are odd. The graphs of $y = \cosh x$ and $y = \operatorname{sech} x$ are even.

63. $\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = \frac{e^{-x} - e^x}{2} = -\sinh x \Rightarrow$ odd

$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^x}{2} = \cosh x \Rightarrow$ even

$\tanh(-x) = \frac{\sinh(-x)}{\cosh(-x)} = -\frac{\sinh x}{\cosh x} = -\tanh x \Rightarrow$ odd

$\coth(-x) = \frac{\cosh(-x)}{\sinh(-x)} = \frac{\cosh x}{-\sinh x} = -\coth x \Rightarrow$ odd

$\operatorname{csch}(-x) = \frac{1}{\sinh(-x)} = \frac{1}{-\sinh x} = -\operatorname{csch} x \Rightarrow$ odd

$\operatorname{sech}(-x) = \frac{1}{\cosh(-x)} = \frac{1}{\cosh x} = \operatorname{sech} x \Rightarrow$ even

64. (a) $f(x) = \cosh x$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

$g(x) = \tanh x$ is increasing on $(-\infty, \infty)$.

(b) $f(x) = \cosh x$ is concave upward on $(-\infty, \infty)$.

$g(x) = \tanh x$ is concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$.

65. $y = \cosh^{-1}(3x)$

$$y' = \frac{3}{\sqrt{9x^2 - 1}}$$

66. $y = \operatorname{csch}^{-1}(1 - x)$

$$y' = \frac{-(-1)}{|1 - x|\sqrt{1 + (1 - x)^2}} = \frac{1}{|1 - x|\sqrt{x^2 - 2x + 2}}$$

67. $y = \tanh^{-1}\sqrt{x}$

$$y' = \frac{1}{1 - (\sqrt{x})^2}\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{2\sqrt{x}(1 - x)}$$

68. $f(x) = \coth^{-1}(x^2)$

$$f'(x) = \frac{1}{1 - (x^2)^2}(2x) = \frac{2x}{1 - x^4}$$

69. $y = \sinh^{-1}(\tan x)$

$$y' = \frac{1}{\sqrt{\tan^2 x + 1}}(\sec^2 x) = |\sec x|$$

70. $y = \tanh^{-1}(\sin 2x)$

$$y' = \frac{1}{1 - \sin^2 2x}(2\cos 2x) = 2\sec 2x$$

71. $y = \operatorname{sech}^{-1}(\sin x), 0 < x < \pi/2$

$$y' = \frac{-\cos x}{\sin x\sqrt{1 - \sin^2 x}} = \frac{-\cos x}{\sin x(\cos x)} = -\csc x$$

72. $y = \coth^{-1}(e^{2x})$

$$y' = \frac{2e^{2x}}{1 - (e^{2x})^2} = \frac{2e^{2x}}{1 - e^{4x}}$$

73. $y = 2x\sinh^{-1}(2x) - \sqrt{1 + 4x^2}$

$$y' = 2x\left(\frac{2}{\sqrt{1 + 4x^2}}\right) + 2\sinh^{-1}(2x) - \frac{4x}{\sqrt{1 + 4x^2}} = 2\sinh^{-1}(2x)$$

74. $y = x\tanh^{-1}x + \ln\sqrt{1 - x^2}$

$$= x\tanh^{-1}x + \frac{1}{2}\ln(1 - x^2)$$

$$y' = x\left(\frac{1}{1 - x^2}\right) + \tanh^{-1}x + \frac{-x}{1 - x^2} = \tan^{-1}x$$

75. $$\int \frac{1}{3 - 9x^2}\,dx = \frac{1}{3}\int \frac{1}{3 - (3x)^2}(3)\,dx = \frac{1}{3}\frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3} + 3x}{\sqrt{3} - 3x}\right| + C = \frac{\sqrt{3}}{18}\ln\left|\frac{1 + \sqrt{3}x}{1 - \sqrt{3}x}\right| + C$$

76. $$\int \frac{1}{2x\sqrt{1 - 4x^2}}\,dx = \frac{1}{2}\int \frac{1}{2x\sqrt{1 - (2x)^2}}(2)\,dx = -\frac{1}{2}\ln\left[\frac{1 + \sqrt{1 - 4x^2}}{|2x|}\right] + C$$

77. $$\int \frac{1}{\sqrt{1 + e^{2x}}}\,dx = \int \frac{e^x}{e^x\sqrt{1 + (e^x)^2}}\,dx$$

$$= -\operatorname{csch}^{-1}(e^x) + C$$

$$= -\ln\left(\frac{1 + \sqrt{1 + e^{2x}}}{e^x}\right) + C$$

$$= \ln\left(\frac{e^x}{1 + \sqrt{1 + e^{2x}}}\right) + C$$

$$= \ln\left(\frac{-e^x + e^x\sqrt{1 + e^{2x}}}{e^{2x}}\right) + C$$

$$= \ln\left(\sqrt{1 + e^{2x}} - 1\right) - x + C$$

78. $\int \frac{x}{9 - x^4}\,dx = -\frac{1}{2}\int \frac{-2x}{9 - (x^2)^2}\,dx$

$= -\frac{1}{2}\left(\frac{1}{6}\right)\ln\left|\frac{3 - x^2}{3 + x^2}\right| + C$

$= -\frac{1}{12}\ln\left|\frac{3 - x^2}{3 + x^2}\right| + C$

79. Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$.

$\int \frac{1}{\sqrt{x}\sqrt{1 + x}}\,dx = 2\int \frac{1}{\sqrt{1 + (\sqrt{x})^2}}\left(\frac{1}{2\sqrt{x}}\right)dx$

$= 2\sinh^{-1}\sqrt{x} + C$

$= 2\ln\left(\sqrt{x} + \sqrt{1 + x}\right) + C$

80. Let $u = x^{3/2}$, $du = \frac{3}{2}\sqrt{x}\,dx$.

$\int \frac{\sqrt{x}}{\sqrt{1 + x^3}}\,dx = \frac{2}{3}\int \frac{1}{\sqrt{1 + (x^{3/2})^2}}\left(\frac{3}{2}\sqrt{x}\right)dx$

$= \frac{2}{3}\sinh^{-1}(x^{3/2}) + C$

$= \frac{2}{3}\ln\left(x^{3/2} + \sqrt{1 + x^3}\right) + C$

81. $\int \frac{-1}{4x - x^2}\,dx = \int \frac{1}{(x - 2)^2 - 4}\,dx$

$= \frac{1}{4}\ln\left|\frac{(x - 2) - 2}{(x - 2) + 2}\right|$

$= \frac{1}{4}\ln\left|\frac{x - 4}{x}\right| + C$

82. $\int \frac{dx}{(x + 2)\sqrt{x^2 + 4x + 8}} = \int \frac{dx}{(x + 2)\sqrt{(x + 2)^2 + 4}} = -\frac{1}{2}\ln\left(\frac{2 + \sqrt{(x + 2)^2 + 4}}{|x + 2|}\right) + C$

83. $\int_3^7 \frac{1}{\sqrt{x^2 - 4}}\,dx = \left[\ln\left(x + \sqrt{x^2 - 4}\right)\right]_3^7 = \ln\left(7 + \sqrt{45}\right) - \ln\left(3 + \sqrt{5}\right) = \ln\left(\frac{7 + \sqrt{45}}{3 + \sqrt{5}}\right) = \ln\left(\frac{\sqrt{5} + 3}{2}\right)$

84. $\int_1^3 \frac{1}{x\sqrt{4 + x^2}}\,dx = \left[-\frac{1}{2}\ln\left(\frac{2 + \sqrt{4 + x^2}}{|x|}\right)\right]_1^3 = -\frac{1}{2}\ln\left(\frac{2 + \sqrt{13}}{3}\right) + \frac{1}{2}\ln\left(2 + \sqrt{5}\right)$

85. $\int_{-1}^1 \frac{1}{16 - 9x^2}\,dx = \frac{1}{3}\int_{-1}^1 \frac{1}{4^2 - (3x)^2}(3)\,dx$

$= \left[\frac{1}{3}\frac{1}{4}\frac{1}{2}\ln\left|\frac{4 + 3x}{4 - 3x}\right|\right]_{-1}^1$

$= \frac{1}{24}\left[\ln(7) - \ln\left(\frac{1}{7}\right)\right]$

$= \frac{1}{24}[\ln 7 - \ln 1 + \ln 7] = \frac{1}{12}\ln 7$

86. $\int_0^1 \frac{1}{\sqrt{25x^2 + 1}}\,dx = \frac{1}{5}\int_0^1 \frac{1}{\sqrt{(5x)^2 + 1}}(5)\,dx$

$= \left[\frac{1}{5}\ln\left(5x + \sqrt{25x^2 + 1}\right)\right]_0^1$

$= \frac{1}{5}\ln\left(5 + \sqrt{26}\right)$

87. $y = \int \frac{x^3 - 21x}{5 + 4x - x^2}\,dx = \int\left(-x - 4 + \frac{20}{5 + 4x - x^2}\right)dx$

$= \int(-x - 4)\,dx + 20\int \frac{1}{3^2 - (x - 2)^2}\,dx$

$= -\frac{x^2}{2} - 4x + \frac{20}{6}\ln\left|\frac{3 + (x - 2)}{3 - (x - 2)}\right| + C$

$= -\frac{x^2}{2} - 4x + \frac{10}{3}\ln\left|\frac{1 + x}{5 - x}\right| + C$

$= \frac{-x^2}{2} - 4x - \frac{10}{3}\ln\left|\frac{5 - x}{x + 1}\right| + C$

88. $y = \int \frac{1 - 2x}{4x - x^2}\,dx = \int \frac{4 - 2x}{4x - x^2}\,dx + 3\int \frac{1}{(x - 2)^2 - 4}\,dx$

$$= \ln\left|4x - x^2\right| + \frac{3}{4}\ln\left|\frac{(x - 2) - 2}{(x - 2) + 2}\right| + C$$

$$= \ln\left|4x - x^2\right| + \frac{3}{4}\ln\left|\frac{x - 4}{x}\right| + C$$

89. $A = 2\int_0^4 \operatorname{sech}\frac{x}{2}\,dx$

$$= 2\int_0^4 \frac{2}{e^{x/2} + e^{-x/2}}\,dx$$

$$= 4\int_0^4 \frac{e^{x/2}}{\left(e^{x/2}\right)^2 + 1}\,dx$$

$$= \left[8\arctan\left(e^{x/2}\right)\right]_0^4$$

$$= 8\arctan\left(e^2\right) - 2\pi \approx 5.207$$

90. $A = \int_0^2 \tanh 2x\,dx = \int_0^2 \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx$

$$= \frac{1}{2}\int_0^2 \frac{1}{e^{2x} + e^{-2x}}(2)\left(e^{2x} - e^{-2x}\right)dx$$

$$= \left[\frac{1}{2}\ln\left(e^{2x} + e^{-2x}\right)\right]_0^2 = \frac{1}{2}\ln\left(e^4 + e^{-4}\right) - \frac{1}{2}\ln 2$$

$$= \ln\sqrt{\frac{e^4 + e^{-4}}{2}} \approx 1.654$$

91. $A = \int_0^2 \frac{5x}{\sqrt{x^4 + 1}}\,dx$

$$= \frac{5}{2}\int_0^2 \frac{2x}{\sqrt{\left(x^2\right)^2 + 1}}\,dx$$

$$= \left[\frac{5}{2}\ln\left(x^2 + \sqrt{x^4 + 1}\right)\right]_0^2$$

$$= \frac{5}{2}\ln\left(4 + \sqrt{17}\right) \approx 5.237$$

92. $A = \int_1^2 \frac{6}{x\sqrt{9 - x^2}}\,dx = 6\left[\left(-\frac{1}{3}\ln\frac{3 + \sqrt{9 - x^2}}{|x|}\right)\right]_1^2$

$$= -2\left(\ln\frac{3 + \sqrt{5}}{2} - \ln\frac{3 + \sqrt{8}}{1}\right)$$

$$= -2\ln\frac{\left(3 + \sqrt{5}\right)/2}{\left(3 + \sqrt{8}\right)}$$

$$= 2\ln\frac{6 + 4\sqrt{2}}{3 + \sqrt{5}} \approx 1.6006$$

93. (a) $y = a\operatorname{sech}^{-1}\frac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0$

