CHAPTER 4
Integration

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CHAPTER 4
Integration

Section 4.1 Antiderivatives and Indefinite Integration

1. A function F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

2. Yes. For example, $F_1(x) = x$ and $F_2(x) = x + 2$ are both antiderivates of $f(x) = 1$.

3. The particular solution results from knowing the value of $y = F(x)$ for one value of x. Using the initial condition in the general solution, you can solve for C to obtain the particular solution.

4. A general solution of a differential equation contains an arbitrary constant, C. A particular solution results from applying the initial condition to determine the value of C.

5. $\dfrac{d}{dx}\left(\dfrac{2}{x^3} + C\right) = \dfrac{d}{dx}\left(2x^{-3} + C\right) = -6x^{-4} = \dfrac{-6}{x^4}$

6. $\dfrac{d}{dx}\left(2x^4 - \dfrac{1}{2x} + C\right) = \dfrac{d}{dx}\left(2x^4 - \dfrac{1}{2}x^{-1} + C\right)$

$= 8x^3 + \dfrac{1}{2}x^{-2} = 8x^3 + \dfrac{1}{2x^2}$

$\dfrac{d}{dx}\left(2x^4 - \dfrac{1}{2x} + C\right) = \dfrac{d}{dx}\left(2x^4 - \dfrac{1}{2}x^{-1} + C\right)$

$= 8x^3 + \dfrac{1}{2}x^{-2} = 8x^3 + \dfrac{1}{2x^2}$

7. $\dfrac{dy}{dt} = 9t^2$

$y = 3t^3 + C$

Check: $\dfrac{d}{dt}\left[3t^3 + C\right] = 9t^2$

8. $\dfrac{dy}{dt} = 5$

$y = 5t + C$

Check: $\dfrac{d}{dt}\left[5t + C\right] = 5$

9. $\dfrac{dy}{dx} = x^{3/2}$

$y = \dfrac{2}{5}x^{5/2} + C$

Check: $\dfrac{d}{dx}\left[\dfrac{2}{5}x^{5/2} + C\right] = x^{3/2}$

10. $\dfrac{dy}{dx} = 2x^{-3}$

$y = \dfrac{2x^{-2}}{-2} + C = \dfrac{-1}{x^2} + C$

Check: $\dfrac{d}{dx}\left[\dfrac{-1}{x^2} + C\right] = 2x^{-3}$

	Given	*Rewrite*	*Integrate*	*Simplify*
11.	$\int \sqrt[3]{x}\,dx$	$\int x^{1/3}\,dx$	$\dfrac{x^{4/3}}{4/3} + C$	$\dfrac{3}{4}x^{4/3} + C$
12.	$\int \dfrac{1}{4x^2}\,dx$	$\dfrac{1}{4}\int x^{-2}\,dx$	$\dfrac{1}{4}\dfrac{x^{-1}}{-1} + C$	$-\dfrac{1}{4x} + C$
13.	$\int \dfrac{1}{x\sqrt{x}}\,dx$	$\int x^{-3/2}\,dx$	$\dfrac{x^{-1/2}}{-1/2} + C$	$-\dfrac{2}{\sqrt{x}} + C$
14.	$\int \dfrac{1}{(3x)^2}\,dx$	$\dfrac{1}{9}\int x^{-2}\,dx$	$\dfrac{1}{9}\left(\dfrac{x^{-1}}{-1}\right) + C$	$\dfrac{-1}{9x} + C$

15. $\int (x + 7)\,dx = \dfrac{x^2}{2} + 7x + C$

Check: $\dfrac{d}{dx}\left[\dfrac{x^2}{2} + 7x + C\right] = x + 7$

16. $\int (13 - x)\,dx = 13x - \dfrac{x^2}{2} + C$

Check: $\dfrac{d}{dx}\left[13x - \dfrac{x^2}{2} + C\right] = 13 - x$

17. $\int (x^5 + 1)\,dx = \dfrac{x^6}{6} + x + C$

Check: $\dfrac{d}{dx}\left(\dfrac{x^6}{6} + x + C\right) = x^5 + 1$

18. $\int (9x^8 - 2x - 6)\,dx = \dfrac{9x^9}{9} - \dfrac{2}{2}x^2 - 6x + C$

$= x^9 - x^2 - 6x + C$

Check: $\dfrac{d}{dx}(x^9 - x^2 - 6x + C) = 9x^8 - 2x - 6$

19. $\int (x^{3/2} + 2x + 1)\,dx = \dfrac{2}{5}x^{5/2} + x^2 + x + C$

Check: $\dfrac{d}{dx}\left(\dfrac{2}{5}x^{5/2} + x^2 + x + C\right) = x^{3/2} + 2x + 1$

20. $\int \left(\sqrt{x} + \dfrac{1}{2\sqrt{x}}\right) dx = \int \left(x^{1/2} + \dfrac{1}{2}x^{-1/2}\right) dx$

$= \dfrac{x^{3/2}}{3/2} + \dfrac{1}{2}\left(\dfrac{x^{1/2}}{1/2}\right) + C$

$= \dfrac{2}{3}x^{3/2} + x^{1/2} + C$

Check: $\dfrac{d}{dx}\left(\dfrac{2}{3}x^{3/2} + x^{1/2} + C\right) = x^{1/2} + \dfrac{1}{2}x^{-1/2}$

$= \sqrt{x} + \dfrac{1}{2\sqrt{x}}$

21. $\int \sqrt[3]{x^2}\,dx = \int x^{2/3}\,dx = \dfrac{x^{5/3}}{5/3} + C = \dfrac{3}{5}x^{5/3} + C$

Check: $\dfrac{d}{dx}\left(\dfrac{3}{5}x^{5/3} + C\right) = x^{2/3} = \sqrt[3]{x^2}$

22. $\int \left(\sqrt[4]{x^3} + 1\right) dx = \int (x^{3/4} + 1)\,dx = \dfrac{4}{7}x^{7/4} + x + C$

Check: $\dfrac{d}{dx}\left(\dfrac{4}{7}x^{7/4} + x + C\right) = x^{3/4} + 1 = \sqrt[4]{x^3} + 1$

23. $\int \dfrac{1}{x^5}\,dx = \int x^{-5}\,dx = \dfrac{x^{-4}}{-4} + C = \dfrac{-1}{4x^4} + C$

Check: $\dfrac{d}{dx}\left(\dfrac{-1}{4x^4} + C\right) = \dfrac{d}{dx}\left(-\dfrac{1}{4}x^{-4} + C\right)$

$= -\dfrac{1}{4}(-4x^{-5}) = \dfrac{1}{x^5}$

24. $\int \left(2 - \dfrac{3}{x^{10}}\right) dx = \int (2 - 3x^{-10})\,dx$

$= 2x - 3\left(\dfrac{x^{-9}}{-9}\right) + C$

$= 2x + \dfrac{1}{3x^9} + C$

Check: $\dfrac{d}{dx}\left(2x + \dfrac{1}{3x^9} + C\right) = \dfrac{d}{dx}\left(2x + \dfrac{1}{3}x^{-9} + C\right)$

$= 2 - 3x^{-10} = 2 - \dfrac{3}{x^{10}}$

25. $\int \dfrac{x + 6}{\sqrt{x}}\,dx = \int (x^{1/2} + 6x^{-1/2})\,dx$

$= \dfrac{x^{3/2}}{3/2} + 6\dfrac{x^{1/2}}{1/2} + C$

$= \dfrac{2}{3}x^{3/2} + 12x^{1/2} + C$

$= \dfrac{2}{3}x^{1/2}(x + 18) + C$

Check: $\dfrac{d}{dx}\left(\dfrac{2}{3}x^{3/2} + 12x^{1/2} + C\right)$

$= \dfrac{2}{3}\left(\dfrac{3}{2}x^{1/2}\right) + 12\left(\dfrac{1}{2}x^{-1/2}\right)$

$= x^{1/2} + 6x^{-1/2} = \dfrac{x + 6}{\sqrt{x}}$

26. $\int \dfrac{x^4 - 3x^2 + 5}{x^4}\,dx = \int (1 - 3x^{-2} + 5x^{-4})\,dx$

$= x - \dfrac{3x^{-1}}{-1} + \dfrac{5x^{-3}}{-3} + C$

$= x + \dfrac{3}{x} - \dfrac{5}{3x^3} + C$

Check:

$\dfrac{d}{dx}\left[x + \dfrac{3}{x} - \dfrac{5}{3x^3} + C\right] = \dfrac{d}{dx}\left[x + 3x^{-1} - \dfrac{5}{3}x^{-3} + C\right]$

$= 1 - 3x^{-2} + 5x^{-4}$

$= 1 - \dfrac{3}{x^2} + \dfrac{5}{x^4}$

$= \dfrac{x^4 - 3x^2 + 5}{x^4}$

27. $\int (x + 1)(3x - 2)\,dx = \int (3x^2 + x - 2)\,dx$

$= x^3 + \dfrac{1}{2}x^2 - 2x + C$

Check: $\dfrac{d}{dx}\left(x^3 + \dfrac{1}{2}x^2 - 2x + C\right) = 3x^2 + x - 2$

$= (x + 1)(3x - 2)$

28. $\int (4t^2 + 3)^2\, dt = \int (16t^4 + 24t^2 + 9)\, dt$

$= \dfrac{16t^5}{5} + 8t^3 + 9t + C$

Check: $\dfrac{d}{dt}\left(\dfrac{16t^5}{5} + 8t^3 + 9t + C\right) = 16t^4 + 24t^2 + 9$

$= (4t^2 + 3)^2$

29. $\int (5\cos x + 4\sin x)\, dx = 5\sin x - 4\cos x + C$

Check:
$\dfrac{d}{dx}(5\sin x - 4\cos x + C) = 5\cos x + 4\sin x$

30. $\int (\sin x - 6\cos x)\, dx = -\cos x - 6\sin x + C$

Check: $\dfrac{d}{dx}(-\cos x - 6\sin x + C) = \sin x - 6\cos x$

31. $\int (\csc x\cot x - 2x)\, dx = -\csc x - x^2 + C$

Check: $\dfrac{d}{dx}(-\csc x - x^2 + C) = \csc x\cot x - 2x$

32. $\int (\theta^2 + \sec^2\theta)\, d\theta = \dfrac{1}{3}\theta^3 + \tan\theta + C$

Check: $\dfrac{d}{d\theta}\left(\dfrac{1}{3}\theta^3 + \tan\theta + C\right) = \theta^2 + \sec^2\theta$

33. $\int (\sec^2\theta - \sin\theta)\, d\theta = \tan\theta + \cos\theta + C$

Check: $\dfrac{d}{d\theta}(\tan\theta + \cos\theta + C) = \sec^2\theta - \sin\theta$

34. $\int (\sec y)(\tan y - \sec y)\, dy = \int (\sec y\tan y - \sec^2 y)\, dy$

$= \sec y - \tan y + C$

Check:
$\dfrac{d}{dy}(\sec y - \tan y + C) = \sec y\tan y - \sec^2 y$

$= (\sec y)(\tan y - \sec y)$

35. $\int (\tan^2 y + 1)\, dy = \int \sec^2 y\, dy = \tan y + C$

Check: $\dfrac{d}{dy}(\tan y + C) = \sec^2 y = \tan^2 y + 1$

36. $\int (4x - \csc^2 x)\, dx = 2x^2 + \cot x + C$

Check: $\dfrac{d}{dx}(2x^2 + \cot x + C) = 4x - \csc^2 x$

37. $f'(x) = 6x,\ f(0) = 8$

$f(x) = \int 6x\, dx = 3x^2 + C$

$f(0) = 8 = 3(0)^2 + C \Rightarrow C = 8$

$f(x) = 3x^2 + 8$

38. $g'(x) = 4x^2,\ g(-1) = 3$

$g(x) = \int 4x^2\, dx = \dfrac{4}{3}x^3 + C$

$g(-1) = 3 = -\dfrac{4}{3} + C \Rightarrow C = \dfrac{13}{3}$

$g(x) = \dfrac{4}{3}x^3 + \dfrac{13}{3}$

39. $h'(x) = 7x^6 + 5,\ h(1) = -1$

$h(x) = \int (7x^6 + 5)\, dx = x^7 + 5x + C$

$h(1) = -1 = 1 + 5 + C \Rightarrow C = -7$

$h(x) = x^7 + 5x - 7$

40. $f'(s) = 10s - 12s^3,\ f(3) = 2$

$f(s) = \int (10s - 12s^3)\, ds = 5s^2 - 3s^4 + C$

$f(3) = 2 = 5(3)^2 - 3(3)^4 + C = 45 - 243 + C \Rightarrow C = 200$

$f(s) = 5s^2 - 3s^4 + 200$

41. $f''(x) = 2$

$f'(2) = 5$

$f(2) = 10$

$f'(x) = \int 2\, dx = 2x + C_1$

$f'(2) = 4 + C_1 = 5 \Rightarrow C_1 = 1$

$f'(x) = 2x + 1$

$f(x) = \int (2x + 1)\, dx = x^2 + x + C_2$

$f(2) = 6 + C_2 = 10 \Rightarrow C_2 = 4$

$f(x) = x^2 + x + 4$

42. $f''(x) = 3x^2,\ f'(-1) = -2,\ f(2) = 3$

$f'(x) = \int 3x^2\, dx = x^3 + C$

$f'(-1) = -1 + C = -2 \Rightarrow C = -1$

$f'(x) = x^3 - 1$

$f(x) = \int (x^3 - 1)\, dx = \dfrac{x^4}{4} - x + C_1$

$f(2) = \dfrac{16}{4} - 2 + C_1 = 4 - 2 + C_1 = 3 \Rightarrow C_1 = 1$

$f(x) = \dfrac{x^4}{4} - x + 1$

43. $f''(x) = x^{-3/2}$

$f'(4) = 2$

$f(0) = 0$

$f'(x) = \int x^{-3/2}\,dx = -2x^{-1/2} + C_1 = -\dfrac{2}{\sqrt{x}} + C_1$

$f'(4) = -\dfrac{2}{2} + C_1 = 2 \Rightarrow C_1 = 3$

$f'(x) = -\dfrac{2}{\sqrt{x}} + 3$

$f(x) = \int \left(-2x^{-1/2} + 3\right) dx = -4x^{1/2} + 3x + C_2$

$f(0) = 0 + 0 + C_2 = 0 \Rightarrow C_2 = 0$

$f(x) = -4x^{1/2} + 3x = -4\sqrt{x} + 3x$

44. $f''(x) = \sin x$

$f'(0) = 1$

$f(0) = 6$

$f'(x) = \int \sin x\,dx = -\cos x + C_1$

$f'(0) = -1 + C_1 = 1 \Rightarrow C_1 = 2$

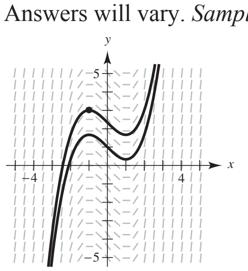
$f'(x) = -\cos x + 2$

$f(x) = \int (-\cos x + 2)\,dx = -\sin x + 2x + C_2$

$f(0) = 0 + 0 + C_2 = 6 \Rightarrow C_2 = 6$

$f(x) = -\sin x + 2x + 6$

45. (a) Answers will vary. *Sample answer.*



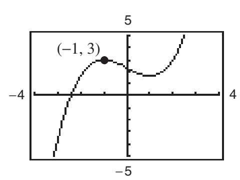
(b) $\dfrac{dy}{dx} = x^2 - 1, (-1, 3)$

$y = \dfrac{x^3}{3} - x + C$

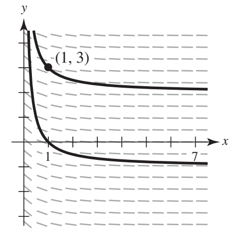
$3 = \dfrac{(-1)^3}{3} - (-1) + C$

$C = \dfrac{7}{3}$

$y = \dfrac{x^3}{3} - x + \dfrac{7}{3}$



46. (a) Answers will vary. *Sample answer:*

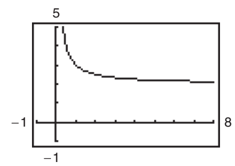


(b) $\dfrac{dy}{dx} = \dfrac{-1}{x^2}, x > 0, (1, 3)$

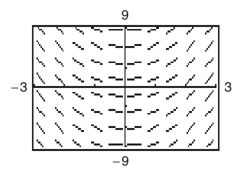
$y = \int -\dfrac{1}{x^2}\,dx = \int -x^{-2}\,dx = \dfrac{-x^{-1}}{-1} + C = \dfrac{1}{x} + C$

$3 = \dfrac{1}{1} + C \Rightarrow C = 2$

$y = \dfrac{1}{x} + 2$



47. (a)



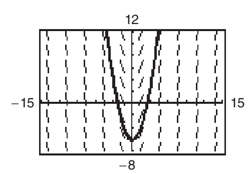
(b) $\dfrac{dy}{dx} = 2x, (-2, -2)$

$y = \int 2x\,dx = x^2 + C$

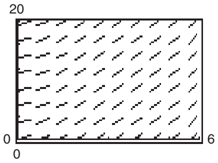
$-2 = (-2)^2 + C = 4 + C \Rightarrow C = -6$

$y = x^2 - 6$

(c)



48. (a)



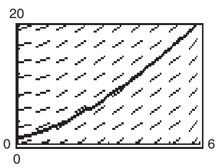
(b) $\dfrac{dy}{dx} = 2\sqrt{x},\ (4, 12)$

$$y = \int 2x^{1/2}\,dx = \frac{4}{3}x^{3/2} + C$$

$$12 = \frac{4}{3}(4)^{3/2} + C = \frac{4}{3}(8) + C = \frac{32}{3} + C \Rightarrow C = \frac{4}{3}$$

$$y = \frac{4}{3}x^{3/2} + \frac{4}{3}$$

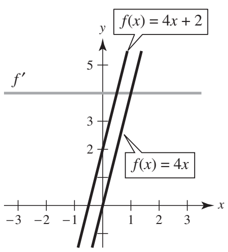
(c)



49. $f'(x) = 4$

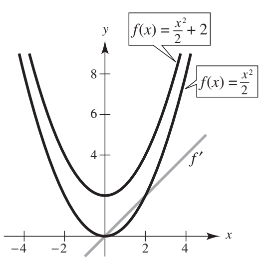
$f(x) = 4x + C$

Answers will vary.



50. $f'(x) = x$

$$f(x) = \frac{x^2}{2} + C$$



Answers will vary.

51. $f(x) = \tan^2 x \Rightarrow f'(x) = 2\tan x \cdot \sec^2 x$

$g(x) = \sec^2 x \Rightarrow g'(x) = 2\sec x \cdot \sec x \tan x = f'(x)$

The derivatives are the same, so f and g differ by a constant. In fact, $\tan^2 x + 1 = \sec^2 x$.

52. $f(0) = -4$. Graph of f' is given.

(a) $f'(4) \approx -1.0$

(b) No, $f(5) < f(4)$ because f is decreasing on $[4, 5]$.

(c) f is a maximum at $x = 3.5$ because $f'(3.5) \approx 0$ and the First Derivative Test.

(d) f is concave upward when f' is increasing on $(-\infty, 1)$ and $(5, \infty)$. f is concave downward on $(1, 5)$. Points of inflection at $x = 1, 5$.

53. $f''(x) = 2x$

$f'(x) = x^2 + C$

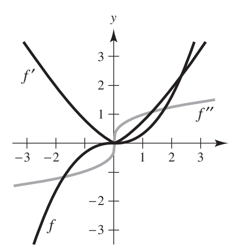
$f'(2) = 0 \Rightarrow 4 + C = 0 \Rightarrow C = -4$

$$f(x) = \frac{x^3}{3} - 4x + C_1$$

$$f(2) = 0 \Rightarrow \frac{8}{3} - 8 + C_1 = 0 \Rightarrow C_1 = \frac{16}{3}$$

$$f(x) = \frac{x^3}{3} - 4x + \frac{16}{3}$$

54. Because f'' is negative on $(-\infty, 0)$, f' is decreasing on $(-\infty, 0)$. Because f'' is positive on $(0, \infty)$, f' is increasing on $(0, \infty)$. f' has a relative minimum at $(0, 0)$. Because f' is positive on $(-\infty, \infty)$, f is increasing on $(-\infty, \infty)$.



55. (a) $h(t) = \int (1.5t + 5)\,dt = 0.75t^2 + 5t + C$

$h(0) = 0 + 0 + C = 12 \Rightarrow C = 12$

$h(t) = 0.75t^2 + 5t + 12$

(b) $h(6) = 0.75(6)^2 + 5(6) + 12 = 69$ cm

56. $\dfrac{dP}{dt} = k\sqrt{t},\ 0 \le t \le 10$

$$P(t) = \int kt^{1/2}\,dt = \frac{2}{3}kt^{3/2} + C$$

$$P(0) = 0 + C = 500 \Rightarrow C = 500$$

$$P(1) = \frac{2}{3}k + 500 = 600 \Rightarrow k = 150$$

$$P(t) = \frac{2}{3}(150)t^{3/2} + 500 = 100t^{3/2} + 500$$

$$P(7) = 100(7)^{3/2} + 500 \approx 2352 \text{ bacteria}$$

57. $a(t) = -32 \text{ ft/sec}^2$

$$v(t) = \int -32\,dt = -32t + C_1$$

$$v(0) = 60 = C_1$$

$$s(t) = \int(-32t + 60)\,dt = -16t^2 + 60t + C_2$$

$$s(0) = 6 = C_2$$

$$s(t) = -16t^2 + 60t + 6, \text{ Position function}$$

The ball reaches its maximum height when

$$v(t) = -32t + 60 = 0$$

$$32t = 60$$

$$t = \tfrac{15}{8} \text{ seconds.}$$

$$s\left(\tfrac{15}{8}\right) = -16\left(\tfrac{15}{8}\right)^2 + 60\left(\tfrac{15}{8}\right) + 6 = 62.25 \text{ feet}$$

58. $a(t) = -32 \text{ ft/sec}^2$

$$v(t) = \int -32\,dt = -32t + C_1$$

$$v(0) = 0 + C_1 = V_0 \Rightarrow C_1 = V_0$$

$$s'(t) = -32t + V_0$$

$$s(t) = \int(-32t + V_0)dt = -16t^2 + V_0t + C_2$$

$$s(0) = 0 + 0 + C_2 = S_0 \Rightarrow C_2 = S_0$$

$$s(t) = -16t^2 + V_0t + S_0$$

$s'(t) = -32t + v_0 = 0$ when $t = \dfrac{v_0}{32}$ = time to reach maximum height.

$$s\left(\frac{v_0}{32}\right) = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) = 550$$

$$-\frac{v_0^2}{64} + \frac{v_0^2}{32} = 550$$

$$v_0^2 = 35{,}200$$

$$v_0 \approx 187.617 \text{ ft/sec}$$

59. $v_0 = 16 \text{ ft/sec}$

$s_0 = 64 \text{ ft}$

(a) $s(t) = -16t^2 + 16t + 64 = 0$

$$-16(t^2 - t - 4) = 0$$

$$t = \frac{1 \pm \sqrt{17}}{2}$$

Choosing the positive value,

$$t = \frac{1 + \sqrt{17}}{2} \approx 2.562 \text{ seconds.}$$

(b) $v(t) = s'(t) = -32t + 16$

$$v\left(\frac{1 + \sqrt{17}}{2}\right) = -32\left(\frac{1 + \sqrt{17}}{2}\right) + 16$$

$$= -16\sqrt{17} \approx -65.970 \text{ ft/sec}$$

60. $a(t) = -9.8$

$$v(t) = \int -9.8\,dt = -9.8t + C_1$$

$$v(0) = v_0 = C_1 \Rightarrow v(t) = -9.8t + v_0$$

$$f(t) = \int(-9.8t + v_0)\,dt = -4.9t^2 + v_0t + C_2$$

$$f(0) = s_0 = C_2 \Rightarrow f(t) = -4.9t^2 + v_0t + s_0$$

So, $f(t) = -4.9t^2 + 10t + 2.$

$v(t) = -9.8t + 10 = 0$ (Maximum height when $v = 0$.)

$$9.8t = 10$$

$$t = \frac{10}{9.8}$$

$$f\left(\frac{10}{9.8}\right) \approx 7.1 \text{ m}$$

61. From Exercise 60, $f(t) = -4.9t^2 + v_0t + 2.$ If

$$f(t) = 200 = -4.9t^2 + v_0t + 2,$$

then

$$v(t) = -9.8t + v_0 = 0$$

for this t value. So, $t = v_0/9.8$ and you solve

$$-4.9\left(\frac{v_0}{9.8}\right)^2 + v_0\left(\frac{v_0}{9.8}\right) + 2 = 200$$

$$\frac{-4.9v_0^2}{(9.8)^2} + \left(\frac{v_0^2}{9.8}\right) = 198$$

$$-4.9v_0^2 + 9.8v_0^2 = (9.8)^2 198$$

$$4.9v_0^2 = (9.8)^2 198$$

$$v_0^2 = 3880.8$$

$$v_0 \approx 62.3 \text{ m/sec.}$$

62. From Exercise 60, $f(t) = -4.9t^2 + 1800$. (Using the canyon floor as position 0.)

$$f(t) = 0 = -4.9t^2 + 1800$$
$$4.9t^2 = 1800$$
$$t^2 = \frac{1800}{4.9} \Rightarrow t \approx 9.2 \text{ sec}$$

63. $a = -1.6$

$v(t) = \int -1.6\, dt = -1.6t + v_0 = -1.6t$, because the stone was dropped, $v_0 = 0$.

$$s(t) = \int (-1.6t)\, dt = -0.8t^2 + s_0$$
$$s(20) = 0 \Rightarrow -0.8(20)^2 + s_0 = 0$$
$$s_0 = 320$$

So, the height of the cliff is 320 meters.

$$v(t) = -1.6t$$
$$v(20) = -32 \text{ m/sec}$$

64. $\int v\, dv = -GM \int \frac{1}{y^2}\, dy$

$$\frac{1}{2}v^2 = \frac{GM}{y} + C$$

When $y = R$, $v = v_0$.

$$\frac{1}{2}v_0^2 = \frac{GM}{R} + C$$
$$C = \frac{1}{2}v_0^2 - \frac{GM}{R}$$
$$\frac{1}{2}v^2 = \frac{GM}{y} + \frac{1}{2}v_0^2 - \frac{GM}{R}$$
$$v^2 = \frac{2GM}{y} + v_0^2 - \frac{2GM}{R}$$
$$v^2 = v_0^2 + 2GM\left(\frac{1}{y} - \frac{1}{R}\right)$$

65. $x(t) = t^3 - 6t^2 + 9t - 2, \quad 0 \le t \le 5$

(a) $v(t) = x'(t) = 3t^2 - 12t + 9$
$= 3(t^2 - 4t + 3) = 3(t - 1)(t - 3)$
$a(t) = v'(t) = 6t - 12 = 6(t - 2)$

(b) $v(t) > 0$ when $0 < t < 1$ or $3 < t < 5$.

(c) $a(t) = 6(t - 2) = 0$ when $t = 2$.
$v(2) = 3(1)(-1) = -3$

66. $x(t) = (t - 1)(t - 3)^2 \quad 0 \le t \le 5$
$= t^3 - 7t^2 + 15t - 9$

(a) $v(t) = x'(t) = 3t^2 - 14t + 15 = (3t - 5)(t - 3)$
$a(t) = v'(t) = 6t - 14$

(b) $v(t) > 0$ when $0 < t < \frac{5}{3}$ and $3 < t < 5$.