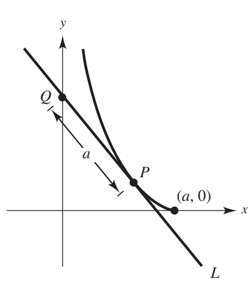
$$\frac{dy}{dx} = \frac{-1}{(x/a)\sqrt{1 - \left(x^2/a^2\right)}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{-a^2}{x\sqrt{a^2 - x^2}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{x^2 - a^2}{x\sqrt{a^2 - x^2}} = \frac{-\sqrt{a^2 - x^2}}{x}$$

(b) Equation of tangent line through $P = (x_0, y_0)$: $y - a\operatorname{sech}^{-1}\frac{x_0}{a} + \sqrt{a^2 - x_0^2} = -\frac{\sqrt{a^2 - x_0^2}}{x_0}(x - x_0)$

When $x = 0$, $y = a\operatorname{sech}^{-1}\frac{x_0}{a} - \sqrt{a^2 - x_0^2} + \sqrt{a^2 - x_0^2} = a\operatorname{sech}^{-1}\frac{x_0}{a}$.

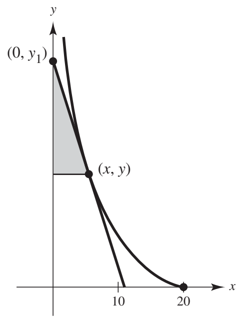
So, Q is the point $\left[0, a\operatorname{sech}^{-1}(x_0/a)\right]$.

Distance from P to Q: $d = \sqrt{(x_0 - 0)^2 + \left(y_0 - a\operatorname{sech}^{-1}(x_0/a)\right)} = \sqrt{x_0^2 + \left(-\sqrt{a^2 - x_0^2}\right)^2} = \sqrt{a^2} = a$

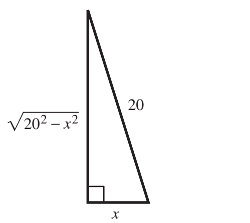


94. In Example 5, $a = 20$. From Exercise 93(a),

$$y' = \frac{-\sqrt{20^2 - x^2}}{x}.$$



The slope of the line connecting (x, y) and $(0, y_1)$ can be determined by analyzing the shaded triangle. From Exercise 93(b), the hypotenuse is a.



$$m = -\frac{\sqrt{20^2 - x^2}}{x} = y'$$

Hence, the boat is always pointing toward the person

95. Let $u = \tanh^{-1} x$, $-1 < x < 1$

$$\tanh u = x.$$

$$\frac{\sinh u}{\cosh u} = \frac{e^u - e^{-u}}{e^u + e^{-u}} = x$$

$$e^u - e^{-u} = xe^u + xe^{-u}$$

$$e^{2u} - 1 = xe^{2u} + x$$

$$e^{2u}(1 - x) = 1 + x$$

$$e^{2u} = \frac{1 + x}{1 - x}$$

$$2u = \ln\left(\frac{1 + x}{1 - x}\right)$$

$$u = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right), \quad -1 < x < 1$$

96. Let $u = \sinh^{-1} t$. Then

$$\sinh u = \frac{e^u - e^{-u}}{2} = t$$

$$e^u - e^{-u} = 2t$$

$$e^{2u} - 2te^u - 1 = 0$$

$$e^u = \frac{2t \pm \sqrt{4t^2 + 4}}{2}$$

$$= t \pm \sqrt{t^2 + 1}$$

$$= t + \sqrt{t^2 + 1} \quad \left(\text{because } e^u > 0\right)$$

$$u = \ln\left(t + \sqrt{t^2 + 1}\right)$$

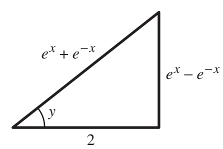
97. Let $y = \arcsin(\tanh x)$. Then,

$$\sin y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ and}$$

$$\tan y = \frac{e^x - e^{-x}}{2} = \sinh x.$$

So, $y = \arctan(\sinh x)$. Therefore,

$$\arctan(\sinh x) = \arcsin(\tanh x).$$



98.
$$\int_{-b}^{b} e^{xt}\,dt = \left[\frac{e^{xt}}{x}\right]_{-b}^{b}$$

$$= \frac{e^{xb}}{x} - \frac{e^{-xb}}{x}$$

$$= \frac{2}{x}\left[\frac{e^{xb} - e^{-xb}}{2}\right]$$

$$= \frac{2}{x}\sinh(xb)$$

99.
$$y = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$y' = \frac{e^x - e^{-x}}{2} = \sinh x$$

100.
$$y = \coth x = \frac{\cosh x}{\sinh x}$$

$$y' = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\operatorname{csch}^2 x$$

101.
$$y = \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

$$y' = -2\left(e^x + e^{-x}\right)^{-2}\left(e^x - e^{-x}\right) = \left(\frac{-2}{e^x + e^{-x}}\right)\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = -\operatorname{sech} x \tanh x$$

102.
$$y = \cosh^{-1} x$$
$$\cosh y = x$$
$$(\sinh y)(y') = 1$$
$$y' = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

103.
$$y = \sinh^{-1} x$$
$$\sinh y = x$$
$$(\cosh y)y' = 1$$
$$y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{\sinh^2 y + 1}} = \frac{1}{\sqrt{x^2 + 1}}$$

104.
$$y = \operatorname{sech}^{-1} x$$
$$\operatorname{sech} y = x$$
$$-(\operatorname{sech} y)(\tanh y)y' = 1$$
$$y' = \frac{-1}{(\operatorname{sech} y)(\tanh y)} = \frac{-1}{(\operatorname{sech} y)\sqrt{1 - \operatorname{sech}^2 y}} = \frac{-1}{x\sqrt{1 - x^2}}$$

105. $y = c \cosh \dfrac{x}{c}$

Let $P(x_1, y_1)$ be a point on the catenary.

$$y' = \sinh \frac{x}{c}$$

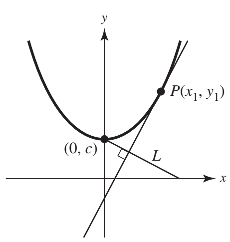
The slope at P is $\sinh(x_1/c)$. The equation of line L is

$$y - c = \frac{-1}{\sinh(x_1/c)}(x - 0).$$

When $y = 0$, $c = \dfrac{x}{\sinh(x_1/c)} \Rightarrow x = c \sinh\left(\dfrac{x_1}{c}\right)$. The length of L is

$$\sqrt{c^2 \sinh^2\left(\frac{x_1}{c}\right) + c^2} = c \cdot \cosh\frac{x_1}{c} = y_1,$$

the ordinate y_1 of the point P.

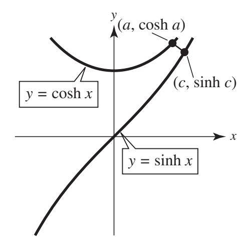


106. There is no such common normal. To see this, assume there is a common normal.

$y = \cosh x \Rightarrow y' = \sinh x$

Normal line at $(a, \cosh a)$ is

$$y - \cosh a = \frac{-1}{\sinh a}(x - a).$$



Similarly,

$$y - \sinh c = \frac{-1}{\cosh c}(x - c)$$

is normal at $(c, \sinh c)$. Also,

$$\frac{-1}{\sinh a} = \frac{-1}{\cosh c} \Rightarrow \cosh c = \sinh a.$$

The slope between the points is $\dfrac{\sinh c - \cosh a}{c - a}$.

Therefore, $-\dfrac{a - c}{\cosh a - \sinh c} = \cosh c = \sinh a.$

$\cosh c > 0 \Rightarrow a > 0$

$\sinh x < \cosh x$ for all

$x \Rightarrow \sinh c < \cosh c = \sinh a < \cosh a$. So, $c < a$.

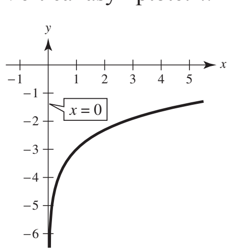
But, $-\dfrac{a - c}{\cosh a - \sinh c} < 0$, a contradiction.

Review Exercises for Chapter 5

1. $f(x) = \ln x - 3$

Vertical shift 3 units downward

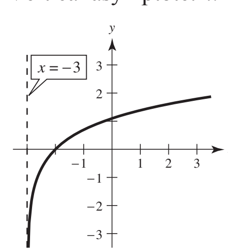
Vertical asymptote: $x = 0$



2. $f(x) = \ln(x + 3)$

Horizontal shift 3 units to the left.

Vertical asymptote: $x = -3$



3. (a) $\ln 20 = \ln(4 \cdot 5) = \ln 4 + \ln 5 \approx 1.3863 + 1.6094 = 2.9957$

(b) $\ln \frac{4}{5} = \ln 4 - \ln 5 \approx 1.3863 - 1.6094 = -0.2231$

(c) $\ln 625 = \ln 5^4 = 4 \ln 5 \approx 4(1.6094) = 6.4376$

(d) $\ln \sqrt{5} = \ln 5^{1/2} = \frac{1}{2} \ln 5 \approx \frac{1}{2}(1.6094) = 0.8047$

4. (a) $\ln 0.0625 = \ln\left(4^{-2}\right) = -2 \ln 4 \approx -2(1.3863) = -2.7726$

(b) $\ln \frac{5}{4} = \ln 5 - \ln 4 \approx 1.6094 - 1.3863 = 0.2231$

(c) $\ln 16 = \ln 4^2 = 2 \ln 4 \approx 2(1.3863) = 2.7726$

(d) $\ln \sqrt[3]{80} = \frac{1}{3} \ln(4^2 \cdot 5) = \frac{1}{3}(2 \ln 4 + \ln 5) \approx \frac{1}{3}[2(1.3863) + 1.6094] \approx 1.4607$

5. $$\ln \sqrt[5]{\frac{4x^2-1}{4x^2+1}} = \frac{1}{5} \ln \frac{(2x-1)(2x+1)}{4x^2+1}$$
$$= \frac{1}{5}\left[\ln(2x-1) + \ln(2x+1) - \ln\left(4x^2+1\right)\right]$$