(c) $a(t) = 6t - 14 = 0$ when $t = \frac{7}{3}$.
$v\left(\frac{7}{3}\right) = \left(3\left(\frac{7}{3}\right) - 5\right)\left(\frac{7}{3} - 3\right) = 2\left(-\frac{2}{3}\right) = -\frac{4}{3}$

67. $v(t) = \frac{1}{\sqrt{t}} = t^{-1/2} \quad t > 0$

$$x(t) = \int v(t) dt = 2t^{1/2} + C$$
$$x(1) = 4 = 2(1) + C \Rightarrow C = 2$$

Position function: $x(t) = 2t^{1/2} + 2$

Acceleration function: $a(t) = v'(t) = -\frac{1}{2}t^{-3/2} = \frac{-1}{2t^{3/2}}$

68. (a) $a(t) = \cos t$
$v(t) = \int a(t)\, dt$
$= \int \cos t\, dt$
$= \sin t + C_1 = \sin t$ (because $v_0 = 0$)
$x(t) = \int v(t)\, dt = \int \sin t\, dt = -\cos t + C_2$
$x(0) = 3 = -\cos(0) + C_2 = -1 + C_2 \Rightarrow C_2 = 4$
$x(t) = -\cos t + 4$

(b) $v(t) = 0 = \sin t$ for $t = k\pi$, $k = 0, 1, 2, \ldots$

69. (a) $v(0) = 25 \text{ km/h} = 25 \cdot \frac{1000}{3600} = \frac{250}{36} \text{ m/sec}$
$v(13) = 80 \text{ km/h} = 80 \cdot \frac{1000}{3600} = \frac{800}{36} \text{ m/sec}$
$a(t) = a$ (constant acceleration)
$v(t) = at + C$
$v(0) = \frac{250}{36} \Rightarrow v(t) = at + \frac{250}{36}$
$v(13) = \frac{800}{36} = 13a + \frac{250}{36}$
$\frac{550}{36} = 13a$
$a = \frac{550}{468} = \frac{275}{234} \approx 1.175 \text{ m/sec}^2$

(b) $s(t) = a\frac{t^2}{2} + \frac{250}{36}t \quad (s(0) = 0)$

$$s(13) = \frac{275}{234}\frac{(13)^2}{2} + \frac{250}{36}(13) \approx 189.58 \text{ m}$$

70. $v(0) = 45 \text{ mi/h} = 66 \text{ ft/sec}$

$30 \text{ mi/h} = 44 \text{ ft/sec}$

$15 \text{ mi/h} = 22 \text{ ft/sec}$

$a(t) = -a$

$v(t) = -at + 66$

$s(t) = -\frac{a}{2}t^2 + 66t \text{ (Let } s(0) = 0.)$

$v(t) = 0$ after car moves 132 ft.

$-at + 66 = 0$ when $t = \frac{66}{a}$.

$s\left(\frac{66}{a}\right) = -\frac{a}{2}\left(\frac{66}{a}\right)^2 + 66\left(\frac{66}{a}\right)$

$= 132$ when $a = \frac{33}{2} = 16.5$.

$a(t) = -16.5$

$v(t) = -16.5t + 66$

$s(t) = -8.25t^2 + 66t$

(a) $-16.5t + 66 = 44$

$t = \frac{22}{16.5} \approx 1.333$

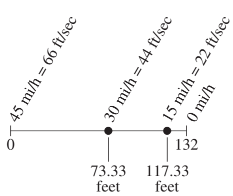
$s\left(\frac{22}{16.5}\right) \approx 73.33 \text{ ft}$

(b) $-16.5t + 66 = 22$

$t = \frac{44}{16.5} \approx 2.667$

$s\left(\frac{44}{16.5}\right) \approx 117.33 \text{ ft}$

(c)



It takes 1.333 seconds to reduce the speed from 45 mi/h to 30 mi/h, 1.333 seconds to reduce the speed from 30 mi/h to 15 mi/h, and 1.333 seconds to reduce the speed from 15 mi/h to 0 mi/h. Each time, less distance is needed to reach the next speed reduction.

71. Truck: $v(t) = 30$

$s(t) = 30t \text{ (Let } s(0) = 0.)$

Automobile: $a(t) = 6$

$v(t) = 6t \text{ (Let } v(0) = 0.)$

$s(t) = 3t^2 \text{ (Let } s(0) = 0.)$

At the point where the automobile overtakes the truck:

$30t = 3t^2$

$0 = 3t^2 - 30t$

$0 = 3t(t - 10)$ when $t = 10$ sec.

(a) $s(10) = 3(10)^2 = 300 \text{ ft}$

(b) $v(10) = 6(10) = 60 \text{ ft/sec} \approx 41 \text{ mi/h}$ $kt = 160$

72. $a(t) = k$

$v(t) = kt$

$s(t) = \frac{k}{2}t^2$ because $v(0) = s(0) = 0$.

At the time of lift-off, $kt = 160$ and $(k/2)t^2 = 0.7$.

Because $(k/2)t^2 = 0.7$,

$t = \sqrt{\frac{1.4}{k}}$

$v\left(\sqrt{\frac{1.4}{k}}\right) = k\sqrt{\frac{1.4}{k}} = 160$

$1.4k = 160^2 \Rightarrow k = \frac{160^2}{1.4}$

$\approx 18{,}285.714 \text{ mi/h}^2$

$\approx 7.45 \text{ ft/sec}^2$.

73. False. f has an infinite number of antiderivatives, each differing by a constant.

74. True

75. True

76. True

77. True

78. False. For example, $\int x \cdot x \, dx \neq \int x \, dx \cdot \int x \, dx$ because $\frac{x^3}{3} + C \neq \left(\frac{x^2}{2} + C_1\right)\left(\frac{x^2}{2} + C_2\right)$.

79. $\frac{d}{dx}\left[\left[s(x)\right]^2 + \left[c(x)\right]^2\right] = 2s(x)s'(x) + 2c(x)c'(x)$

$= 2s(x)c(x) - 2c(x)s(x) = 0$

So, $\left[s(x)\right]^2 + \left[c(x)\right]^2 = k$ for some constant k. Because, $s(0) = 0$ and $c(0) = 1$, $k = 1$.

Therefore, $\left[s(x)\right]^2 + \left[c(x)\right]^2 = 1$. [Note that $s(x) = \sin x$ and $c(x) = \cos x$ satisfy these properties.]

80. Note that $\frac{d}{dx}\left(\cos x^2\right) = \left(-\sin x^2\right)(2x) = -2x \sin x^2$.

Hence, $f(x) = \int \left(-2x \sin x^2\right) dx = \cos x^2 + C$.

81. $f(x + y) = f(x)f(y) - g(x)g(y)$

$g(x + y) = f(x)g(y) + g(x)f(y)$

$f'(0) = 0$

[Note: $f(x) = \cos x$ and $g(x) = \sin x$ satisfy these conditions]

$f'(x + y) = f(x)f'(y) - g(x)g'(y)$ (Differentiate with respect to y)

$g'(x + y) = f(x)g'(y) + g(x)f'(y)$ (Differentiate with respect to y)

Letting $y = 0$, $f'(x) = f(x)f'(0) - g(x)g'(0) = -g(x)g'(0)$

$g'(x) = f(x)g'(0) + g(x)f'(0) = f(x)g'(0)$

So, $2f(x)f'(x) = -2f(x)g(x)g'(0)$

$2g(x)g'(x) = 2g(x)f(x)g'(0)$.

Adding, $2f(x)f'(x) + 2g(x)g'(x) = 0$.

Integrating, $f(x)^2 + g(x)^2 = C$.

Clearly $C \neq 0$, for if $C = 0$, then $f(x)^2 = -g(x)^2 \Rightarrow f(x) = g(x) = 0$, which contradicts that f, g are nonconstant.

Now, $C = f(x + y)^2 + g(x + y)^2 = \left(f(x)f(y) - g(x)g(y)\right)^2 + \left(f(x)g(y) + g(x)f(y)\right)^2$

$= f(x)^2 f(y)^2 + g(x)^2 g(y)^2 + f(x)^2 g(y)^2 + g(x)^2 f(y)^2$

$= \left[f(x)^2 + g(x)^2\right]\left[f(y)^2 + g(y)^2\right] = C^2$

So, $C = 1$ and you have $f(x)^2 + g(x)^2 = 1$.

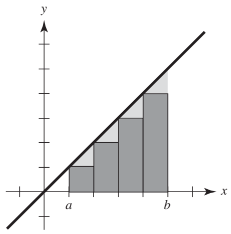
Section 4.2 Area

1. For $\sum_{i=3}^{8}(i - 4)$, the index of summation is i, the upper bound of summation is 8, and the lower bound of summation is 3.

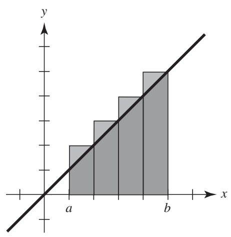
2. (a) From Theorem 4.2, part 2, you see that $n = 5$. That is, $\sum_{i=1}^{5} i = \frac{5(5 + 1)}{2}$.

(b) From Theorem 4.2, part 3, you see that $n = 20$. That is, $\sum_{i=1}^{20} i^2 = \frac{20(20 + 1)\left[2(20) + 1\right]}{6}$.

3. You can use the line $y = x$ bounded by $x = a$ and $x = b$. The sum of the areas of these inscribed rectangles is the lower sum.



The sum of the areas of these circumscribed rectangles is the upper sum.



You can see that the rectangles do not contain all of the area in the first graph and the rectangles in the second graph cover more than the area of the region. The exact value of the area lies between these two sums.

4. Answers will vary. See definition on page 264. The area of a region bounded above by f and below by the x-axis is given by

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

where $x_{i-1} \le c_i < x_i$ and $\Delta x = \dfrac{b-a}{2}$.

5. $\displaystyle\sum_{i=1}^{6}(3i+2) = 3\sum_{i=1}^{6} i + \sum_{i=1}^{6} 2 = 3(1+2+3+4+5+6) + 12 = 75$

6. $\displaystyle\sum_{k=3}^{9}(k^2+1) = (3^2+1) + (4^2+1) + \dots + (9^2+1) = 287$

7. $\displaystyle\sum_{k=0}^{4}\frac{1}{k^2+1} = 1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} = \frac{158}{85}$

8. $$\sum_{j=2}^{5}\frac{1}{2j} = \frac{1}{2(2)} + \frac{1}{2(3)} + \frac{1}{2(4)} + \frac{1}{2(5)}$$
$$= \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} = \frac{77}{120}$$

9. $\displaystyle\sum_{k=0}^{7} c = c + c + c + c + c + c + c + c = 8c$

10. $\displaystyle\sum_{i=1}^{4}\left[(i-1)^2 + (i+1)^3\right] = (0+8) + (1+27) + (4+64) + (9+125) = 238$

11. $\displaystyle\sum_{i=1}^{11}\frac{1}{5i}$

12. $\displaystyle\sum_{j=1}^{11}\frac{6}{2+j}$

13. $\displaystyle\sum_{j=1}^{6}\left[7\left(\frac{j}{6}\right) + 5\right]$

14. $\displaystyle\sum_{j=1}^{4}\left[1 - \left(\frac{j}{4}\right)^2\right]$

15. $\frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3 - \left(\frac{2i}{n}\right)\right]$

16. $\frac{3}{n}\sum_{i=1}^{n}\left[2\left(1 + \frac{3i}{n}\right)^2\right]$

17. $\sum_{i=1}^{12} 7 = 7(12) = 84$

18. $\sum_{i=1}^{20} (-8) = -8\sum_{i=1}^{20} 1 = (-8)(20) = -160$

19. $\sum_{i=1}^{24} 4i = 4\sum_{i=1}^{24} i = 4\left[\frac{24(25)}{2}\right] = 1200$

20. $\sum_{i=1}^{16}(5i - 4) = 5\sum_{i=1}^{16} i - 4(16) = 5\left[\frac{16(17)}{2}\right] - 64 = 616$

21. $\sum_{i=1}^{20}(i - 1)^2 = \sum_{i=1}^{19} i^2 = \left[\frac{19(20)(39)}{6}\right] = 2470$

22. $\sum_{i=1}^{10}(i^2 - 1) = \sum_{i=1}^{10} i^2 - \sum_{i=1}^{10} 1 = \left[\frac{10(11)(21)}{6}\right] - 10 = 375$

23.
$$\begin{aligned}\sum_{i=1}^{7} i(i + 3)^2 &= \sum_{i=1}^{7} i(i^2 + 6i + 9)\\ &= \sum_{i=1}^{7}(i^3 + 6i^2 + 9i)\\ &= \frac{7^2(7 + 1)^2}{4} + 6\cdot\frac{7(7 + 1)(14 + 1)}{6} + 9\cdot\frac{7(7 + 1)}{2}\\ &= 784 + 840 + 252\\ &= 1876\end{aligned}$$

24.
$$\begin{aligned}\sum_{i=1}^{25}(i^3 - 2i) &= \sum_{i=1}^{25} i^3 - 2\sum_{i=1}^{25} i\\ &= \frac{(25)^2(26)^2}{4} - 2\frac{25(26)}{2}\\ &= 105{,}625 - 650\\ &= 104{,}975\end{aligned}$$

25. $\sum_{i=1}^{n}\frac{2i + 1}{n^2} = \frac{1}{n^2}\sum_{i=1}^{n}(2i + 1) = \frac{1}{n^2}\left[2\frac{n(n + 1)}{2} + n\right] = \frac{n + 2}{n} = 1 + \frac{2}{n} = S(n)$

$$\begin{aligned}S(10) &= \frac{12}{10} = 1.2\\ S(100) &= 1.02\\ S(1000) &= 1.002\\ S(10{,}000) &= 1.0002\end{aligned}$$

26.
$$\begin{aligned}\sum_{j=1}^{n}\frac{7j + 4}{n^2} &= \frac{1}{n^2}\sum_{j=1}^{n}(7j + 4)\\ &= \frac{1}{n^2}\left[7\frac{n(n + 1)}{2} + 4n\right]\\ &= \frac{7n^2 + 7n}{2n^2} + \frac{4n}{n^2} = \frac{7n + 15}{2n} = S(n)\end{aligned}$$

$$\begin{aligned}S(10) &= \tfrac{17}{4} = 4.25\\ S(100) &= 3.575\\ S(1000) &= 3.5075\\ S(10{,}000) &= 3.50075\end{aligned}$$

27. $\displaystyle\sum_{k=1}^{n} \frac{6k(k-1)}{n^3} = \frac{6}{n^3}\sum_{k=1}^{n}\left(k^2 - k\right) = \frac{6}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right]$

$$= \frac{6}{n^2}\left[\frac{2n^2 + 3n + 1 - 3n - 3}{6}\right] = \frac{1}{n^2}\left[2n^2 - 2\right] = 2 - \frac{2}{n^2} = S(n)$$

$S(10) = 1.98$
$S(100) = 1.9998$
$S(1000) = 1.999998$
$S(10{,}000) = 1.99999998$

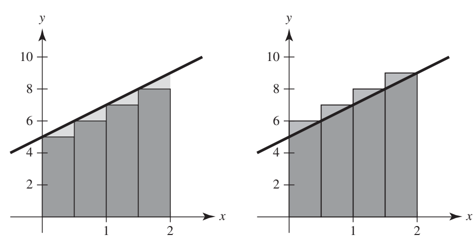
28. $\displaystyle\sum_{i=1}^{n} \frac{2i^3 - 3i}{n^4} = \frac{1}{n^4}\sum_{i=1}^{n}\left(2i^3 - 3i\right)$

$$= \frac{1}{n^4}\left[2\frac{n^2(n+1)^2}{4} - 3\frac{n(n+1)}{2}\right]$$

$$= \frac{(n+1)^2}{2n^2} - \frac{3(n+1)}{2n^3} = \frac{n^3 + 2n^2 - 2n - 3}{2n^3} = S(n)$$

$S(10) = 0.5885$
$S(100) = 0.5098985$
$S(1000) = 0.5009989985$
$S(10{,}000) = 0.50009999$

29.



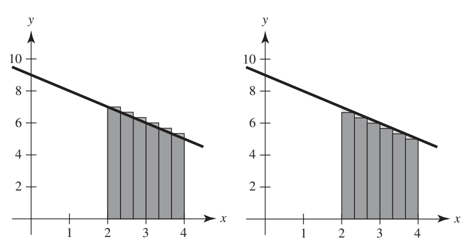
$\Delta x = \dfrac{2 - 0}{4} = \dfrac{1}{2}$

Left endpoints: Area $\approx \frac{1}{2}[5 + 6 + 7 + 8] = \frac{26}{2} = 13$

Right endpoints: Area $\approx \frac{1}{2}[6 + 7 + 8 + 9] = \frac{30}{2} = 15$

$13 < \text{Area} < 15$

30.



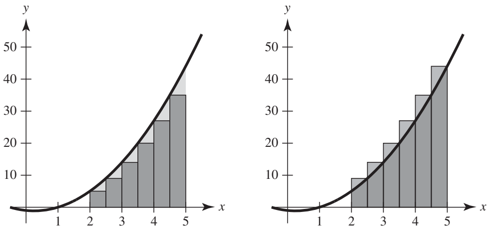
$\Delta x = \dfrac{4 - 2}{6} = \dfrac{1}{3}$

Left endpoints: Area $\approx \dfrac{1}{3}\left[7 + \dfrac{20}{3} + \dfrac{19}{3} + 6 + \dfrac{17}{3} + \dfrac{16}{3}\right] = \dfrac{37}{3} \approx 12.333$

Right endpoints: Area $\approx \dfrac{1}{3}\left[\dfrac{20}{3} + \dfrac{19}{3} + 6 + \dfrac{17}{3} + \dfrac{16}{3} + \dfrac{15}{3}\right] = \dfrac{35}{3} \approx 11.667$

$\dfrac{35}{3} < \text{Area} < \dfrac{37}{3}$

31.



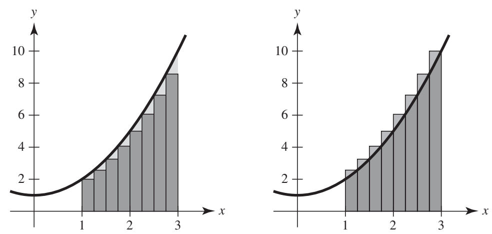
$\Delta x = \dfrac{5-2}{6} = \dfrac{1}{2}$

Left endpoints: Area $\approx \frac{1}{2}[5 + 9 + 14 + 20 + 27 + 35] = 55$

Right endpoints: Area $\approx \frac{1}{2}[9 + 14 + 20 + 27 + 35 + 44] = \frac{149}{2} = 74.5$

$55 < \text{Area} < 74.5$

32.



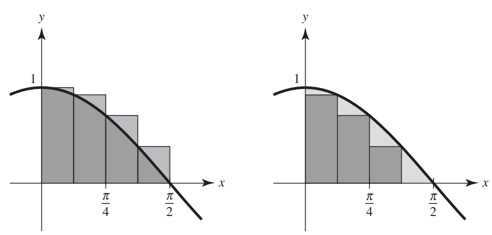
$\Delta x = \dfrac{3-1}{8} = \dfrac{1}{4}$

Left endpoints: Area $\approx \frac{1}{4}\left[2 + \frac{41}{16} + \frac{13}{4} + \frac{65}{16} + 5 + \frac{97}{16} + \frac{29}{4} + \frac{137}{16}\right] = \frac{155}{16} = 9.6875$

Right endpoint: Area $\approx \frac{1}{4}\left[\frac{41}{16} + \frac{13}{4} + \frac{65}{16} + 5 + \frac{97}{16} + \frac{29}{4} + \frac{137}{16} + 10\right] = 11.6875$

$9.6875 < \text{Area} < 11.6875$

33.



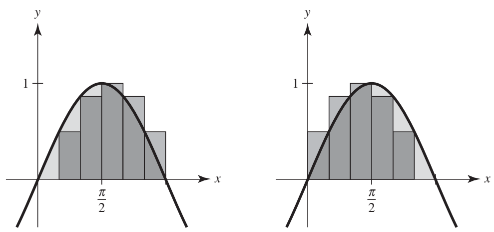
$$\Delta x = \frac{\frac{\pi}{2} - 0}{4} = \frac{\pi}{8}$$

Left endpoints: Area $\approx \dfrac{\pi}{8}\left[\cos(0) + \cos\left(\dfrac{\pi}{8}\right) + \cos\left(\dfrac{\pi}{4}\right) + \cos\left(\dfrac{3\pi}{8}\right)\right] \approx 1.1835$

Right endpoints: Area $\approx \dfrac{\pi}{8}\left[\cos\left(\dfrac{\pi}{8}\right) + \cos\left(\dfrac{\pi}{4}\right) + \cos\left(\dfrac{3\pi}{8}\right) + \cos\left(\dfrac{\pi}{2}\right)\right] \approx 0.7908$

$0.7908 < \text{Area} < 1.1835$

34.



$$\Delta x = \frac{\pi - 0}{6} = \frac{\pi}{6}$$

Left endpoints: Area $\approx \frac{\pi}{6}\left[\sin 0 + \sin\frac{\pi}{6} + \sin\frac{\pi}{3} + \sin\frac{\pi}{2} + \sin\frac{2\pi}{3} + \sin\frac{5\pi}{6}\right] \approx 1.9541$

Right endpoints: Area $\approx \frac{\pi}{6}\left[\sin\frac{\pi}{6} + \sin\frac{\pi}{3} + \sin\frac{\pi}{2} + \sin\frac{2\pi}{3} + \sin\frac{5\pi}{6} + \sin\pi\right] \approx 1.9541$

By symmetry, the answers are the same. The exact area (2) is larger.

35. $S = \left[3 + 4 + \frac{9}{2} + 5\right](1) = \frac{33}{2} = 16.5$

$s = \left[1 + 3 + 4 + \frac{9}{2}\right](1) = \frac{25}{2} = 12.5$

36. $S = [5 + 5 + 4 + 2](1) = 16$

$s = [4 + 4 + 2 + 0](1) = 10$

37. $S(4) = \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) + \sqrt{1}\left(\frac{1}{4}\right) = \frac{1 + \sqrt{2} + \sqrt{3} + 2}{8} \approx 0.768$

$s(4) = 0\left(\frac{1}{4}\right) + \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) = \frac{1 + \sqrt{2} + \sqrt{3}}{8} \approx 0.518$

38. $S(8) = \left(\sqrt{\frac{1}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{3}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{1} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{5}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{3}{2}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{7}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{2} + 2\right)\frac{1}{4}$

$= \frac{1}{4}\left(16 + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{5}}{2} + \frac{\sqrt{6}}{2} + \frac{\sqrt{7}}{2} + \sqrt{2}\right) \approx 6.038$

$s(8) = (0 + 2)\frac{1}{4} + \left(\sqrt{\frac{1}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2\right)\frac{1}{4} + \dots + \left(\sqrt{\frac{7}{4}} + 2\right)\frac{1}{4} \approx 5.685$

39. $S(5) = 1\left(\frac{1}{5}\right) + \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \approx 0.746$

$s(5) = \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5}\right) = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \approx 0.646$

40. $S(5) = 1\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right)$

$= \frac{1}{5}\left[1 + \frac{\sqrt{24}}{5} + \frac{\sqrt{21}}{5} + \frac{\sqrt{16}}{5} + \frac{\sqrt{9}}{5}\right] \approx 0.859$

$s(5) = \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) + 0 \approx 0.659$

41. $f(x) = 3x, [0, 4]$

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n}$$

Left endpoints for lower sum: $m_i = 0 + (i-1)\Delta x = (i-1)\frac{4}{n}$

Right endpoints for upper sum: $M_i = i\Delta x = i\left(\frac{4}{n}\right)$

$$s(n) = \sum_{i=1}^{n} f(m_i)\Delta x = \sum_{i=1}^{n} 3(i-1)\frac{4}{n}\left(\frac{4}{n}\right)$$

$$= \frac{48}{n^2}\sum_{i=1}^{n}(i-1) = \frac{48}{n^2}\left[\frac{n(n+1)}{2} - n\right]$$

$$= \frac{48}{n^2}\left(\frac{n^2}{2} - \frac{n}{2}\right) = 24 - \frac{24}{n}$$

$$S(n) = \sum_{i=1}^{n} f(M_i)\Delta x = \sum_{i=1}^{n} 3i\left(\frac{4}{n}\right)\left(\frac{4}{n}\right)$$

$$= \frac{48}{n^2}\sum_{i=1}^{n} i = \frac{48}{n^2}\cdot\frac{n(n+1)}{2}$$

$$= 24 + \frac{24}{n}$$

42. $f(x) = 6 - 2x, [1, 2]$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

Right endpoints for lower sum: $m_i = 1 + i\Delta x = 1 + i\left(\frac{1}{n}\right)$

Left endpoints for upper sum: $M_i = 1 + (i-1)\Delta x = 1 + (i-1)\left(\frac{1}{n}\right)$

$$s(n) = \sum_{i=1}^{n} f(m_i)\Delta x = \sum_{i=1}^{n}\left[6 - 2\left(1 + i\left(\frac{1}{n}\right)\right)\right]\left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n}\left(4 - \frac{2i}{n}\right)\left(\frac{1}{n}\right) = \frac{1}{n}\sum_{i=1}^{n} 4 - \frac{2}{n^2}\sum_{i=1}^{n} i$$

$$= \frac{1}{n}(4n) - \frac{2}{n^2}\left[\frac{n(n+1)}{2}\right] = 4 - \frac{1}{n^2}\left(n^2 + n\right) = 3 - \frac{1}{n}$$

$$S(n) = \sum_{i=1}^{n} f(M_i)\Delta x = \sum_{i=1}^{n}\left[6 - 2\left(1 + (i-1)\left(\frac{1}{n}\right)\right)\right]\left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n}\left[4 - \frac{2(i-1)}{n}\right]\left(\frac{1}{n}\right) = \frac{1}{n}\sum_{i=1}^{n} 4 - \frac{2}{n^2}\sum_{i=1}^{n} i + \frac{2}{n^2}\sum_{i=1}^{n} 1$$

$$= \frac{1}{n}(4n) - \frac{2}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{2}{n^2}(n) = 4 - \frac{1}{n^2}\left(n^2 + n\right) + \frac{2}{n} = 3 + \frac{1}{n}$$

43. $f(x) = 5x^2, [0, 1]$

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

Right endpoints for upper sum: $M_i = i\Delta x = i\left(\frac{1}{n}\right)$

Left endpoints for lower sum: $m_i = (i-1)\Delta x = (i-1)\left(\frac{1}{n}\right)$

$$s(n) = \sum_{i=1}^{n} f(m_i)\Delta x = \sum_{i=1}^{n} 5\left[(i-1)\frac{1}{n}\right]^2\left(\frac{1}{n}\right)$$

$$= \frac{5}{n^3}\sum_{i=1}^{n}(i-1)^2 = \frac{5}{n^3}\sum_{i=1}^{n}\left(i^2 - 2i + 1\right)$$

$$= \frac{5}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} + n\right]$$

$$= \frac{5}{n^3}\left[\frac{2n^3 + 3n^2 + n}{6} - \left(n^2 + n\right) + n\right] = \frac{5}{6}\left[2 - \frac{3}{n} + \frac{1}{n^2}\right]$$

$$S(n) = \sum_{i=1}^{n} f(M_i)\Delta x = \sum_{i=1}^{n} 5\left(i\frac{1}{n}\right)^2\left(\frac{1}{n}\right)$$

$$= \frac{5}{n^3}\sum_{i=1}^{n} i^2 = \frac{5}{n^3}\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{5}{6}\left(\frac{2n^3 + 3n^2 + n}{n^3}\right) = \frac{5}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