6. $\ln\left[\left(x^2+1\right)(x-1)\right] = \ln\left(x^2+1\right) + \ln(x-1)$

7. $\ln 3 + \frac{1}{3}\ln\left(4 - x^2\right) - \ln x = \ln 3 + \ln \sqrt[3]{4 - x^2} - \ln x = \ln\left(\frac{3\sqrt[3]{4-x^2}}{x}\right)$

8. $$3\left[\ln x - 2\ln\left(x^2+1\right)\right] + 2\ln 5 = 3\ln x - 6\ln\left(x^2+1\right) + \ln 5^2$$
$$= \ln x^3 - \ln\left(x^2+1\right)^6 + \ln 25 = \ln\left[\frac{25x^3}{\left(x^2+1\right)^6}\right]$$

9. $g(x) = \ln\sqrt{2x} = \ln(2x)^{1/2} = \frac{1}{2}\ln 2x$

$g'(x) = \frac{1}{2}\frac{1}{2x}(2) = \frac{1}{2x}$

10. $f(x) = \ln\left(3x^2 + 2x\right)$

$f'(x) = \frac{1}{3x^2+2x}(6x+2) = \frac{6x+2}{3x^2+2x}$

11. $f(x) = x\sqrt{\ln x}$

$$f'(x) = \left(\frac{x}{2}\right)(\ln x)^{-1/2}\left(\frac{1}{x}\right) + \sqrt{\ln x}$$
$$= \frac{1}{2\sqrt{\ln x}} + \sqrt{\ln x} = \frac{1 + 2\ln x}{2\sqrt{\ln x}}$$

12. $f(x) = \left[\ln(2x)\right]^3$

$f'(x) = 3\left[\ln(2x)\right]^2 \frac{1}{2x}(2) = \frac{3}{x}(\ln 2x)^2$

13. $$y = \ln\sqrt{\frac{x^2+4}{x^2-4}} = \frac{1}{2}\left[\ln\left(x^2+4\right) - \ln\left(x^2-4\right)\right]$$
$$y' = \frac{1}{2}\left[\frac{1}{x^2+4} \cdot 2x - \frac{1}{x^2-4} \cdot 2x\right]$$
$$= \frac{1}{2}\left[\frac{2x^3 - 8x - 2x^3 - 8x}{x^4 - 16}\right]$$
$$= -\frac{8x}{x^4 - 16}$$

14. $y = \ln\left(\frac{4x}{x-6}\right) = \ln 4 + \ln x - \ln(x - 6)$

$y' = \frac{1}{x} - \frac{1}{x-6} = \frac{6}{x(6-x)}$

15. $y = \dfrac{1}{\ln(1-7x)} = \left[\ln(1-7x)\right]^{-1}$

$y' = -\left[\ln(1-7x)\right]^{-2}\dfrac{1}{1-7x}(-7)$

$= \dfrac{7}{(1-7x)\left[\ln(1-7x)\right]^2}$

16. $y = \dfrac{\ln 5x}{1-x}$

$y' = \dfrac{(1-x)\frac{1}{x} - \ln 5x(-1)}{(1-x)^2} = \dfrac{\ln 5x + \frac{1}{x} - 1}{(1-x)^2}$

17. $y = \ln(2+x) + \dfrac{2}{2+x}, \quad (-1, 2)$

$y' = \dfrac{1}{2+x} - \dfrac{2}{(2+x)^2}$

$y'(-1) = 1 - 2 = -1$

Tangent line: $y - 2 = -1(x+1)$

$y = -x + 1$

18. $y = 2x^2 + \ln x^2 = 2x^2 + 2\ln x, (1, 2)$

$y' = 4x + \dfrac{2}{x}$

$y'(1) = 4 + 2 = 6$

Tangent line: $y - 2 = 6(x-1)$

$y = 6x - 4$

19. $y = x^2\sqrt{x-1}, x > 1$

$\ln y = \ln\left[x^2(x-1)^{1/2}\right] = 2\ln x + \dfrac{1}{2}\ln(x-1)$

$\dfrac{y'}{y} = \dfrac{2}{x} + \dfrac{1}{2(x-1)} = \dfrac{4(x-1)+x}{2x(x-1)} = \dfrac{5x-4}{2x(x-1)}$

$y' = y\left[\dfrac{5x-4}{2x(x-1)}\right] = x^2\sqrt{x-1}\left[\dfrac{5x-4}{2x(x-1)}\right] = \dfrac{5x^2-4x}{2\sqrt{x-1}}$

20. $y = \dfrac{x+2}{\sqrt{3x-2}}, x > \dfrac{2}{3}$

$\ln y = \ln\left[\dfrac{x+2}{(3x-2)^{1/2}}\right] = \ln(x+2) - \dfrac{1}{2}\ln(3x-2)$

$\dfrac{y'}{y} = \dfrac{1}{x+2} - \dfrac{3}{6x-4} = \dfrac{6x-4-3x-6}{(x+2)(6x-4)} = \dfrac{3x-10}{(x+2)(6x-4)}$

$y' = y\left[\dfrac{3x-10}{(x+2)(6x-4)}\right] = \dfrac{x+2}{\sqrt{3x-2}}\left[\dfrac{3x-10}{(x+2)(6x-4)}\right] = \dfrac{3x-10}{2(3x-2)^{3/2}}$

21. $u = 7x - 2, du = 7\,dx$

$\displaystyle\int \frac{1}{7x-2}\,dx = \frac{1}{7}\int\frac{1}{7x-2}(7)\,dx = \frac{1}{7}\ln|7x-2| + C$

22. $\displaystyle\int \frac{x^2}{x^3+1}\,dx = \frac{1}{3}\int\frac{1}{x^3+1}(3x^2)\,dx = \frac{1}{3}\ln|x^3+1| + C$

23. $\displaystyle\int \frac{\sin x}{1+\cos x}\,dx = -\int\frac{-\sin x}{1+\cos x}\,dx = -\ln|1+\cos x| + C$

24. $u = \ln x, du = \dfrac{1}{x}\,dx$

$\displaystyle\int \frac{\ln\sqrt{x}}{x}\,dx = \frac{1}{2}\int(\ln x)\left(\frac{1}{x}\right)dx = \frac{1}{4}(\ln x)^2 + C$

25. $\displaystyle\int \frac{x^2-6x+1}{x^2+1}\,dx = \int\frac{x^2+1-6x}{x^2+1}\,dx$

$\displaystyle= \int\left(1 - \frac{6x}{x^2+1}\right)dx$

$= x - 3\ln(x^2+1) + C$

26. Let $u = 2\sqrt{x} + 5, du = \dfrac{1}{\sqrt{x}}\,dx.$

$\displaystyle\int\frac{dx}{\sqrt{x}(2\sqrt{x}+5)} = \int\frac{1}{2\sqrt{x}+5}\cdot\frac{1}{\sqrt{x}}\,dx$

$= \ln\left(2\sqrt{x}+5\right) + C$

27. $\int_1^4 \frac{2x+1}{2x}\,dx = \int_1^4 \left(1 + \frac{1}{2x}\right)dx$

$= \left[x + \frac{1}{2}\ln|x|\right]_1^4$

$= 4 + \frac{1}{2}\ln 4 - 1 = 3 + \ln 2$

28. $\int_1^e \frac{\ln x}{x}\,dx = \int_1^e (\ln x)^1\left(\frac{1}{x}\right)dx = \left[\frac{1}{2}(\ln x)^2\right]_1^e = \frac{1}{2}$

29. $\int_0^{\pi/3} \sec\theta\,d\theta = \left[\ln|\sec\theta + \tan\theta|\right]_0^{\pi/3} = \ln\left(2 + \sqrt{3}\right)$

30. $\int_0^{\pi} \tan\frac{\theta}{3}\,d\theta = 3\int_0^{\pi}\tan\frac{\theta}{3}\left(\frac{1}{3}\right)d\theta$

$= \left[-3\ln\left|\cos\frac{\theta}{3}\right|\right]_0^{\pi}$

$= -3\ln\left(\frac{1}{2}\right) + 3\ln(1)$

$= 3\ln 2$

31. $A = \int_3^5 \frac{6x^2}{x^3 - 2}\,dx$

Let $u = x^3 - 2$, $du = 3x^2\,dx$.

$A = 2\int_3^5 \frac{1}{x^3 - 2}\left(3x^2\,dx\right)$

$= 2\left[\ln\left(x^3 - 2\right)\right]_3^5$

$= 2\ln 123 - 2\ln 25 = 2\ln\frac{123}{25} \approx 3.187$

32. $A = \int_2^6 \left(x + \csc\frac{\pi x}{12}\right)dx$

$= \left[\frac{x^2}{2} - \frac{12}{\pi}\ln\left|\csc\frac{\pi x}{12} + \cot\frac{\pi x}{12}\right|\right]_2^6$

$= \left[18 - \frac{12}{\pi}\ln(1 + 0)\right] - \left[2 - \frac{12}{\pi}\ln\left(2 + \sqrt{3}\right)\right]$

$= 16 + \frac{12}{\pi}\ln\left(2 + \sqrt{3}\right) \approx 21.0304$

33. (a) $f(x) = \frac{1}{2}x - 3$

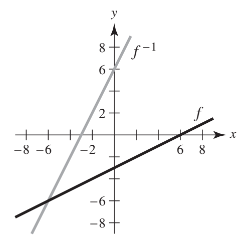
$y = \frac{1}{2}x - 3$

$2(y + 3) = x$

$2(x + 3) = y$

$f^{-1}(x) = 2x + 6$

(b)



(c) $f^{-1}(f(x)) = f^{-1}\left(\frac{1}{2}x - 3\right) = 2\left(\frac{1}{2}x - 3\right) + 6 = x$

$f\left(f^{-1}(x)\right) = f(2x + 6) = \frac{1}{2}(2x + 6) - 3 = x$

(d) Domain f: all real numbers; Range f: all real numbers

Domain f^{-1}: all real numbers; Range f^{-1}: all real numbers

34. (a) $f(x) = 5x - 7$

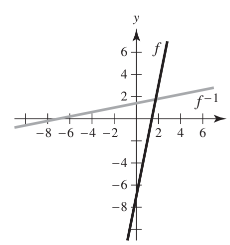
$y = 5x - 7$

$\frac{y + 7}{5} = x$

$\frac{x + 7}{5} = y$

$f^{-1}(x) = \frac{x + 7}{5}$

(b)



(c) $f^{-1}(f(x)) = f^{-1}(5x - 7) = \frac{(5x - 7) + 7}{5} = x$

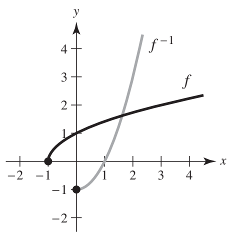
$f\left(f^{-1}(x)\right) = f\left(\frac{x + 7}{5}\right) = 5\left(\frac{x + 7}{5}\right) - 7 = x$

(d) Domain f: all real numbers; Range f: all real numbers

Domain f^{-1}: all real numbers; Range f^{-1}: all real numbers

35. (a)
$$f(x) = \sqrt{x+1}$$
$$y = \sqrt{x+1}$$
$$y^2 - 1 = x$$
$$x^2 - 1 = y$$
$$f^{-1}(x) = x^2 - 1, \quad x \geq 0$$

(b)



(c) $f^{-1}(f(x)) = f^{-1}\left(\sqrt{x+1}\right) = \sqrt{\left(x^2-1\right)^2} - 1 = x$

$f(f^{-1}(x)) = f\left(x^2 - 1\right) = \sqrt{\left(x^2-1\right)+1}$

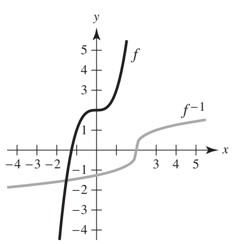
$= \sqrt{x^2} = x$ for $x \geq 0$.

(d) Domain f: $x \geq -1$; Range f: $y \geq 0$

Domain f^{-1}: $x \geq 0$; Range f^{-1}: $y \geq -1$

36. (a)
$$f(x) = x^3 + 2$$
$$y = x^3 + 2$$
$$\sqrt[3]{y-2} = x$$
$$\sqrt[3]{x-2} = y$$
$$f^{-1}(x) = \sqrt[3]{x-2}$$

(b)



(c) $f^{-1}(f(x)) = f^{-1}\left(x^3+2\right) = \sqrt[3]{\left(x^3+2\right)-2} = x$

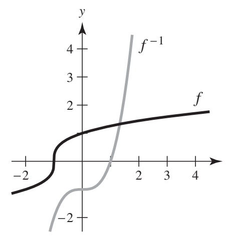
$f(f^{-1}(x)) = f\left(\sqrt[3]{x-2}\right) = \left(\sqrt[3]{x-2}\right)^3 + 2 = x$

(d) Domain f: all real numbers; Range f: all real numbers

Domain f^{-1}: all real numbers; Range f^{-1}: all real numbers

37. (a)
$$f(x) = \sqrt[3]{x+1}$$
$$y = \sqrt[3]{x+1}$$
$$y^3 - 1 = x$$
$$x^3 - 1 = y$$
$$f^{-1}(x) = x^3 - 1$$

(b)



(c) $f^{-1}(f(x)) = f^{-1}\left(\sqrt[3]{x+1}\right)$

$= \left(\sqrt[3]{x+1}\right)^3 - 1 = x$

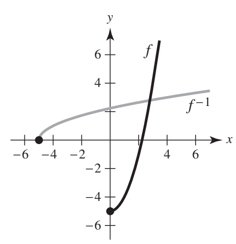
$f(f^{-1}(x)) = f\left(x^3-1\right) = \sqrt[3]{\left(x^3-1\right)+1} = x$

(d) Domain f: all real numbers; Range f: all real numbers

Domain f^{-1}: all real numbers; Range f^{-1}: all real numbers

38. (a)
$$f(x) = x^2 - 5, \quad x \geq 0$$
$$y = x^2 - 5$$
$$\sqrt{y+5} = x$$
$$\sqrt{x+5} = y$$
$$f^{-1}(x) = \sqrt{x+5}$$

(b)



(c) $f^{-1}(f(x)) = f^{-1}\left(x^2-5\right)$

$= \sqrt{\left(x^2-5\right)+5} = x$ for $x \geq 0$.

$f(f^{-1}(x)) = f\left(\sqrt{x+5}\right) = \left(\sqrt{x+5}\right)^2 - 5 = x$

(d) Domain f: $x \geq 0$; Range f: $y \geq -5$

Domain f^{-1}: $x \geq -5$; Range f^{-1}: $y \geq 0$

39. $f(x) = x^3 + 2, \quad a = -1$

$f'(x) = 3x^2 > 0$

f is monotonic (increasing) on $(-\infty, \infty)$ therefore f has an inverse.

$f(-3^{1/3}) = -1 \Rightarrow f^{-1}(-1) = -3^{1/3}$

$f'(-3^{1/3}) = 3^{2/3}$

$$(f^{-1})'(-1) = \frac{1}{f'(f^{-1}(-1))} = \frac{1}{f'(-3^{1/3})} = \frac{1}{3(3^{2/3})} = \frac{1}{3^{5/3}}$$

40. $f(x) = x\sqrt{x-3}, \quad a = 4$

$$f'(x) = \frac{1}{2}x\frac{1}{\sqrt{x-3}} + \sqrt{x-3} > 0$$

f is monotonic (increasing) on $[3, \infty)$ therefore f has an inverse.

$f(4) = 4 \Rightarrow f^{-1}(4) = 4$

$f'(4) = 2 + 1 = 3$

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(4)} = \frac{1}{3}$$

41. $f(x) = \tan x, \quad a = \dfrac{\sqrt{3}}{3}, -\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}$

$$f'(x) = \sec^2 x > 0 \text{ on } \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$$

f is monotonic (increasing) on $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$ therefore f has an inverse.

$$f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3} \Rightarrow f^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

$$f'\left(\frac{\pi}{6}\right) = \frac{4}{3}$$

$$(f^{-1})\left(\frac{\sqrt{3}}{3}\right) = \frac{1}{f'\left(f^{-1}\left(\frac{\sqrt{3}}{3}\right)\right)} = \frac{1}{f'\left(\frac{\pi}{6}\right)} = \frac{1}{\left(\frac{4}{3}\right)} = \frac{3}{4}$$

42. $f(x) = \cos x, \quad a = 0, 0 \le x \le \pi$

$f'(x) = -\sin x < \text{on } (0, \pi)$

f is monotonic (decreasing) on $[0, \pi]$ therefore f has an inverse.

$$f\left(\frac{\pi}{2}\right) = 0 \Rightarrow f^{-1}(0) = \frac{\pi}{2}$$

$$f'\left(\frac{\pi}{2}\right) = -1$$

$$(f^{-1})(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'\left(\frac{\pi}{2}\right)} = \frac{1}{-1} = -1$$

43.

$$\begin{aligned} e^{3x} &= 30 \\ \ln e^{3x} &= \ln 30 \\ 3x &= \ln 30 \\ x &= \frac{1}{3}\ln 30 \approx 1.134 \end{aligned}$$

44.

$$\begin{aligned} -4 + 3e^{-2x} &= 6 \\ 3e^{-2x} &= 10 \\ e^{-2x} &= \tfrac{10}{3} \\ \ln e^{-2x} &= \ln \tfrac{10}{3} \\ -2x &= \ln \tfrac{10}{3} \\ x &= -\tfrac{1}{2}\ln \tfrac{10}{3} \approx -0.602 \end{aligned}$$

45.

$$\begin{aligned} \ln\sqrt{x+1} &= 2 \\ \sqrt{x+1} &= e^2 \\ x + 1 &= e^4 \\ x &= e^4 - 1 \approx 53.598 \end{aligned}$$

46.

$$\begin{aligned} \ln x + \ln(x-3) &= 0 \\ \ln x(x-3) &= 0 \\ x(x-3) &= e^0 \\ x^2 - 3x - 1 &= 0 \\ x &= \frac{3 \pm \sqrt{13}}{2} \\ x &= \frac{3 + \sqrt{13}}{2} \end{aligned}$$

only because $\dfrac{3 - \sqrt{13}}{2} < 0.$

47. $g(t) = t^2e^t$

$g'(t) = t^2e^t + 2te^t = te^t(t + 2)$

48.

$$\begin{aligned} g(x) &= \ln\left(\frac{e^x}{1 + e^x}\right) \\ &= \ln e^x - \ln(1 + e^x) = x - \ln(1 + e^x) \end{aligned}$$

$$g'(x) = 1 - \frac{e^x}{1 + e^x} = \frac{1}{1 + e^x}$$

49. $y = \sqrt{e^{2x} + e^{-2x}}$

$$y' = \frac{1}{2}(e^{2x} + e^{-2x})^{-1/2}(2e^{2x} - 2e^{-2x}) = \frac{e^{2x} - e^{-2x}}{\sqrt{e^{2x} + e^{-2x}}}$$

50. $h(z) = e^{-z^2/2}$

$h'(z) = -ze^{-z^2/2}$

51. $g(x) = \dfrac{x^3}{e^{2x}}$

$$g'(x) = \frac{e^{2x}(3x^2) - x^3(2e^{2x})}{(e^{2x})^2}$$

$$= \frac{3x^2 - 2x^3}{e^{2x}}$$

52. $y = 3e^{-3/t}$

$$y' = 3e^{-3/t}(3t^{-2}) = \frac{9e^{-3/t}}{t^2}$$

53. $f(x) = e^{6x}, (0, 1)$

$f'(x) = 6e^{6x}$

$f'(0) = 6e^0 = 6$

Tangent line: $y - 1 = 6(x - 0)$

$y = 6x + 1$

54. $h(x) = -xe^{2-x}, \ (2, -2)$

$h'(x) = (-x)(-1)e^{2-x} + (-e)^{2-x}$

$= e^{2-x}(x - 1)$

$h'(2) = 1$

Tangent line: $y + 2 = 1(x - 2)$

$y = x - 4$

55. $f(x) = (x + 1)e^{-x}$

$f'(x) = -(x + 1)e^{-x} + e^{-x} = -xe^{-x}$

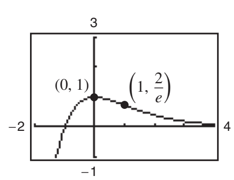
$f'(x) = 0$ when $x = 0$.

$f''(x) = xe^{-x} - e^{-x} = (x - 1)e^{-x}$

$f''(x) = 0$ when $x = 1$.

Relative maximum: $(0, 1)$

Point of inflection: $\left(1, \dfrac{2}{e}\right)$



56. $g(x) = \dfrac{1}{\sqrt{2\pi}} e^{-(x-5)^2/2}$

$$g'(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-5)^2/2}\left[-(x - 5)\right]$$

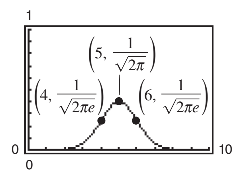
$g'(x) = 0$ when $x = 5$.

$$g''(x) = \frac{1}{\sqrt{2\pi}}(x - 4)(x - 6)e^{-(x-5)^2/2}$$

$g''(x) = 0$ when $x = 4$ and $x = 6$.

Relative maximum: $\left(5, \dfrac{1}{\sqrt{2\pi}}\right) \approx (5, 0.3989)$

Points of inflection: $(4, 0.2420), (6, 0.2420)$



57. $\displaystyle\int xe^{1-x^2}\,dx = -\frac{1}{2}\int e^{1-x^2}(-2x)\,dx = -\frac{1}{2}e^{1-x^2} + C$

58. Let $u = x^3 + 1$, $du = 3x^2dx$.

$$\int x^2e^{x^3+1}dx = \frac{1}{3}\int e^{x^3+1}(3x^2)\,dx = \frac{1}{3}e^{x^3+1} + C$$

59. $\displaystyle\int \frac{e^{4x} - e^{2x} + 1}{e^x}\,dx = \int\left(e^{3x} - e^x + e^{-x}\right)dx$

$$= \frac{1}{3}e^{3x} - e^x - e^{-x} + C$$

$$= \frac{e^{4x} - 3e^{2x} - 3}{3e^x} + C$$

60. Let $u = e^{2x} + e^{-2x}$, $du = \left(2e^{2x} - e^{-2x}\right)dx$.

$$\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx = \frac{1}{2}\int \frac{2e^{2x} - 2e^{-2x}}{e^{2x} + e^{-2x}}\,dx$$

$$= \frac{1}{2}\ln\left(e^{2x} + e^{-2x}\right) + C$$

61. $\displaystyle\int_0^1 xe^{-3x^2}\,dx = -\frac{1}{6}\int_0^1 e^{-3x^2}(-6x\,dx)$

$$= \left[-\frac{1}{6}e^{-3x^2}\right]_0^1$$

$$= -\frac{1}{6}\left[e^{-3} - 1\right]$$

$$= \frac{1}{6}\left(1 - \frac{1}{e^3}\right) \approx 0.158$$

62. $\int_{1/2}^{2} \frac{e^{1/x}}{x^2}\,dx$

Let $u = \frac{1}{x}$, $du = \frac{-1}{x^2}\,dx$.