44. $f(x) = 9 - x^2, [0, 2]$

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

Left endpoints for upper sum: $M_i = (i-1)\Delta x = (i-1)\left(\frac{2}{n}\right)$

Right endpoints for lower sum: $m_i = i\Delta x = i\left(\frac{2}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f(M_i)\Delta x = \sum_{i=1}^{n}\left[9 - \left((i-1)\frac{2}{n}\right)^2\right]\left(\frac{2}{n}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} 18 - \frac{8}{n^3}\sum_{i=1}^{n}(i-1)^2 = \frac{1}{n}(18n) - \frac{8}{n^3}\left(\frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}\right)$$

$$= 18 - \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} = \frac{46}{3} + \frac{4}{n} + \frac{4}{3n^2}$$

$$s(n) = \sum_{i=1}^{n} f(m_i)\Delta x = \sum_{i=1}^{n}\left[9 - \left(i\frac{2}{n}\right)^2\right]\left(\frac{2}{n}\right)$$

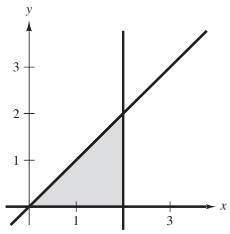
$$= \frac{1}{n}\sum_{i=1}^{n} 18 - \frac{8}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{1}{n}(18n) - \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 18 - \frac{4}{3n^3}\left(2n^3 + 3n^2 + n\right) = 18 - \left(\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}\right)$$

$$= \frac{46}{3} - \frac{4}{n} - \frac{4}{3n^2}$$

45. (a)



(b) $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$

Endpoints: $0 < 1\left(\dfrac{2}{n}\right) < 2\left(\dfrac{2}{n}\right) < \ldots < (n-1)\left(\dfrac{2}{n}\right) < n\left(\dfrac{2}{n}\right) = 2$

(c) Because $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} f\left(\frac{2i-2}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\frac{2}{n} = \sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

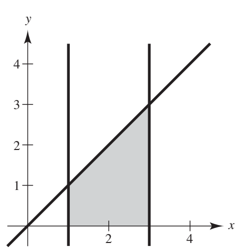
(e)

x	5	10	50	100
$s(n)$	1.6	1.8	1.96	1.98
$S(n)$	2.4	2.2	2.04	2.02

(f) $$\lim_{n\to\infty}\sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n}(i-1) = \lim_{n\to\infty}\frac{4}{n^2}\left[\frac{n(n+1)}{2} - n\right] = \lim_{n\to\infty}\left[\frac{2(n+1)}{n} - \frac{4}{n}\right] = 2$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty}\left(\frac{4}{n^2}\right)\frac{n(n+1)}{2} = \lim_{n\to\infty}\frac{2(n+1)}{n} = 2$$

46. (a)



(b) $\Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}$

Endpoints:

$$1 < 1 + \frac{2}{n} < 1 + \frac{4}{n} < \cdots < 1 + \frac{2n}{n} = 3$$

$$1 < 1 + 1\left(\frac{2}{n}\right) < 1 + 2\left(\frac{2}{n}\right) < \cdots < 1 + (n-1)\left(\frac{2}{n}\right) < 1 + n\left(\frac{2}{n}\right)$$

(c) Because $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x = \sum_{i=1}^{n} f\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(e)

x	5	10	50	100
$s(n)$	3.6	3.8	3.96	3.98
$S(n)$	4.4	4.2	4.04	4.02

(f) $$\lim_{n\to\infty}\sum_{i=1}^{n}\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\left(\frac{2}{n}\right)\left[n + \frac{2}{n}\left(\frac{n(n+1)}{2} - n\right)\right]$$

$$= \lim_{n\to\infty}\left[2 + \frac{2n+2}{n} - \frac{4}{n}\right] = \lim_{n\to\infty}\left[4 - \frac{2}{n}\right] = 4$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{2}{n}\left[n + \left(\frac{2}{n}\right)\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[2 + \frac{2(n+1)}{n}\right] = \lim_{n\to\infty}\left[4 + \frac{2}{n}\right] = 4$$

47. $y = -4x + 5$ on $[0, 1]$. $\left(\textbf{Note}\text{: } \Delta x = \frac{1}{n}\right)$

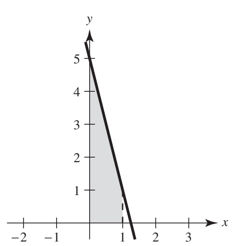
$$s(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[-4\left(\frac{i}{n}\right) + 5\right]\left(\frac{1}{n}\right)$$

$$= -\frac{4}{n^2}\sum_{i=1}^{n} i + 5$$

$$= -\frac{4}{n^2}\frac{n(n+1)}{2} + 5$$

$$= -2\left(1 + \frac{1}{n}\right) + 5$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 3$$



48. $y = 3x - 2$ on $[2, 5]$. $\left(\textbf{Note: } \Delta x = \frac{5-2}{n} = \frac{3}{n}\right)$

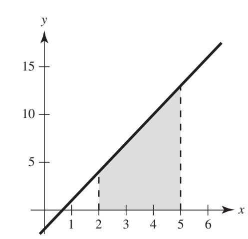
$$S(n) = \sum_{i=1}^{n} f\left(2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \sum_{i=1}^{n}\left[3\left(2 + \frac{3i}{n}\right) - 2\right]\left(\frac{3}{n}\right)$$

$$= 18 + 3\left(\frac{3}{n}\right)^2\sum_{i=1}^{n} i - 6$$

$$= 12 + \frac{27}{n^2}\left(\frac{(n+1)n}{2}\right) = 12 + \frac{27}{2}\left(1 + \frac{1}{n}\right)$$

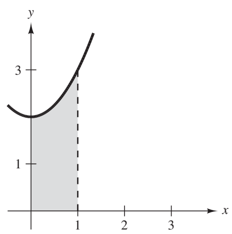
$$\text{Area} = \lim_{n\to\infty} S(n) = 12 + \frac{27}{2} = \frac{51}{2}$$



49. $y = x^2 + 2$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \frac{1}{n}\right)$

$$\begin{aligned} S(n) &= \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) \\ &= \sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^2 + 2\right]\left(\frac{1}{n}\right) \\ &= \left[\frac{1}{n^3}\sum_{i=1}^{n} i^2\right] + 2 \\ &= \frac{n(n+1)(2n+1)}{6n^3} + 2 = \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 2 \end{aligned}$$

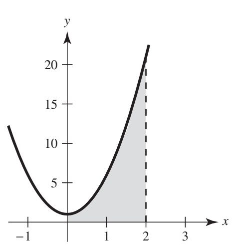
$$\text{Area} = \lim_{n\to\infty} S(n) = \frac{7}{3}$$



50. $y = 5x^2 + 1$ on $[0, 2]$. $\left(\textbf{Note: } \Delta x = \frac{2-0}{n} = \frac{2}{n}\right)$

$$\begin{aligned} s(n) &= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[5\left(\frac{2i}{n}\right)^2 + 1\right]\left(\frac{2}{n}\right) \\ &= \frac{40}{n^3}\sum_{i=1}^{n} i^2 + \frac{2}{n}\sum_{i=1}^{n} 1 \\ &= \frac{40}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{2}{n}(n) \\ &= \frac{20}{3n^2}(n+1)(2n+1) + 2 \end{aligned}$$

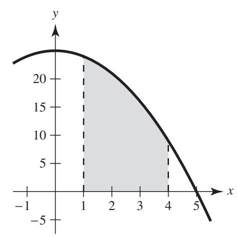
$$\text{Area} = \lim_{n\to\infty} S(n) = \frac{40}{3} + 2 = \frac{46}{3}$$



51. $y = 25 - x^2$ on $[1, 4]$. $\left(\textbf{Note: } \Delta x = \frac{3}{n}\right)$

$$\begin{aligned} s(n) &= \sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left[25 - \left(1 + \frac{3i}{n}\right)^2\right]\left(\frac{3}{n}\right) \\ &= \frac{3}{n}\sum_{i=1}^{n}\left[24 - \frac{9i^2}{n^2} - \frac{6i}{n}\right] \\ &= \frac{3}{n}\left[24n - \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{6}{n}\frac{n(n+1)}{2}\right] \\ &= 72 - \frac{9}{2n^2}(n+1)(2n+1) - \frac{9}{n}(n+1) \end{aligned}$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 72 - 9 - 9 = 54$$

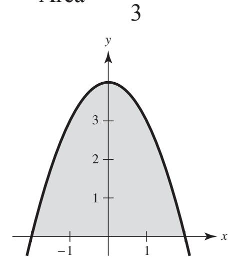


52. $y = 4 - x^2$ on $[-2, 2]$. Find area of region over the interval $[0, 2]$. $\left(\textbf{Note: } \Delta x = \frac{2}{n}\right)$

$$\begin{aligned} s(n) &= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n}\left[4 - \left(\frac{2i}{n}\right)^2\right]\left(\frac{2}{n}\right) \\ &= 8 - \frac{8}{n^3}\sum_{i=1}^{n} i^2 \\ &= 8 - \frac{8n(n+1)(2n+1)}{6n^3} = 8 - \frac{4}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \end{aligned}$$

$$\frac{1}{2}\text{Area} = \lim_{n\to\infty} s(n) = 8 - \frac{8}{3} = \frac{16}{3}$$

$$\text{Area} = \frac{32}{3}$$



53. $y = 27 - x^3$ on $[1, 3]$. $\left(\textbf{Note: } \Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}\right)$

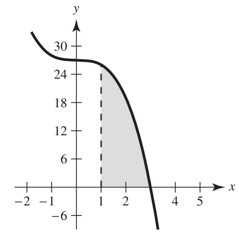
$$s(n) = \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[27 - \left(1 + \frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right)$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left[26 - \frac{8i^3}{n^3} - \frac{12i^2}{n^2} - \frac{6i}{n}\right]$$

$$= \frac{2}{n}\left[26n - \frac{8}{n^3}\frac{n^2(n+1)^2}{4} - \frac{12}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{6}{n}\frac{n(n+1)}{2}\right]$$

$$= 52 - \frac{4}{n^2}(n+1)^2 - \frac{4}{n^2}(n+1)(2n+1) - \frac{6n+1}{n}$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 52 - 4 - 8 - 6 = 34$$



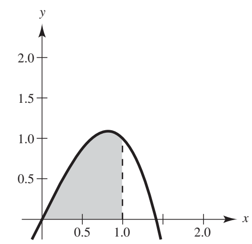
54. $y = 2x - x^3$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}\right)$

Because y both increases and decreases on $[0, 1]$, $T(n)$ is neither an upper nor lower sum.

$$T(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[2\left(\frac{i}{n}\right) - \left(\frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right)$$

$$= \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{n(n+1)}{n^2} - \frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] = 1 + \frac{1}{n} - \frac{1}{4} - \frac{2}{4n} - \frac{1}{4n^2}$$

$$\text{Area} = \lim_{n\to\infty} T(n) = 1 - \frac{1}{4} = \frac{3}{4}$$



55. $y = x^2 - x^3$ on $[-1, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-(-1)}{n} = \dfrac{2}{n}\right)$

Because y both increases and decreases on $[-1, 1]$, $T(n)$ is neither an upper nor a lower sum.

$$T(n) = \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[\left(-1 + \frac{2i}{n}\right)^2 - \left(-1 + \frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right)$$

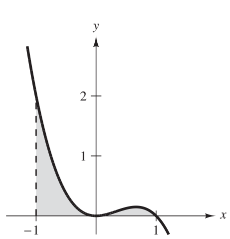
$$= \sum_{i=1}^{n}\left[\left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right)\right]\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left[2 - \frac{10i}{n} + \frac{16i^2}{n^2} - \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right) = \frac{4}{n}\sum_{i=1}^{n} 1 - \frac{20}{n^2}\sum_{i=1}^{n} i + \frac{32}{n^3}\sum_{i=1}^{n} i^2 - \frac{16}{n^4}\sum_{i=1}^{n} i^3$$

$$= \frac{4}{n}(n) - \frac{20}{n^2}\cdot\frac{n(n+1)}{2} + \frac{32}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{16}{n^4}\cdot\frac{n^2(n+1)^2}{4}$$

$$= 4 - 10\left(1 + \frac{1}{n}\right) + \frac{16}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) - 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$$

$$\text{Area} = \lim_{n\to\infty} T(n) = 4 - 10 + \frac{32}{3} - 4 = \frac{2}{3}$$



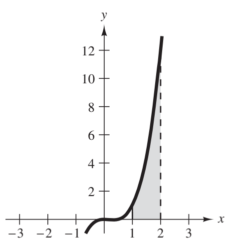
56. $y = 2x^3 - x^2$ on $[1, 2]$. $\left(\textbf{Note: } \Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}\right)$

$$s(n) = \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[2\left(1 + \frac{i}{n}\right)^3 - \left(1 + \frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n}\left(\frac{2i^3}{n^3} + \frac{5i^2}{n^2} + \frac{4i}{n} + 1\right)\left(\frac{1}{n}\right)$$

$$= \frac{2}{n^4} \cdot \frac{n^2(n+1)^2}{4} + \frac{5}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{4}{n^2} \cdot \frac{n(n+1)}{2} + 1$$

$$\text{Area} = \lim_{n\to\infty} s_n = \frac{1}{2} + \frac{5}{3} + 2 + 1 = \frac{31}{6}$$



57. $f(y) = 4y,\ 0 \le y \le 2$ $\left(\textbf{Note: } \Delta y = \dfrac{2-0}{n} = \dfrac{2}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f(m_i)\Delta y$$

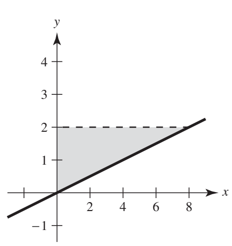
$$= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n} 4\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \frac{16}{n^2}\sum_{i=1}^{n} i$$

$$= \left(\frac{16}{n^2}\right) \cdot \frac{n(n+1)}{2} = \frac{8(n+1)}{n} = 8 + \frac{8}{n}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(8 + \frac{8}{n}\right) = 8$$