$x = \frac{1}{2} \Rightarrow u = 2$, $x = 2 \Rightarrow u = \frac{1}{2}$

$$-\int_{2}^{1/2} e^u\,du = \left[-e^u\right]_2^{1/2} = -e^{1/2} + e^2 = e^2 - \sqrt{e} \approx 5.740$$

63. $\int_{1}^{3} \frac{e^x}{e^x - 1}\,dx$

Let $u = e^x - 1$, $du = e^x dx$.

$$\int_1^3 \frac{e^x}{e^x - 1}\,dx = \left[\ln\left|e^x - 1\right|\right]_1^3$$
$$= \ln\left(e^3 - 1\right) - \ln\left(e - 1\right)$$
$$= \ln\left(\frac{e^3 - 1}{e - 1}\right)$$
$$= \ln\left(e^2 + e + 1\right) \approx 2.408$$

64. $\int_{1/4}^{5} \frac{e^{4x} + 1}{4x + e^{4x}}\,dx$

Let

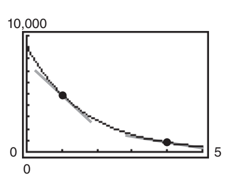
$u = 4x + e^{4x}$, $du = \left(4 + 4e^{4x}\right)dx = 4\left(1 + e^{4x}\right)dx$.

$$\int_{1/4}^{5} \frac{e^{4x} + 1}{4x + e^{4x}}\,dx = \frac{1}{4}\int_{1/4}^{5} \frac{1}{4x + e^{4x}}\left[4\left(1 + e^{4x}\right)dx\right]$$
$$= \left[\frac{1}{4}\ln\left(4x + e^{4x}\right)\right]_{1/4}^{5}$$
$$= \frac{1}{4}\ln\left(20 + e^{20}\right) - \frac{1}{4}\ln(1 + e)$$
$$= \frac{1}{4}\ln\left(\frac{e^{20} + 20}{1 + e}\right) \approx 4.672$$

65. $A = \int_0^2 2e^{-x}\,dx = \left[-2e^{-x}\right]_0^2$

$$= -2e^{-2} + 2 \approx 1.729$$

66. (a), (c)



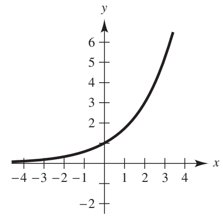
(b) $V = 9000e^{-0.6t}$, $0 \le t \le 5$

$V'(t) = -5400e^{-0.6t}$

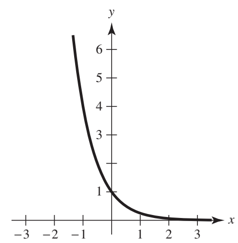
$V'(1) = -2963.58$ dollars/year

$V'(4) = -489.88$ dollars/year

67. $y = 3^{x/2}$



68. $y = \left(\frac{1}{4}\right)^x$



69.
$$4^{1-x} = 52$$
$$(1 - x)\ln 4 = \ln 52$$
$$1 - x = \frac{\ln 52}{\ln 4}$$
$$x = 1 - \frac{\ln 52}{\ln 4} \approx -1.850$$

70.
$$2\left(3^{x+2}\right) = 17$$
$$3^{x+2} = \frac{17}{2}$$
$$(x + 2)\ln 3 = \ln\left(\frac{17}{2}\right)$$
$$x + 2 = \frac{\ln(17/2)}{\ln 3}$$
$$x = -2 + \frac{\ln(17/2)}{\ln 3} \approx -0.052$$

71.
$$\left(1 + \frac{0.03}{12}\right)^{12t} = 3$$
$$12t\ln\left(1 + \frac{0.03}{12}\right) = \ln 3$$
$$t = \frac{\ln 3}{12\ln\left(1 + \frac{0.03}{12}\right)} = \frac{\ln 3}{12\ln(1.0025)}$$
$$\approx 36.666$$

72. $\left(1 + \frac{0.06}{365}\right)^{365t} = 2$

$$365t \ln\left(1 + \frac{0.06}{365}\right) = \ln 2$$

$$t = \frac{\ln 2}{365 \ln\left(1 + \frac{0.06}{365}\right)} \approx 11.553$$

73. $\log_6(x + 1) = 2$

$$6^2 = x + 1$$

$$x = 36 - 1 = 35$$

74. $\log_5 x^2 = 4.1$

$$5^{4.1} = x^2$$

$$x = \pm\sqrt{5^{4.1}} \approx \pm 27.095$$

75. $f(x) = 3^{x-1}$

$$f'(x) = 3^{x-1} \ln 3$$

76. $f(x) = 5^{3x}$

$$f'(x) = 3(\ln 5)\, 5^{3x} = 3(\ln 5)\, 125^x$$

77. $g(t) = \frac{2^{3t}}{t^2}$

$$g'(t) = \frac{t^2\left(2^{3t}\, 3 \ln 2\right) - 2^{3t}\,(2t)}{t^4}$$

$$= \frac{3t \cdot 2^{3t} \ln 2 - 2^{3t}(2)}{t^3}$$

$$= \frac{8^t(3t \ln 2 - 2)}{t^3}$$

$$= \frac{8^t(t \ln 8 - 2)}{t^3}$$

78. $f(x) = x\left(4^{-3x}\right)$

$$f'(x) = x(-3)(\ln 4)4^{-3x} + 4^{-3x}$$

$$= 4^{-3x}(-3x \ln 4 + 1)$$

$$= \frac{1 - 3x \ln 4}{64^x}$$

79. $g(x) = \log_3\sqrt{1 - x} = \frac{1}{2}\log_3(1 - x)$

$$g'(x) = \left(\frac{1}{2}\right)\frac{-1}{(1 - x)\ln 3} = \frac{1}{2(x - 1)\ln 3}$$

80. $h(x) = \log_5 \frac{x}{x - 1} = \log_5 x - \log_5(x - 1)$

$$h'(x) = \frac{1}{\ln 5}\left[\frac{1}{x} - \frac{1}{x - 1}\right] = \frac{1}{\ln 5}\left[\frac{-1}{x(x - 1)}\right]$$

81. $y = x^{2x+1}$

$$\ln y = (2x + 1)\ln x$$

$$\frac{y'}{y} = \frac{2x + 1}{x} + 2 \ln x$$

$$y' = y\left(\frac{2x + 1}{x} + 2 \ln x\right) = x^{2x+1}\left(\frac{2x + 1}{x} + 2 \ln x\right)$$

82. $y = (3x + 5)^x$

$$\ln y = x \ln(3x + 5)$$

$$\frac{y'}{y} = \frac{3x}{3x + 5} + \ln(3x + 5)$$

$$y' = (3x + 5)^x\left[\frac{3x}{3x + 5} + \ln(3x + 5)\right]$$

83. $\int (x + 1)5^{(x+1)^2}\, dx = \left(\frac{1}{2}\right)\frac{1}{\ln 5}5^{(x+1)^2} + C$

84. $\int \frac{2^{-1/t}}{t^2}\, dt = \frac{1}{\ln 2}2^{-1/t} + C$

85. $\int_1^2 6^x\, dx = \left.\frac{6^x}{\ln 6}\right]_1^2 = \frac{1}{\ln 6}(36 - 6) = \frac{30}{\ln 6}$

86. $\int_{-4}^0 9^{x/2}\, dx = \left.\frac{2\left(9^{x/2}\right)}{\ln 9}\right]_{-4}^0$

$$= \frac{2}{\ln 9}\left(1 - 9^{-2}\right)$$

$$= \frac{2}{\ln 9}\left(\frac{80}{81}\right)$$

$$= \frac{80}{81 \ln 3}$$

87. (a) $A = 550\left(1 + \frac{0.01}{12}\right)^{12(11)} = \$613.92$

(b) $10{,}000 = Pe^{0.05(15)} \Rightarrow P = 10{,}000e^{-0.75}$

$$= \$4723.67$$

(c) $2P = Pe^{r(10)}$

$$2 = e^{10r}$$

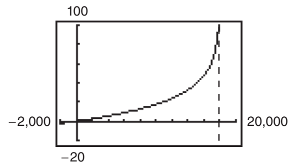
$$\ln 2 = 10r$$

$$r = \frac{\ln 2}{10} \approx 0.0693 \text{ or } 6.93\%$$

88. $t = 50\log_{10}\left(\dfrac{18{,}000}{18{,}000 - h}\right)$

(a) Domain: $0 \le h < 18{,}000$

(b)



Vertical asymptote: $h = 18{,}000$

(c)

$$t = 50\log_{10}\left(\frac{18{,}000}{18{,}000 - h}\right)$$
$$10^{t/50} = \frac{18{,}000}{18{,}000 - h}$$
$$18{,}000 - h = 18{,}000\left(10^{-t/50}\right)$$
$$h = 18{,}000\left(1 - 10^{-t/50}\right)$$

$\dfrac{dh}{dt} = 360\ln 10\left(\dfrac{1}{10}\right)^{t/50}$ is greatest when $t = 0$.

89. $\lim_{x\to 1}\left[\dfrac{(\ln x)^2}{x - 1}\right] = \lim_{x\to 1}\left[\dfrac{2(1/x)\ln x}{1}\right] = 0$

90. $\lim_{x\to 0}\dfrac{\sin \pi x}{\sin 5\pi x} = \lim_{x\to 0}\dfrac{\pi\cos \pi x}{5\pi\cos 5\pi x} = \dfrac{1}{5}$

91. $\lim_{x\to\infty}\dfrac{e^{2x}}{x^2} = \lim_{x\to\infty}\dfrac{2e^{2x}}{2x} = \lim_{x\to\infty}\dfrac{4e^{2x}}{2} = \infty$

92. $\lim_{x\to\infty} xe^{-x^2} = \lim_{x\to\infty}\dfrac{x}{e^{x^2}} = \lim_{x\to\infty}\dfrac{1}{2xe^{x^2}} = 0$

93. $y = \lim_{x\to\infty}(\ln x)^{2/x}$

$\ln y = \lim_{x\to\infty}\dfrac{2\ln(\ln x)}{x} = \lim_{x\to\infty}\left[\dfrac{2/(x\ln x)}{1}\right] = 0$

Because $\ln y = 0$, $y = 1$.

94. $y = \lim_{x\to 1^+}(x - 1)^{\ln x}$

$$\ln y = \lim_{x\to 1^+}\left[(\ln x)\ln(x - 1)\right]$$
$$= \lim_{x\to 1^+}\left[\frac{\ln(x - 1)}{\frac{1}{\ln x}}\right] = \lim_{x\to 1^+}\left[\frac{\frac{1}{x - 1}}{\left(\frac{1}{x}\right)\frac{-1}{\ln^2 x}}\right] = \lim_{x\to 1^+}\left[\frac{\frac{-\ln^2 x}{x - 1}}{x}\right] = \lim_{x\to 1^+}\left[\frac{-2\left(\frac{1}{x}\right)(\ln x)}{\frac{1}{x^2}}\right]$$
$$= \lim_{x\to 1^+} 2x(\ln x) = 0$$

Because $\ln y = 0$, $y = 1$.