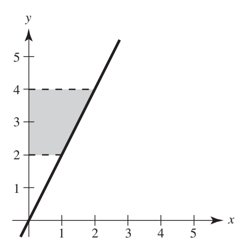
58. $g(y) = \frac{1}{2}y,\ 2 \le y \le 4.$ $\left(\textbf{Note: } \Delta y = \dfrac{4-2}{n} = \dfrac{2}{n}\right)$

$$S(n) = \sum_{i=1}^{n} g\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n} \frac{1}{2}\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \frac{2}{n}\sum_{i=1}^{n}\left(1 + \frac{i}{n}\right)$$

$$= \frac{2}{n}\left[n + \frac{1}{n}\frac{n(n+1)}{2}\right] = 2 + \frac{n+1}{n}$$

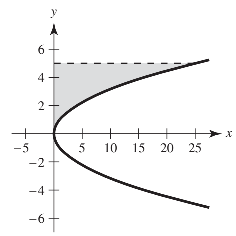
$$\text{Area} = \lim_{n\to\infty} S(n) = 2 + 1 = 3$$



59. $f(y) = y^2, 0 \le y \le 5$ $\left(\textbf{Note: } \Delta y = \frac{5-0}{n} = \frac{5}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f\left(\frac{5i}{n}\right)\left(\frac{5}{n}\right)$$
$$= \sum_{i=1}^{n} \left(\frac{5i}{n}\right)^2\left(\frac{5}{n}\right)$$
$$= \frac{125}{n^3}\sum_{i=1}^{n} i^2$$
$$= \frac{125}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$
$$= \frac{125}{n^2}\left(\frac{2n^2+3n+1}{6}\right) = \frac{125}{3} + \frac{125}{2n} + \frac{125}{6n^2}$$

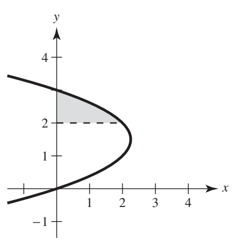
Area $\lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(\frac{125}{3} + \frac{125}{2n} + \frac{125}{6n^2}\right) = \frac{125}{3}$



60. $f(y) = 3y - y^2, 2 \le y \le 3$ $\left(\textbf{Note: } \Delta y = \frac{3-2}{n} = \frac{1}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f\left(2 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \frac{1}{n}\sum_{i=1}^{n}\left[3\left(2+\frac{i}{n}\right) - \left(2+\frac{i}{n}\right)^2\right]$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left(6 + \frac{3i}{n} - 4 - \frac{4i}{n} - \frac{i^2}{n^2}\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left(2 - \frac{i}{n} - \frac{i^2}{n^2}\right)$$
$$= \frac{1}{n}\left[2n - \frac{1}{n}\cdot\frac{n(n+1)}{2} - \frac{1}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$

Area $= \lim_{n\to\infty} S(n) = 2 - \frac{1}{2} - \frac{1}{3} = \frac{7}{6}$



61. $g(y) = 4y^2 - y^3,\ 1 \le y \le 3.\ \left(\text{Note: } \Delta y = \dfrac{3-1}{n} = \dfrac{2}{n}\right)$

$$S(n) = \sum_{i=1}^{n} g\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

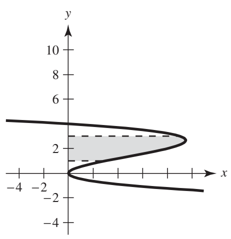
$$= \sum_{i=1}^{n}\left[4\left(1 + \frac{2i}{n}\right)^2 - \left(1 + \frac{2i}{n}\right)^3\right]\frac{2}{n}$$

$$= \frac{2}{n}\sum_{i=1}^{n} 4\left[1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right] - \left[1 + \frac{6i}{n} + \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right]$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left[3 + \frac{10i}{n} + \frac{4i^2}{n^2} - \frac{8i^3}{n^3}\right]$$

$$= \frac{2}{n}\left[3n + \frac{10}{n}\frac{n(n+1)}{2} + \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{8}{n^2}\frac{n^2(n+1)^2}{4}\right]$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 6 + 10 + \frac{8}{3} - 4 = \frac{44}{3}$$



62. $h(y) = y^3 + 1,\ 1 \le y \le 2\ \left(\textbf{Note: } \Delta y = \dfrac{1}{n}\right)$

$$S(n) = \sum_{i=1}^{n} h\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right)$$

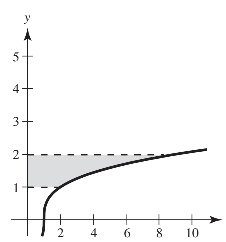
$$= \sum_{i=1}^{n}\left[\left(1 + \frac{i}{n}\right)^3 + 1\right]\frac{1}{n}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(2 + \frac{i^3}{n^3} + \frac{3i^2}{n^2} + \frac{3i}{n}\right)$$

$$= \frac{1}{n}\left[2n + \frac{1}{n^3}\frac{n^2(n+1)^2}{4} + \frac{3}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}\frac{3n(n+1)}{2n}\right]$$

$$= 2 + \frac{(n+1)^2}{n^2 4} + \frac{1}{2}\frac{(n+1)(2n+1)}{n^2} + \frac{3(n+1)}{2n}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 2 + \frac{1}{4} + 1 + \frac{3}{2} = \frac{19}{4}$$



63. $f(x) = x^2 + 3,\ 0 \le x \le 2,\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$\Delta x = \dfrac{1}{2},\ c_1 = \dfrac{1}{4},\ c_2 = \dfrac{3}{4},\ c_3 = \dfrac{5}{4},\ c_4 = \dfrac{7}{4}$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{4}\left[c_i^2 + 3\right]\left(\frac{1}{2}\right) = \frac{1}{2}\left[\left(\frac{1}{16} + 3\right) + \left(\frac{9}{16} + 3\right) + \left(\frac{25}{16} + 3\right) + \left(\frac{49}{16} + 3\right)\right] = \frac{69}{8}$$

64. $f(x) = x^2 + 4x,\ 0 \le x \le 4,\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$\Delta x = 1,\ c_1 = \dfrac{1}{2},\ c_2 = \dfrac{3}{2},\ c_3 = \dfrac{5}{2},\ c_4 = \dfrac{7}{2}$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}\left[c_i^2 + 4c_i\right](1) = \left[\left(\frac{1}{4} + 2\right) + \left(\frac{9}{4} + 6\right) + \left(\frac{25}{4} + 10\right) + \left(\frac{49}{4} + 14\right)\right] = 53$$

65. $f(x) = \tan x,\ 0 \le x \le \dfrac{\pi}{4},\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$\Delta x = \dfrac{\pi}{16},\ c_1 = \dfrac{\pi}{32},\ c_2 = \dfrac{3\pi}{32},\ c_3 = \dfrac{5\pi}{32},\ c_4 = \dfrac{7\pi}{32}$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{4}(\tan c_i)\left(\frac{\pi}{16}\right) = \frac{\pi}{16}\left(\tan\frac{\pi}{32} + \tan\frac{3\pi}{32} + \tan\frac{5\pi}{32} + \tan\frac{7\pi}{32}\right) \approx 0.345$$

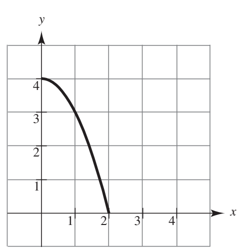
66. $f(x) = \cos x,\ 0 \le x \le \dfrac{\pi}{2},\ n = 4$

Let $c_i = \dfrac{x_i + x_{i+1}}{2}$.

$\Delta x = \dfrac{\pi}{8},\ c_1 = \dfrac{\pi}{16},\ c_2 = \dfrac{3\pi}{16},\ c_3 = \dfrac{5\pi}{16},\ c_4 = \dfrac{7\pi}{16}$

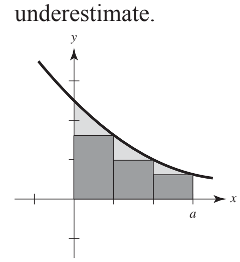
$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{4}\cos(ci)\left(\frac{\pi}{8}\right) = \frac{\pi}{8}\left(\cos\frac{\pi}{16} + \cos\frac{3\pi}{16} + \cos\frac{5\pi}{16} + \cos\frac{7\pi}{16}\right) \approx 1.006$$

67.



(b) $A \approx 6$ square units

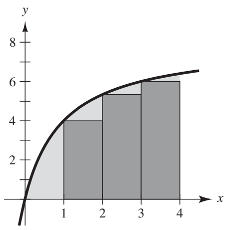
68. The rectangles are inscribed and therefore the area is an underestimate.



69. In general, an overestimate on one side of the midpoint compensates for an underestimate on the other side.

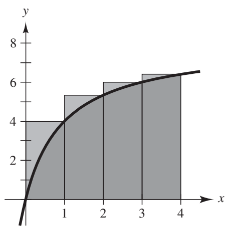
70. Yes. Consider a linear function such as $f(x) = -x + 2$ on the interval $[0, 2]$.

71. (a)



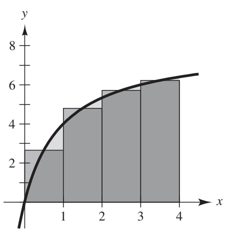
Lower sum: $s(4) = 0 + 4 + 5\frac{1}{3} + 6 = 15\frac{1}{3} = \frac{46}{3} \approx 15.333$

(b)



Upper sum: $S(4) = 4 + 5\frac{1}{3} + 6 + 6\frac{2}{5} = 21\frac{11}{15} = \frac{326}{15} \approx 21.733$

(c)



Midpoint Rule: $M(4) = 2\frac{2}{3} + 4\frac{4}{5} + 5\frac{5}{7} + 6\frac{2}{9} = \frac{6112}{315} \approx 19.403$

(d) In each case, $\Delta x = 4/n$. The lower sum uses left end-points, $(i - 1)(4/n)$. The upper sum uses right endpoints, $(i)(4/n)$. The Midpoint Rule uses midpoints, $\left(i - \frac{1}{2}\right)(4/n)$.

(e)

N	4	8	20	100	200
$s(n)$	15.333	17.368	18.459	18.995	19.06
$S(n)$	21.733	20.568	19.739	19.251	19.188
$M(n)$	19.403	19.201	19.137	19.125	19.125

(f) $s(n)$ increases because the lower sum approaches the exact value as n increases. $S(n)$decreases because the upper sum approaches the exact value as n increases. Because of the shape of the graph, the lower sum is always smaller than the exact value, whereas the upper sum is always larger.

72. (a) Left endpoint of first subinterval is 1.

Left endpoint of last subinterval is $4 - \frac{1}{4} = \frac{15}{4}$.

(b) Right endpoint of first subinterval is $1 + \frac{1}{4} = \frac{5}{4}$.

Right endpoint of second subinterval is $1 + \frac{1}{2} = \frac{3}{2}$.

(c) The rectangles lie above the graph.

(d) The heights would be equal to that constant.

73. True. (Theorem 4.2 (2))

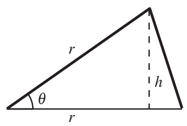
74. True. (Theorem 4.3)

75. Suppose there are n rows and $n + 1$ columns in the figure. The stars on the left total $1 + 2 + \cdots + n$, as do the stars on the right. There are $n(n + 1)$ stars in total, so

$$2[1 + 2 + \cdots + n] = n(n + 1)$$
$$1 + 2 + \cdots + n = \frac{1}{2}(n)(n + 1).$$

76. (a) $\theta = \dfrac{2\pi}{n}$

(b) $\sin\theta = \dfrac{h}{r}$



$h = r\sin\theta$

$$A = \frac{1}{2}bh = \frac{1}{2}r(r\sin\theta) = \frac{1}{2}r^2\sin\theta$$

(c) $$A_n = n\left(\frac{1}{2}r^2\sin\frac{2\pi}{n}\right)$$

$$= \frac{r^2 n}{2}\sin\frac{2\pi}{n} = \pi r^2\left(\frac{\sin(2\pi/n)}{2\pi/n}\right)$$

Let $x = 2\pi/n$. As $n \to \infty$, $x \to 0$.

$$\lim_{n\to\infty} A_n = \lim_{x\to 0}\pi r^2\left(\frac{\sin x}{x}\right) = \pi r^2(1) = \pi r^2$$

77. If n is even, the number of seats is

$$\sum_{i=1}^{n/2} 2i = 2\frac{(n/2)(n/2+1)}{2} = \frac{n^2+2n}{4}.$$

For example, if $n = 8$, then the number of seats is

$$8 + 6 + 4 + 2 = 20 = \frac{8^2 + 2(8)}{4}.$$

If n is odd, the number of seats can be calculated from the case above:

$$\frac{(n+1)^2 + 2(n+1)}{4} - \frac{n+1}{2} = \frac{(n+1)^2}{4}.$$

For example, if $n = 9$, then the number of seats is

$$(10 + 8 + 6 + 4 + 2) - (5) = 30 - 5 = 25.$$

78. (a) $$\sum_{i=1}^{n} 2i = n(n+1)$$

The formula is true for $n = 1$: $2 = 1(1+1) = 2$.

Assume that the formula is true for $n = k$: $\displaystyle\sum_{i=1}^{k} 2i = k(k+1)$.

Then you have $$\sum_{i=1}^{k+1} 2i = \sum_{i=1}^{k} 2i + 2(k+1) = k(k+1) + 2(k+1) = (k+1)(k+2)$$

which shows that the formula is true for $n = k + 1$.

(b) $$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

The formula is true for $n = 1$ because $1^3 = \dfrac{1^2(1+1)^2}{4} = \dfrac{4}{4} = 1$.

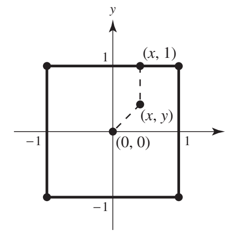
Assume that the formula is true for $n = k$: $\displaystyle\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$.