95. $\lim_{n\to\infty} 1000\left(1 + \dfrac{0.09}{n}\right)^n = 1000\lim_{n\to\infty}\left(1 + \dfrac{0.09}{n}\right)^n$

Let $y = \lim_{n\to\infty}\left(1 + \dfrac{0.09}{n}\right)^n$.

$$\ln y = \lim_{n\to\infty} n\ln\left(1 + \frac{0.09}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1 + \frac{0.09}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{\frac{-0.09/n^2}{1 + (0.09/n)}}{-\frac{1}{n^2}}\right) = \lim_{n\to\infty}\frac{0.09}{1 + \left(\frac{0.09}{n}\right)} = 0.09$$

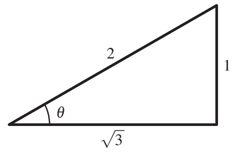
So, $\ln y = 0.09 \Rightarrow y = e^{0.09}$ and $\lim_{n\to\infty} 1000\left(1 + \dfrac{0.09}{n}\right)^n = 1000e^{0.09} \approx 1094.17$.

96. $\lim_{x\to\infty}\left(1 + \dfrac{4}{x}\right)^x = \lim_{x\to\infty}\left(1 + \dfrac{4}{x}\right)^{\left(\frac{x}{4}\right)4} = e^4$

97. (a) Let $\theta = \arcsin \frac{1}{2}$

$\sin \theta = \frac{1}{2}$

$\sin\left(\arcsin \frac{1}{2}\right) = \sin \theta = \frac{1}{2}.$



(b) Let $\theta = \arcsin \frac{1}{2}$

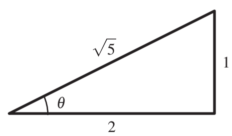
$\sin \theta = \frac{1}{2}$

$\cos\left(\arcsin \frac{1}{2}\right) = \cos \theta = \frac{\sqrt{3}}{2}.$

98. (a) Let $\theta = \operatorname{arccot} 2$

$\cot \theta = 2$

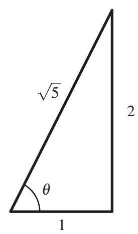
$\tan(\operatorname{arccot} 2) = \tan \theta = \frac{1}{2}.$



(b) Let $\theta = \operatorname{arcsec} \sqrt{5}$

$\sec \theta = \sqrt{5}$

$\cos\left(\operatorname{arcsec} \sqrt{5}\right) = \cos \theta = \frac{1}{\sqrt{5}}.$



99. $y = \operatorname{arccsc} 2x^2$

$$y' = \frac{-4x}{\left|2x^2\right|\sqrt{\left(2x^2\right)^2 - 1}} = \frac{-4x}{2x^2\sqrt{4x^4 - 1}} = \frac{-2}{x\sqrt{4x^4 - 1}}$$

100. $y = \frac{1}{2} \arctan e^{2x}$

$$y' = \frac{1}{2}\left(\frac{1}{1 + e^{4x}}\right)\left(2e^{2x}\right) = \frac{e^{2x}}{1 + e^{4x}}$$

101. $y = x \operatorname{arcsec} x$

$$y' = \frac{x}{|x|\sqrt{x^2 - 1}} + \operatorname{arcsec} x$$

102. $y = \sqrt{x^2 - 4} - 2 \operatorname{arcsec} \frac{x}{2}, \quad 2 < x < 4$

$$y' = \frac{x}{\sqrt{x^2 - 4}} - \frac{1}{\left(|x|/2\right)\sqrt{(x/2)^2 - 1}} = \frac{x}{\sqrt{x^2 - 4}} - \frac{4}{|x|\sqrt{x^2 - 4}} = \frac{x^2 - 4}{|x|\sqrt{x^2 - 4}} = \frac{\sqrt{x^2 - 4}}{x}$$

103. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1 - x^2} \arcsin x$

$$y' = \frac{2x \arcsin x}{\sqrt{1 - x^2}} + (\arcsin x)^2 - 2 + \frac{2\sqrt{1 - x^2}}{\sqrt{1 - x^2}} - \frac{2x}{\sqrt{1 - x^2}} \arcsin x = (\arcsin x)^2$$

104. $y = \tan(\arcsin x) = \frac{x}{\sqrt{1 - x^2}}$

$$y' = \frac{\left(1 - x^2\right)^{1/2} + x^2\left(1 - x^2\right)^{-1/2}}{1 - x^2} = \left(1 - x^2\right)^{-3/2}$$

105. Let $u = e^{2x}$, $du = 2e^{2x}\,dx$.

$$\int \frac{1}{e^{2x} + e^{-2x}}\,dx = \int \frac{e^{2x}}{1 + e^{4x}}\,dx = \frac{1}{2}\int \frac{1}{1 + \left(e^{2x}\right)^2}\left(2e^{2x}\right)dx = \frac{1}{2} \arctan\left(e^{2x}\right) + C$$

106. Let $u = 5x$, $du = 5\,dx$.

$$\int \frac{1}{3 + 25x^2}\,dx = \frac{1}{5}\int \frac{1}{\left(\sqrt{3}\right)^2 + (5x)^2}(5)\,dx = \frac{1}{5\sqrt{3}} \arctan \frac{5x}{\sqrt{3}} + C$$

107. Let $u = x^2, du = 2x\,dx.$

$$\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1-(x^2)^2}}(2x)\,dx = \frac{1}{2}\arcsin x^2 + C$$

108. $$\int \frac{1}{x\sqrt{9x^2-49}}\,dx = \int \frac{1}{3x\sqrt{(3x)^2-7^2}}\,3\,dx = \frac{1}{7}\operatorname{arcsec}\frac{|3x|}{7} + C$$

109. Let $u = \arctan\left(\frac{x}{2}\right), du = \frac{2}{4+x^2}\,dx.$

$$\int \frac{\arctan(x/2)}{4+x^2}\,dx = \frac{1}{2}\int\left(\arctan\frac{x}{2}\right)\left(\frac{2}{4+x^2}\right)dx = \frac{1}{4}\left(\arctan\frac{x}{2}\right)^2 + C$$

110. Let $u = \arcsin(2x), du = \frac{2}{\sqrt{1-4x^2}}\,dx.$

$$\int \frac{\arcsin 2x}{\sqrt{1-4x^2}}\,dx = \frac{1}{2}\frac{[\arcsin(2x)]^2}{2} + C = \frac{(\arcsin 2x)^2}{4} + C$$

111. Let $u = 7x, du = 7\,dx.$

$$\begin{aligned}\int_0^{1/7}\frac{dx}{\sqrt{1-49x^2}} &= \frac{1}{7}\int_0^{1/7}\frac{1}{\sqrt{1-(7x)^2}}\,7\,dx\\ &= \frac{1}{7}\left[\arcsin 7x\right]_0^{1/7}\\ &= \frac{1}{7}(\arcsin 1 - \arcsin 0)\\ &= \frac{1}{7}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{14}\end{aligned}$$

112. Let $u = x^3, du = 3x^2\,dx.$

$$\begin{aligned}\int_0^1 \frac{2x^2}{\sqrt{4-x^6}}\,dx &= \frac{2}{3}\int_0^1\frac{1}{\sqrt{4-(x^3)^2}}(3x^2\,dx)\\ &= \left[\frac{2}{3}\arcsin\frac{x^3}{2}\right]_0^1\\ &= \frac{2}{3}\left(\arcsin\frac{1}{2} - \arcsin 0\right)\\ &= \frac{2}{3}\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{9}\end{aligned}$$

113. Let $u = e^{2x}, du = 2e^{2x}\,dx.$

$$\int_{-1}^{2}\frac{10e^{2x}}{25+e^{4x}}\,dx = 5\int_{-1}^{2}\frac{2e^{2x}}{5^2+e^{4x}}\,dx = \left[5\left(\frac{1}{5}\right)\arctan\frac{e^{2x}}{5}\right]_{-1}^{2} = \arctan\frac{e^4}{5} - \arctan\frac{e^{-2}}{5}$$

114. Let $u = \sin x, du = \cos x\,dx.$

$$\int_{\pi/3}^{\pi/2}\frac{\cos x}{\sin x\sqrt{\sin^2 x - 1/4}}\,dx = \left[\frac{1}{(1/2)}\operatorname{arcsec}\frac{|\sin x|}{(1/2)}\right]_{\pi/3}^{\pi/2} = 2\operatorname{arcsec}(2) - 2\operatorname{arcsec}\sqrt{3} = 2\left(\frac{\pi}{3}\right) - 2\operatorname{arcsec}\sqrt{3} \approx 0.1838$$

115. $$\begin{aligned}A &= \int_0^1\frac{4-x}{\sqrt{4-x^2}}\,dx\\ &= 4\int_0^1\frac{1}{\sqrt{4-x^2}}\,dx + \frac{1}{2}\int_0^1(4-x^2)^{-1/2}(-2x)\,dx\\ &= \left[4\arcsin\left(\frac{x}{2}\right) + \sqrt{4-x^2}\right]_0^1 = \left(4\arcsin\left(\frac{1}{2}\right) + \sqrt{3}\right) - 2 = \frac{2\pi}{3} + \sqrt{3} - 2 \approx 1.8264\end{aligned}$$

116. $$A = \int_0^4\frac{6}{16+x^2}\,dx = \left[\frac{6}{4}\arctan\left(\frac{x}{4}\right)\right]_0^4 = \frac{3}{2}(\arctan(1) - \arctan 0) = \frac{3}{2}\left(\frac{\pi}{4}\right) = \frac{3\pi}{8}$$

117. $\cosh^2 x + \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} + \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$= \frac{e^{2x} + e^{-2x}}{2}$$

$$= \cosh 2x$$

118. $\cosh x \cosh y - \sinh x \sinh y = \left(\frac{e^x + e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2}\right) - \left(\frac{e^x - e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2}\right)$

$$= \left(\frac{e^{x+y} + e^{-x+y} + e^{x-y} + e^{-x-y}}{4}\right) - \left(\frac{e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y}}{4}\right)$$

$$= \frac{e^{x-y} + e^{-x+y}}{2}$$

$$= \cosh(x - y)$$

119. $y = \operatorname{sech}(4x - 1)$

$$y' = -\operatorname{sech}(4x - 1)\tanh(4x - 1)(4)$$

$$= -4\operatorname{sech}(4x - 1)\tanh(4x - 1)$$

120. $y = 2x - \cosh\sqrt{x}$

$$y' = 2 - \frac{1}{2\sqrt{x}}\left(\sinh\sqrt{x}\right) = 2 - \frac{\sinh\sqrt{x}}{2\sqrt{x}}$$