Then you have $$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2}{4}\left[k^2 + 4(k+1)\right] = \frac{(k+1)^2}{4}(k+2)^2$$

which shows that the formula is true for $n = k + 1$.

79. Assume that the dartboard has corners at $(\pm 1, \pm 1)$.

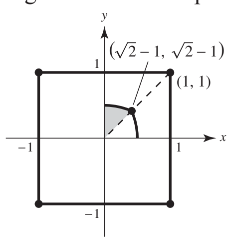
A point (x, y) in the square is closer to the center than the top edge if



$$\sqrt{x^2 + y^2} \le 1 - y$$
$$x^2 + y^2 \le 1 - 2y + y^2$$
$$y \le \tfrac{1}{2}\left(1 - x^2\right).$$

By symmetry, a point (x, y) in the square is closer to the center than the right edge if $x \le \frac{1}{2}\left(1 - y^2\right)$.

In the first quadrant, the parabolas $y = \frac{1}{2}\left(1 - x^2\right)$ and $x = \frac{1}{2}\left(1 - y^2\right)$ intersect at $\left(\sqrt{2} - 1, \sqrt{2} - 1\right)$. There are 8 equal regions that make up the total region, as indicated in the figure.



Area of shaded region $S = \int_0^{\sqrt{2}-1} \left[\frac{1}{2}\left(1 - x^2\right) - x\right] dx = \frac{2\sqrt{2}}{3} - \frac{5}{6}$

Probability $= \frac{8S}{\text{Area square}} = 2\left[\frac{2\sqrt{2}}{3} - \frac{5}{6}\right] = \frac{4\sqrt{2}}{3} - \frac{5}{3}$

Section 4.3 Riemann Sums and Definite Integrals

1. A Riemann sum represents the sum of all of the sub regions for a function f defined on the interval $[a, b]$.

2. For a continuous and nonnegative function f on the interval $[a, b]$, the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by the definite integral $\int_a^b f(x)\, dx$.

3. $f(x) = \sqrt{x},\ y = 0,\ x = 0,\ x = 3,\ c_i = \dfrac{3i^2}{n^2}$

$$\Delta x_i = \frac{3i^2}{n^2} - \frac{3(i - 1)^2}{n^2} = \frac{3}{n^2}(2i - 1)$$

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{3i^2}{n^2}}\, \frac{3}{n^2}(2i - 1) \\
&= \lim_{n\to\infty} \frac{3\sqrt{3}}{n^3} \sum_{i=1}^{n} \left(2i^2 - i\right) \\
&= \lim_{n\to\infty} \frac{3\sqrt{3}}{n^3}\left[2\frac{n(n + 1)(2n + 1)}{6} - \frac{n(n + 1)}{2}\right] \\
&= \lim_{n\to\infty} 3\sqrt{3}\left[\frac{(n + 1)(2n + 1)}{3n^2} - \frac{n + 1}{2n^2}\right] \\
&= 3\sqrt{3}\left[\frac{2}{3} - 0\right] = 2\sqrt{3} \approx 3.464
\end{aligned}$$

4. $f(x) = \sqrt[3]{x},\ y = 0,\ x = 0,\ x = 1,\ c_i = \dfrac{i^3}{n^3}$

$$\Delta x_i = \frac{i^3}{n^3} - \frac{(i-1)^3}{n^3} = \frac{3i^2 - 3i + 1}{n^3}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt[3]{\frac{i^3}{n^3}}\left[\frac{3i^2 - 3i + 1}{n^3}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}\left(3i^3 - 3i^2 + i\right)$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left[3\left(\frac{n^2(n+1)^2}{4}\right) - 3\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{3n^4 + 6n^3 + 3n^2}{4} - \frac{2n^3 + 3n^2 + n}{2} + \frac{n^2 + n}{2}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{3n^4}{4} + \frac{n^3}{2} - \frac{n^2}{4}\right] = \lim_{n\to\infty}\left[\frac{3}{4} + \frac{1}{2n} - \frac{1}{4n^2}\right] = \frac{3}{4}$$

5. $y = 8$ on $[2, 6]$. $\left(\textbf{Note: } \Delta x = \dfrac{6-2}{n} = \dfrac{4}{n},\ \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(2 + \frac{4i}{n}\right)\left(\frac{4}{n}\right) = \sum_{i=1}^{n} 8\left(\frac{4}{n}\right) = \sum_{i=1}^{n}\frac{32}{n} = \frac{1}{n}\sum_{i=1}^{n} 32 = \frac{1}{n}(32n) = 32$$

$$\int_2^6 8\,dx = \lim_{n\to\infty} 32 = 32$$

6. $y = x$ on $[-2, 3]$. $\left(\textbf{Note: } \Delta x = \dfrac{3-(-2)}{n} = \dfrac{5}{n},\ \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right)$$

$$= \sum_{i=1}^{n}\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right) = -10 + \frac{25}{n^2}\sum_{i=1}^{n} i = -10 + \left(\frac{25}{n^2}\right)\frac{n(n+1)}{2} = -10 + \frac{25}{2}\left(1 + \frac{1}{n}\right) = \frac{5}{2} + \frac{25}{2n}$$

$$\int_{-2}^{3} x\,dx = \lim_{n\to\infty}\left(\frac{5}{2} + \frac{25}{2n}\right) = \frac{5}{2}$$

7. $y = x^3$ on $[-1, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-(-1)}{n} = \dfrac{2}{n},\ \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left(-1 + \frac{2i}{n}\right)^3\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left[-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right)$$

$$= -2 + \frac{12}{n^2}\sum_{i=1}^{n} i - \frac{24}{n^3}\sum_{i=1}^{n} i^2 + \frac{16}{n^4}\sum_{i=1}^{n} i^3$$

$$= -2 + 6\left(1 + \frac{1}{n}\right) - 4\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{2}{n}$$

$$\int_{-1}^{1} x^3\,dx = \lim_{n\to\infty}\frac{2}{n} = 0$$

8. $y = 4x^2$ on $[1, 4]$. $\left(\textbf{Note: } \Delta x = \dfrac{4-1}{n} = \dfrac{3}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\begin{aligned}
\sum_{i=1}^{n} f(c_i)\Delta x_i &= \sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)\\
&= \sum_{i=1}^{n} 4\left(1 + \frac{3i}{n}\right)^2\left(\frac{3}{n}\right)\\
&= \frac{12}{n}\sum_{i=1}^{n}\left(1 + \frac{6i}{n} + \frac{9i^2}{n^2}\right)\\
&= \frac{12}{n}\left[n + \frac{6}{n}\frac{n(n+1)}{2} + \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right]\\
&= 12 + 36\frac{n+1}{n} + 18\frac{(n+1)(2n+1)}{n^2}\\
\int_1^4 4x^2\,dx &= \lim_{n\to\infty}\left[12 + \frac{36(n+1)}{n} + \frac{18(n+1)(2n+1)}{n^2}\right]\\
&= 12 + 36 + 36 = 84
\end{aligned}$$

9. $y = x^2 + 1$ on $[1, 2]$. $\left(\textbf{Note: } \Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\begin{aligned}
\sum_{i=1}^{n} f(c_i)\,\Delta x_i &= \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right)\\
&= \sum_{i=1}^{n}\left[\left(1 + \frac{i}{n}\right)^2 + 1\right]\left(\frac{1}{n}\right)\\
&= \sum_{i=1}^{n}\left[1 + \frac{2i}{n} + \frac{i^2}{n^2} + 1\right]\left(\frac{1}{n}\right)\\
&= 2 + \frac{2}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2 = 2 + \left(1 + \frac{1}{n}\right) + \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}\\
\int_1^2 \left(x^2 + 1\right)dx &= \lim_{n\to\infty}\left(\frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}\right) = \frac{10}{3}
\end{aligned}$$

10. $y = 2x^2 + 3$ on $[-2, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-(-2)}{n} = \dfrac{3}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\begin{aligned}
\sum_{i=1}^{n} f(c_i)\Delta x_i &= \sum_{i=1}^{n} f\left(-2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)\\
&= \sum_{i=1}^{n}\left[2\left(-2 + \frac{3i}{n}\right)^2 + 3\right]\left(\frac{3}{n}\right)\\
&= \frac{3}{n}\sum_{i=1}^{n}\left[2\left(4 - \frac{12i}{n} + \frac{9i^2}{n^2}\right) + 3\right]\\
&= \frac{3}{n}\sum_{i=1}^{n}\left[11 - \frac{24i}{n} + \frac{18i^2}{n^2}\right]\\
&= \frac{3}{n}\left[11n - \frac{24}{n}\frac{n(n+1)}{2} + \frac{18}{n^2}\frac{n(n+1)(2n+1)}{6}\right] = 33 - 36\frac{n+1}{n} + 9\frac{(n+1)(2n+1)}{n^2}\\
\int_{-2}^{1}\left(2x^2 + 3\right)dx &= \lim_{n\to\infty}\left[33 - 36\frac{n+1}{n} + 9\frac{(n+1)(2n+1)}{n^2}\right] = 33 - 36 + 18 = 15
\end{aligned}$$

11. $\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(3c_i + 10)\,\Delta x_i = \int_{-1}^{5}(3x + 10)\,dx$

on the interval $[-1, 5]$.

12. $\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\sqrt{c_i^2 + 4}\,\Delta x_i = \int_{0}^{3}\sqrt{x^2 + 4}\,dx$

on the interval $[0, 3]$.

13. $\int_0^4 5\,dx$

14. $\int_0^2 (6 - 3x)\,dx$

15. $\int_{-4}^4 (4 - |x|)\,dx$

16. $\int_0^2 x^2\,dx$

17. $\int_{-5}^5 (25 - x^2)\,dx$

18. $\int_{-1}^1 \frac{4}{x^2 + 2}\,dx$

19. $\int_0^{\pi/2} \cos x\,dx$

20. $\int_0^{\pi/4} \tan x\,dx$

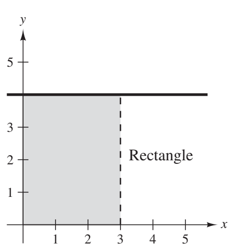
21. $\int_0^2 y^3\,dy$

22. $\int_0^2 (y - 2)^2\,dy$

23. Rectangle

$A = bh = 3(4)$

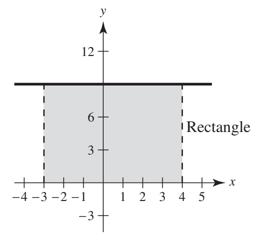
$A = \int_0^3 4\,dx = 12$



24. Rectangle

$A = bh = 7(9) = 63$

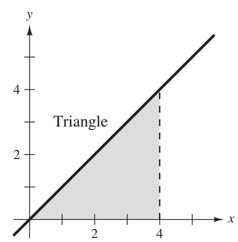
$A = \int_{-3}^4 9\,dx = 63$



25. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(4)(4) = 8$

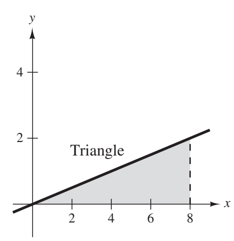
$A = \int_0^4 x\,dx = 8$



26. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(8)(2) = 8$

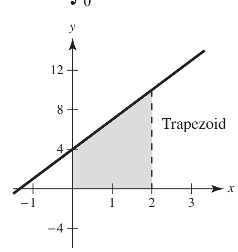
$A = \int_0^8 \frac{x}{4}\,dx = 8$



27. Trapezoid

$A = \frac{b_1 + b_2}{2}h = \left(\frac{4 + 10}{2}\right)2 = 14$

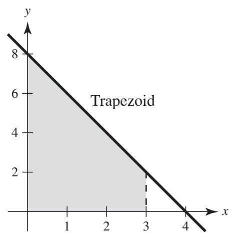
$A = \int_0^2 (3x + 4)\,dx = 14$



28. Trapezoid

$$A = \frac{b_1 + b_2}{2}h = \frac{8 + 2}{2}(3) = 15$$

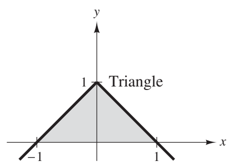
$$A = \int_0^3 (8 - 2x)\,dx = 15$$



29. Triangle

$$A = \tfrac{1}{2}bh = \tfrac{1}{2}(2)(1) = 1$$

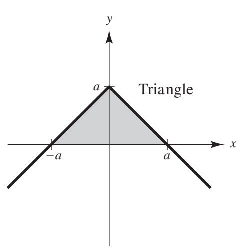
$$A = \int_{-1}^1 \left(1 - |x|\right) dx = 1$$



30. Triangle

$$A = \tfrac{1}{2}bh = \tfrac{1}{2}(2a)\,a = a^2$$

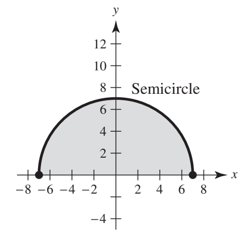
$$A = \int_{-a}^a \left(a - |x|\right) dx = a^2$$



31. Semicircle

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(7)^2 = \frac{49\pi}{2}$$

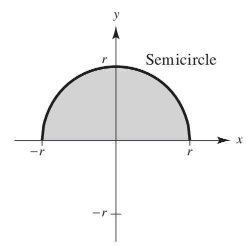
$$A = \int_{-7}^7 \sqrt{49 - x^2}\,dx = \frac{49\pi}{2}$$



32. Semicircle

$$A = \tfrac{1}{2}\pi r^2$$

$$A = \int_{-r}^r \sqrt{r^2 - x^2}\,dx = \tfrac{1}{2}\pi r^2$$



In Exercises 33–40,

$$\int_2^6 x^3\,dx = 320,\ \int_2^6 x\,dx = 16,\ \int_2^6 dx = 4.$$

33. $\int_6^2 x^3\,dx = -\int_2^6 x^3\,dx = -320$

34. $\int_2^2 x\,dx = 0$

35. $\int_2^6 \frac{1}{4}x^3\,dx = \frac{1}{4}\int_2^6 x^3\,dx = \frac{1}{4}(320) = 80$