121. $y = \coth(8x^2)$

$$y' = -\left(\operatorname{csch}^2(8x^2)\right)(16x)$$

$$= -16x\operatorname{csch}^2(8x^2)$$

122. $y = \ln(\coth x)$

$$y' = \frac{1}{\coth x}\left(-\operatorname{csch}^2 x\right)$$

$$= -\frac{\sinh x}{\cosh x}\frac{1}{\sinh^2 x}$$

$$= -\frac{1}{\cosh x \sinh x}$$

123. $y = \sinh^{-1}(4x)$

$$y' = \frac{4}{\sqrt{(4x)^2 + 1}} = \frac{4}{\sqrt{16x^2 + 1}}$$

124. $y = x\tanh^{-1} 2x$

$$y' = x\left(\frac{2}{1 - 4x^2}\right) + \tanh^{-1} 2x = \frac{2x}{1 - 4x^2} + \tanh^{-1} 2x$$

125. Let $u = x^3$, $du = 3x^2\,dx$.

$$\int x^2\left(\operatorname{sech} x^3\right)^2 dx = \frac{1}{3}\int\left(\operatorname{sech} x^3\right)^2\left(3x^2\right)dx$$

$$= \frac{1}{3}\tanh x^3 + C$$

126. $\int \sinh 6x\,dx = \frac{1}{6}\cosh 6x + C$

127. Let $u = \tanh x$, $du = \operatorname{sech}^2 x\,dx$.

$$\int \frac{\operatorname{sech}^2 x}{\tanh x}\,dx$$

$$\int \frac{1}{\tanh x}\operatorname{sech}^2 x\,dx = \ln\left|\tanh x\right| + C$$

128. Let $u = \operatorname{csch}(3x)$, $du = -3\operatorname{csch}(3x)\coth(3x)\,dx$.

$$\int \operatorname{csch}^4(3x)\coth(3x)\,dx$$

$$-\frac{1}{3}\int \operatorname{csch}^3(3x)\left(-3\operatorname{csch}(3x)\coth(3x)\right)dx = -\frac{\operatorname{csch}^4(3x)}{12} + C$$

129. Let $u = \frac{2}{3}x$, $du = \frac{2}{3}\,dx$.

$$\int \frac{1}{9 - 4x^2}\,dx = \int \frac{1/9}{1 - \left(\frac{4}{9}x^2\right)}\,dx = \frac{1}{6}\tanh^{-1}\left(\frac{2}{3}x\right) + C$$

Alternate solution: $\int \frac{1}{3^2 - (2x)^2}\,dx = \frac{1}{12}\ln\left|\frac{3 + 2x}{3 - 2x}\right| + C$

130. Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{x}{\sqrt{x^4 - 1}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{(x^2)^2 - 1}}(2x)\,dx = \frac{1}{2}\ln\left(x^2 + \sqrt{x^4 - 1}\right) + C$$

131.
$$\int_1^2 \operatorname{sech} 2x \tanh 2x\,dx = \left[-\frac{1}{2}\operatorname{sech} 2x\right]_1^2 = -\frac{1}{2}\operatorname{sech} 4 + \frac{1}{2}\operatorname{sech} 2$$

132.
$$\int_0^1 \sinh^2 x\,dx = \int_0^1 \left(\frac{e^x - e^{-x}}{2}\right)^2 dx$$
$$= \int_0^1 \frac{e^{2x} - 2 + e^{-2x}}{4}\,dx$$
$$= \frac{1}{4}\left[\frac{e^{2x}}{2} - 2x - \frac{e^{-2x}}{2}\right]_0^1$$
$$= \frac{1}{4}\left[\left(\frac{e^2}{2} - 2 - \frac{e^{-2}}{2}\right) - \left(\frac{1}{2} - \frac{1}{2}\right)\right]$$
$$= \frac{1}{8}\left[e^2 - 4 - e^{-2}\right]$$

133. Let $u = 3x$, $du = 3\,dx$.

$$\int_0^1 \frac{3}{\sqrt{9x^2 + 16}}\,dx = \left[\ln\left(3x + \sqrt{9x^2 + 16}\right)\right]_0^1$$
$$= \ln\left(3 + \sqrt{9 + 16}\right) - \ln\left(0 + \sqrt{16}\right)$$
$$= \ln(3 + 5) - \ln 4$$
$$= \ln\left(\frac{8}{4}\right) = \ln 2$$

134. Let $u = 2x$, $du = 2\,dx$, $a = 7$.

$$\int_{-1}^0 \frac{1}{49 - 4x^2}\,dx = \left[\frac{1}{14}\ln\left|\frac{7 + 2x}{7 - 2x}\right|\right]_{-1}^0$$
$$= \frac{1}{14}\left[\ln(1) - \ln\frac{5}{9}\right]$$
$$= -\frac{1}{14}\ln\frac{5}{9}$$
$$= \frac{1}{14}\ln\frac{9}{5}$$

Problem Solving for Chapter 5

1. $f(x) = \dfrac{a + bx}{1 + cx}$

$f(0) = e^0 = 1 \Rightarrow a = 1$

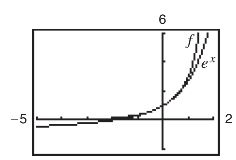
$f(x) = \dfrac{1 + bx}{1 + cx}$

$f'(x) = \dfrac{b - c}{(1 + cx)^2}$

$f'(0) = 1 = b - c$

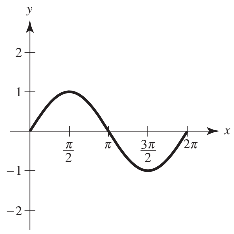
$f''(x) = \dfrac{2c(c - b)}{(1 + cx)^3}$

$f''(0) = 1 = 2c(c - b)$



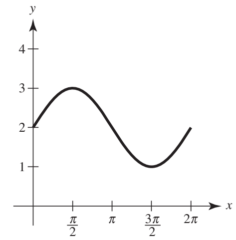
Since $b - c = 1$, $1 = 2c(-1) \Rightarrow c = -\frac{1}{2}$ and therefore, $b = \frac{1}{2}$. So, $f(x) = \dfrac{1 + \left(\frac{1}{2}x\right)}{1 - \left(\frac{1}{2}x\right)}$.

2. (a)



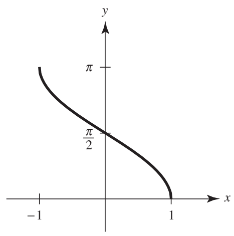
$$\int_0^{\pi} \sin x\,dx = -\int_{\pi}^{2\pi} \sin x\,dx \Rightarrow \int_0^{2\pi} \sin x\,dx = 0$$

(b)



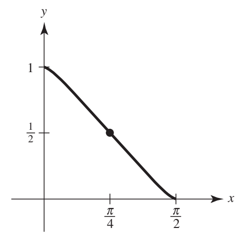
$$\int_0^{2\pi} (\sin x + 2)\,dx = 2(2\pi) = 4\pi$$

(c)



$$\int_{-1}^{1} \arccos x\,dx = 2\left(\frac{\pi}{2}\right) = \pi$$

(d)



$y = \dfrac{1}{1 + (\tan x)^{\sqrt{2}}}$ symmetric with respect to point $\left(\dfrac{\pi}{4}, \dfrac{1}{2}\right)$.

$$\int_0^{\pi/2} \frac{1}{1 + (\tan x)^{\sqrt{2}}}\,dx = \frac{\pi}{2}\left(\frac{1}{2}\right) = \frac{\pi}{4}$$

3.

$$\lim_{x\to\infty}\left(\frac{x + c}{x - c}\right)^x = 9$$

$$\lim_{x\to\infty} x \ln\left(\frac{x + c}{x - c}\right) = \ln 9$$

$$\lim_{x\to\infty} \frac{\ln(x + c) - \ln(x - c)}{1/x} = \ln 9$$

$$\lim_{x\to\infty} \frac{\dfrac{1}{x + c} - \dfrac{1}{x - c}}{-\dfrac{1}{x^2}} = \ln 9$$

$$\lim_{x\to\infty} \frac{-2c}{(x + c)(x - c)}(-x^2) = \ln 9$$

$$\lim_{x\to\infty}\left(\frac{2cx^2}{x^2 - c^2}\right) = \ln 9$$

$$2c = \ln 9$$

$$2c = 2 \ln 3$$

$$c = \ln 3$$

4.

$$\lim_{x\to\infty}\left(\frac{x - c}{x + c}\right)^x = \frac{1}{4}$$

$$\lim_{x\to\infty} x \ln\left(\frac{x - c}{x + c}\right) = \ln \frac{1}{4}$$

$$\lim_{x\to\infty} \frac{\ln(x - c) - \ln(x + c)}{1/x} = -\ln 4$$

$$\lim_{x\to\infty} \frac{\dfrac{1}{x - c} - \dfrac{1}{x + c}}{-\dfrac{1}{x^2}} = -\ln 4$$

$$\lim_{x\to\infty} \frac{2c}{(x - c)(x + c)}(-x^2) = -\ln 4$$

$$\lim_{x\to\infty} \frac{2cx^2}{x^2 - c^2} = \ln 4$$

$$2c = \ln 4$$

$$2x = 2 \ln 2$$

$$c = \ln 2$$

5. Using a graphing utility,

(a) $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right) = \infty.$

(b) $\lim_{x\to 0^+}\left(\cot x - \frac{1}{x}\right) = 0.$

(c) $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right)\left(\cot x - \frac{1}{x}\right) \approx -\frac{2}{3}.$

Analytically,

(a) $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right) = \infty + \infty = \infty.$

(b) $$\lim_{x\to 0^+}\left(\cot x - \frac{1}{x}\right) = \lim_{x\to 0^+}\frac{x\cot x - 1}{x} = \lim_{x\to 0^+}\frac{x\cos x - \sin x}{x\sin x}$$
$$= \lim_{x\to 0^+}\frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x}$$
$$= \lim_{x\to 0^+}\frac{-x\sin x}{\sin x + x\cos x}$$
$$= \lim_{x\to 0^+}\frac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x} = 0.$$

(c) $$\left(\cot x + \frac{1}{x}\right)\left(\cot x - \frac{1}{x}\right) = \cot^2 x - \frac{1}{x^2}$$
$$= \frac{x^2\cot^2 x - 1}{x^2}$$

$$\lim_{x\to 0^+}\frac{x^2\cot^2 x - 1}{x^2} = \lim_{x\to 0^+}\frac{2x\cot^2 x - 2x^2\cot x\csc^2 x}{2x}$$
$$= \lim_{x\to 0^+}\frac{\cot^2 x - x\cot x\csc^2 x}{1}$$
$$= \lim_{x\to 0^+}\frac{\cos^2 x\sin x - x\cos x}{\sin^3 x}$$
$$= \lim_{x\to 0^+}\frac{\left(1 - \sin^2 x\right)\sin x - x\cos x}{\sin^3 x}$$
$$= \lim_{x\to 0^+}\frac{\sin x - x\cos x}{\sin^3 x} - 1.$$

Now, $$\lim_{x\to 0^+}\frac{\sin x - x\cos x}{\sin^3 x} = \lim_{x\to 0^+}\frac{\cos x - \cos x + x\sin x}{3\sin^2 x\cos x}$$
$$= \lim_{x\to 0^+}\frac{x}{3\sin x\cdot\cos x}$$
$$= \lim_{x\to 0^+}\left(\frac{x}{\sin x}\right)\frac{1}{3\cos x} = \frac{1}{3}.$$

So, $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right)\left(\cot x - \frac{1}{x}\right) = \frac{1}{3} - 1 = -\frac{2}{3}.$

The form $0\cdot\infty$ is indeterminant.

6. (a) $\dfrac{\text{Area sector}}{\text{Area circle}} = \dfrac{t}{2\pi} \Rightarrow \text{Area sector} = \dfrac{t}{2\pi}(\pi) = \dfrac{t}{2}$

(b) $\text{Area } AOP = \dfrac{1}{2}(\text{base})(\text{height}) - \displaystyle\int_1^{\cosh t} \sqrt{x^2 - 1}\, dx$

$$A(t) = \frac{1}{2}\cosh t \cdot \sinh t - \int_1^{\cosh t} \sqrt{x^2 - 1}\, dx$$

$$A'(t) = \frac{1}{2}\left[\cosh^2 t + \sinh^2 t\right] - \sqrt{\cosh^2 t - 1}\sinh t = \frac{1}{2}\left[\cosh^2 t + \sinh^2 t\right] - \sinh^2 t = \frac{1}{2}\left[\cosh^2 t - \sinh^2 t\right] = \frac{1}{2}$$

$$A(t) = \frac{1}{2}t + C$$

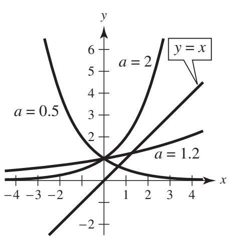
But, $A(0) = C = 0 \Rightarrow C = 0$ So, $A(t) = \dfrac{1}{2}t$ or $t = 2A(t)$.

7. $y = 0.5^x$ and $y = 1.2^x$ intersect $y = x$. $y = 2^x$ does not intersect $y = x$. Suppose $y = x$ is tangent to $y = a^x$ at (x, y).

$a^x = x \Rightarrow a = x^{1/x}$.

$y' = a^x \ln a = 1 \Rightarrow x \ln x^{1/x} = 1 \Rightarrow \ln x = 1 \Rightarrow x = e, a = e^{1/e}$

For $0 < a \le e^{1/e} \approx 1.445$, the curve $y = a^x$ intersects $y = x$.



8. $\sin\theta = \dfrac{PB}{OP} = PB, \cos\theta = OB$

$AQ = \widehat{AP} = \theta$

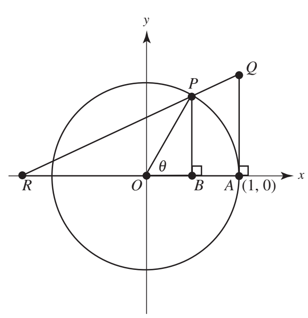
$BR = OR + OB = OR + \cos\theta$

The triangles ΔAQR and ΔBPR are similar:

$$\frac{AR}{AQ} = \frac{BR}{BP} \Rightarrow \frac{OR + 1}{\theta} = \frac{OR + \cos\theta}{\sin\theta}$$

$$\sin\theta(OR) + \sin\theta = (OR)\theta + \theta\cos\theta$$

$$OR = \frac{\theta\cos\theta - \sin\theta}{\sin\theta - \theta}$$



$$\begin{aligned}\lim_{\theta\to 0^+} OR &= \lim_{\theta\to 0^+} \frac{\theta\cos\theta - \sin\theta}{\sin\theta - \theta}\\ &= \lim_{\theta\to 0^+} \frac{-\theta\sin\theta + \cos\theta - \cos\theta}{\cos\theta - 1}\\ &= \lim_{\theta\to 0^+} \frac{-\theta\sin\theta}{\cos\theta - 1}\\ &= \lim_{\theta\to 0^+} \frac{-\sin\theta - \theta\cos\theta}{-\sin\theta}\\ &= \lim_{\theta\to 0^+} \frac{\cos\theta + \cos\theta - \theta\sin\theta}{\cos\theta}\\ &= 2\end{aligned}$$

9. (a) $y = f(x) = \arcsin x$

$\sin y = x$

$$\text{Area } A = \int_{\pi/6}^{\pi/4} \sin y \cdot dy = \left[-\cos y\right]_{\pi/6}^{\pi/4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} = \frac{\sqrt{3} - \sqrt{2}}{2} \approx 0.1589$$

$$\text{Area } B = \left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right) = \frac{\pi}{12} \approx 0.2618$$

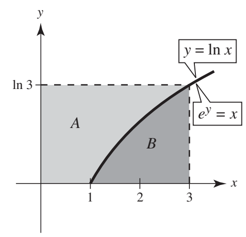
(b) $$\int_{1/2}^{\sqrt{2}/2} \arcsin x\, dx = \text{Area}(C) = \left(\frac{\pi}{4}\right)\left(\frac{\sqrt{2}}{2}\right) - A - B$$

$$= \frac{\pi\sqrt{2}}{8} - \frac{\sqrt{3} - \sqrt{2}}{2} - \frac{\pi}{12} = \pi\left(\frac{\sqrt{2}}{8} - \frac{1}{12}\right) + \frac{\sqrt{2} - \sqrt{3}}{2} \approx 0.1346$$

(c) $$\text{Area } A = \int_0^{\ln 3} e^y\, dy$$

$$= \left[e^y\right]_0^{\ln 3} = 3 - 1 = 2$$

$$\text{Area } B = \int_1^3 \ln x\, dx = 3(\ln 3) - A = 3 \ln 3 - 2 = \ln 27 - 2 \approx 1.2958$$



(d) $\tan y = x$

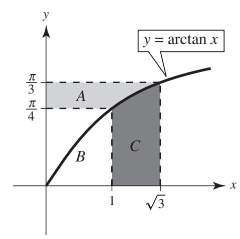
$$\text{Area } A = \int_{\pi/4}^{\pi/3} \tan y\, dy$$

$$= \left[-\ln\left|\cos y\right|\right]_{\pi/4}^{\pi/3}$$

$$= -\ln\frac{1}{2} + \ln\frac{\sqrt{2}}{2} = \ln\sqrt{2} = \frac{1}{2}\ln 2$$

$$\text{Area } C = \int_1^{\sqrt{3}} \arctan x\, dx = \left(\frac{\pi}{3}\right)\left(\sqrt{3}\right) - \frac{1}{2}\ln 2 - \left(\frac{\pi}{4}\right)(1)$$

$$= \frac{\pi}{12}\left(4\sqrt{3} - 3\right) - \frac{1}{2}\ln 2 \approx 0.6818$$



10. $y = \ln x$

$y' = \dfrac{1}{x}$

Tangent line: $y - b = \dfrac{1}{a}(x - a)$

$$y = \frac{1}{a}x + b - 1$$

If $x = 0, c = b - 1$.

So, $b - c = b - (b - 1) = 1$.

11. $y = e^x$

$y' = e^x$

Tangent line: $y - b = e^a(x - a)$

$$y = e^a x - ae^a + b$$

If $y = 0$: $e^a x = ae^a - b$

$$bx = ab - b \quad (b = e^a)$$

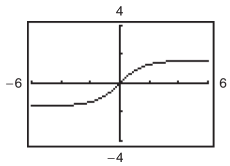
$$x = a - 1$$

$$c = a - 1$$

So, $a - c = a - (a - 1) = 1$.

12. $\text{gd}(x) = \arctan(\sinh x)$

(a)



(b) $\text{gd}(-x) = \arctan(\sinh(-x))$

$= \arctan(-\sinh x)$ (because sinh is odd)

$= -\arctan(\sinh x)$ (because arctan is odd)

$= -\text{gd}(x)$

So, $\text{gd}(x)$ is an odd function.

(c) $\dfrac{d}{dx}(\text{gd}(x)) = \dfrac{1}{1 + \sinh^2 x}\cosh x = \dfrac{\cosh x}{\cosh^2 x} = \dfrac{1}{\cosh x} = \text{sech}\, x > 0$

So, $\text{gd}(x)$ is monotonic (increasing) on $(-\infty, \infty)$

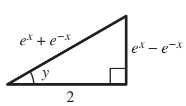
(d) $\dfrac{d^2}{dx^2}(\text{gd}(x)) = \dfrac{d}{dx}(\text{sech}\, x) = -\text{sech}\, x \tanh x$

For $x < 0, -\text{sech}\, x \tanh x > 0$, and for

$x > 0, -\text{sech}\, x \tanh x < 0$. So, $(0, 0)$ is the point of inflection.

(e) Let $y = \arcsin(\tanh x)$. Then

$\sin y = \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, as indicated in the figure.



The third side of the triangle is $\sqrt{(e^x + e^{-x})^2 - (e^x - e^{-x})^2} = \sqrt{4} = 2$.

Finally, $\tan y = \dfrac{e^x - e^{-x}}{2} = \sinh x$ and $y = \arcsin(\tanh x) = \arctan(\sinh x) = \text{gd}(x)$

13. $f(x) = \dfrac{\ln x^n}{x} = \dfrac{n \ln x}{x}, \quad x > e, n > 0$

$$f'(x) = n\left[\frac{x\left(\frac{1}{x}\right) - \ln x}{x^2}\right] = n\frac{1 - \ln x}{x^2}$$

For $x > e$, $\ln x > 1$ and $f'(x) < 0$.

So, f is decreasing for $x > e$.

14. Let $u = \tan x$, $du = \sec^2 x\, dx$.

$$\text{Area} = \int_0^{\pi/4} \frac{1}{\sin^2 x + 4\cos^2 x}\, dx$$

$$= \int_0^{\pi/4} \frac{\sec^2 x}{\tan^2 x + 4}\, dx$$

$$= \int_0^1 \frac{du}{u^2 + 4}$$

$$= \left[\frac{1}{2}\arctan\left(\frac{u}{2}\right)\right]_0^1$$

$$= \frac{1}{2}\arctan\left(\frac{1}{2}\right)$$

15. Let $u = 1 + \sqrt{x}$, $\sqrt{x} = u - 1$, $x = u^2 - 2u + 1$, $dx = (2u - 2)\,du$.

$$\text{Area} = \int_1^4 \frac{1}{\sqrt{x} + x}\,dx = \int_2^3 \frac{2u - 2}{(u - 1) + (u^2 - 2u + 1)}\,du$$

$$= \int_2^3 \frac{2(u - 1)}{u^2 - u}\,du$$

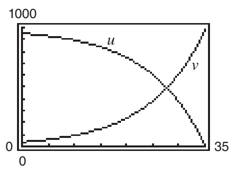
$$= \int_2^3 \frac{2}{u}\,du = \Big[2\ln|u|\Big]_2^3$$

$$= 2\ln 3 - 2\ln 2 = 2\ln\left(\frac{3}{2}\right)$$

$$\approx 0.8109$$

16. (a) $u = 985.93 - \left(985.93 - \frac{(120{,}000)(0.095)}{12}\right)\left(1 + \frac{0.095}{12}\right)^{12t}$

$v = \left(985.93 - \frac{(120{,}000)(0.095)}{12}\right)\left(1 + \frac{0.095}{12}\right)^{12t}$



(b) The larger part goes for interest. The curves intersect when $t \approx 27.7$ years.

(c) The slopes are negatives of each other. Analytically,

$$u = 985.93 - v \Rightarrow \frac{du}{dt} = -\frac{dv}{dt}$$

$u'(15) = -v'(15) = -14.06.$

(d) $t = 12.7$ years

Again, the larger part goes for interest.