36. $\int_2^6 -3x\,dx = -3\int_2^6 x\,dx = -3(16) = -48$

37. $\int_2^6 (x - 14)\,dx = \int_2^6 x\,dx - 14\int_2^6 dx = 16 - 14(4) = -40$

38. $\int_2^6 \left(6x - \frac{1}{8}x^3\right) dx = 6\int_2^6 x\,dx - \frac{1}{8}\int_2^6 x^3 dx = 6(16) - \frac{1}{8}(320) = 56$

39. $\int_2^6 \left(2x^3 - 10x + 7\right) dx = 2\int_2^6 x^3\,dx - 10\int_2^6 x\,dx + 7\int_2^6 dx$

$= 2(320) - 10(16) + 7(4) = 508$

40. $\int_2^6 \left(21 - 5x - x^3\right) dx = 21\int_2^6 dx - 5\int_2^6 x\,dx - \int_2^6 x^3\,dx$

$= 21(4) - 5(16) - 320 = -316$

41. (a) $\int_0^7 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx = 10 + 3 = 13$

(b) $\int_5^0 f(x)\,dx = -\int_0^5 f(x)\,dx = -10$

(c) $\int_5^5 f(x)\,dx = 0$

(d) $\int_0^5 3f(x)\,dx = 3\int_0^5 f(x)\,dx = 3(10) = 30$

42. (a) $\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx = 4 + (-1) = 3$

(b) $\int_6^3 f(x)\,dx = -\int_3^6 f(x)\,dx = -(-1) = 1$

(c) $\int_3^3 f(x)\,dx = 0$

(d) $\int_3^6 -5f(x)\,dx = -5\int_3^6 f(x)\,dx = -5(-1) = 5$

43. (a) $\int_2^6 [f(x) + g(x)]\,dx = \int_2^6 f(x)\,dx + \int_2^6 g(x)\,dx$
$= 10 + (-2) = 8$

(b) $\int_2^6 [g(x) - f(x)]\,dx = \int_2^6 g(x)\,dx - \int_2^6 f(x)\,dx$
$= -2 - 10 = -12$

(c) $\int_2^6 2g(x)\,dx = 2\int_2^6 g(x)\,dx = 2(-2) = -4$

(d) $\int_2^6 3f(x)\,dx = 3\int_2^6 f(x)\,dx = 3(10) = 30$

44. (a) $\int_{-1}^0 f(x)\,dx = \int_{-1}^1 f(x)\,dx - \int_0^1 f(x)\,dx$
$= 0 - 5 = -5$

(b) $\int_0^1 f(x)\,dx - \int_1^0 f(x)\,dx = 5 - (-5) = 10$

(c) $\int_{-1}^1 3f(x)\,dx = 3\int_{-1}^1 f(x)\,dx = 3(0) = 0$

(d) $\int_0^1 3f(x)\,dx = 3\int_0^1 f(x)\,dx = 3(5) = 15$

45. Lower estimate: $[24 + 12 - 4 - 20 - 36](2) = -48$

Upper estimate: $[32 + 24 + 12 - 4 - 20](2) = 88$

46. (a) $[-6 + 8 + 30](2) = 64$ left endpoint estimate

(b) $[8 + 30 + 80](2) = 236$ right endpoint estimate

(c) $[0 + 18 + 50](2) = 136$ midpoint estimate

If f is increasing, then (a) is below the actual value and (b) is above.

47. (a) Quarter circle below x-axis:
$-\frac{1}{4}\pi r^2 = -\frac{1}{4}\pi(2)^2 = -\pi$

(b) Triangle: $\frac{1}{2}bh = \frac{1}{2}(4)(2) = 4$

(c) Triangle + Semicircle below x-axis:
$-\frac{1}{2}(2)(1) - \frac{1}{2}\pi(2)^2 = -(1 + 2\pi)$

(d) Sum of parts (b) and (c): $4 - (1 + 2\pi) = 3 - 2\pi$

(e) Sum of absolute values of (b) and (c):
$4 + (1 + 2\pi) = 5 + 2\pi$

(f) Answers to (d) plus
$2(10) = 20$: $(3 - 2\pi) + 20 = 23 - 2\pi$

48. (a) $\int_0^1 -f(x)\,dx = -\int_0^1 f(x)\,dx = \frac{1}{2}$

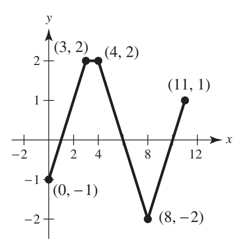
(b) $\int_3^4 3f(x)\,dx = 3(2) = 6$

(c) $\int_0^7 f(x)\,dx = -\frac{1}{2} + \frac{1}{2}(2)(2) + 2 + \frac{1}{2}(2)(2) - \frac{1}{2} = 5$

(d) $\int_5^{11} f(x)\,dx = \frac{1}{2} - \frac{1}{2}(4)(2) + \frac{1}{2} = -3$

(e) $\int_0^{11} f(x)\,dx = -\frac{1}{2} + 2 + 2 + 2 - 4 + \frac{1}{2} = 2$

(f) $\int_4^{10} f(x)\,dx = 2 - 4 = -2$



49. (a) $\int_0^5 [f(x) + 2]\,dx = \int_0^5 f(x)\,dx + \int_0^5 2\,dx$
$= 4 + 10 = 14$

(b) $\int_{-2}^3 f(x + 2)\,dx = \int_0^5 f(x)\,dx = 4$ (Let $u = x + 2$.)

(c) $\int_{-5}^5 f(x)\,dx = 2\int_0^5 f(x)\,dx = 2(4) = 8$ (f even)

(d) $\int_{-5}^5 f(x)\,dx = 0$ (f odd)

50. (a) The left endpoint approximation will be greater than the actual area so,

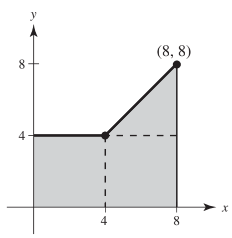
$$\sum_{i=1}^{n} f(x_i)\Delta x > \int_1^5 f(x)\,dx.$$

(b) The right endpoint approximation will be less than the actual area so,

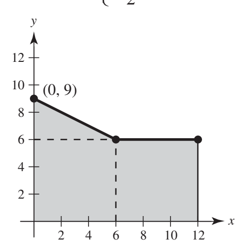
$$\sum_{i=1}^{n} f(x_i)\Delta x < \int_1^5 f(x)\,dx.$$

51. $f(x) = \begin{cases} 4, & x < 4 \\ x, & x \geq 4 \end{cases}$

$$\int_0^8 f(x)\,dx = 4(4) + 4(4) + \tfrac{1}{2}(4)(4) = 40$$

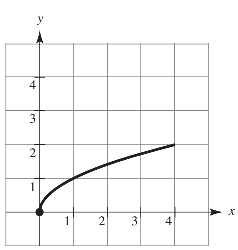


52. $f(x) = \begin{cases} 6, & x > 6 \\ -\frac{1}{2}x + 9, & x \leq 6 \end{cases}$



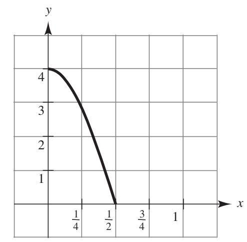
$$\int_0^{12} f(x)\,dx = 6(6) + \tfrac{1}{2}6(3) + 6(6) = 36 + 9 + 36 = 81$$

53.



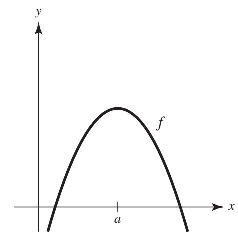
(a) $A \approx 5$ square units

54.



(b) $A \approx \frac{4}{3}$ square units

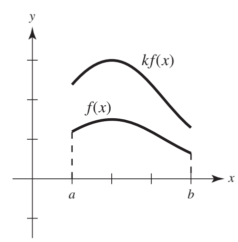
55. Answers will vary. *Sample answer:*



$\int_a^a f(x)\,dx = 0$ because there is no region.

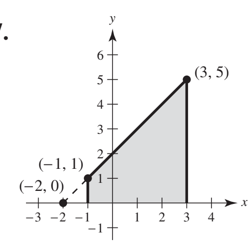
56. Answers will vary. *Sample answer:*

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



We have k times the area under the graph of f.

57.



Method 1, difference of two triangles:

$$\int_{-1}^{3} (x+2)\,dx = \int_{-2}^{3} (x+2)\,dx - \int_{-2}^{1} (x+2)\,dx$$

$$= \frac{1}{2}(5)(5) - \frac{1}{2}(1)(1) = \frac{25}{2} - \frac{1}{2} = 12$$

Method 2, limit definition of area:

$$\Delta x = \frac{4}{n},\ x_i = -1 + i\Delta x$$

$$S(n) = \sum_{i=1}^{n}\left[(-1 + i\Delta x) + 2\right]\Delta x = \sum_{i=1}^{n}\left[1 + i\left(\frac{4}{n}\right)\right]\left(\frac{4}{n}\right)$$

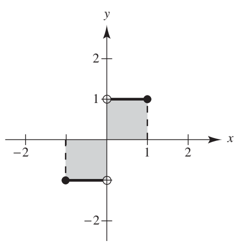
$$= \frac{4}{n}\sum_{i=1}^{n} 1 + \frac{16}{n^2}\sum_{i=1}^{n} i = 4 + \frac{16}{n^2}\cdot\frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} S(n) = 4 + 8 = 12$$

58. $f(x) = |x|/x$ is integrable on $[-1, 1]$, but is not continuous on $[-1, 1]$. There is discontinuity at $x = 0$. To see that

$$\int_{-1}^{1} \frac{|x|}{x}\, dx$$

is integrable, sketch a graph of the region bounded by $f(x) = |x|/x$ and the x-axis for $-1 \le x \le 1$. You see that the integral equals 0.



59. $\int_{-2}^{1} f(x)\, dx + \int_{1}^{5} f(x)\, dx = \int_{-2}^{5} f(x)\, dx$

$a = -2,\ b = 5$

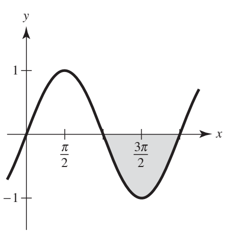
60. $\int_{-3}^{3} f(x)\, dx + \int_{3}^{6} f(x)\, dx - \int_{a}^{b} f(x)\, dx = \int_{-1}^{6} f(x)\, dx$

$$\int_{-3}^{6} f(x)\, dx + \int_{b}^{a} f(x)\, dx = \int_{-1}^{6} f(x)\, dx$$

$a = -3,\ b = -1$

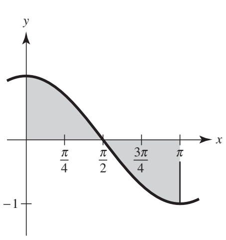
61. Answers will vary. *Sample answer:* $a = \pi,\ b = 2\pi$

$$\int_{\pi}^{2\pi} \sin x\, dx < 0$$



62. Answers will vary. *Sample answer:* $a = 0,\ b = \pi$

$$\int_{0}^{\pi} \cos x\, dx = 0$$



63. True

64. False

Consider $f(x) = x$ and $g(x) = x + 1$.

$$\int_0^1 x\, dx = (\text{Area of triangle}) = \frac{1}{2}(1)(1) = \frac{1}{2}$$

$$\int_0^1 (x+1)\, dx = (\text{Area of trapezoid}) = \frac{1}{2}(1)(2+1) = \frac{3}{2}$$

$$\int_0^1 x(x+1)\, dx = \lim_{n\to\infty}\left[\frac{n(n+1)(2n+1)}{6n^3} + \frac{n(n+1)}{2n^2}\right] = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

$$\left[\int_0^1 f(x)\, dx\right]\cdot\left[\int_0^1 g(x)\, dx\right] = \frac{1}{2}\cdot\frac{3}{2} = \frac{3}{4} \ne \frac{5}{6}$$

65. True

66. True

67. False

$$\int_0^2 (-x)\, dx = -2$$

68. True. The limits of integration are the same.

69. $f(x) = x^2 + 3x,\ [0, 8]$

$x_0 = 0,\ x_1 = 1,\ x_2 = 3,\ x_3 = 7,\ x_4 = 8$

$\Delta x_1 = 1,\ \Delta x_2 = 2,\ \Delta x_3 = 4,\ \Delta x_4 = 1$

$c_1 = 1,\ c_2 = 2,\ c_3 = 5,\ c_4 = 8$

$$\sum_{i=1}^{4} f(c_i)\Delta x = f(1)\Delta x_1 + f(2)\Delta x_2 + f(5)\Delta x_3 + f(8)\Delta x_4 = (4)(1) + (10)(2) + (40)(4) + (88)(1) = 272$$

70. $f(x) = \sin x, [0, 2\pi]$

$$x_0 = 0,\ x_1 = \frac{\pi}{4},\ x_2 = \frac{\pi}{3},\ x_3 = \pi,\ x_4 = 2\pi$$

$$\Delta x_1 = \frac{\pi}{4},\ \Delta x_2 = \frac{\pi}{12},\ \Delta x_3 = \frac{2\pi}{3},\ \Delta x_4 = \pi$$

$$c_1 = \frac{\pi}{6},\ c_2 = \frac{\pi}{3},\ c_3 = \frac{2\pi}{3},\ c_4 = \frac{3\pi}{2}$$

$$\sum_{i=1}^{4} f(c_i)\,\Delta x_i = f\left(\frac{\pi}{6}\right)\Delta x_1 + f\left(\frac{\pi}{3}\right)\Delta x_2 + f\left(\frac{2\pi}{3}\right)\Delta x_3 + f\left(\frac{3\pi}{2}\right)\Delta x_4$$

$$= \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{12}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{2\pi}{3}\right) + (-1)(\pi) \approx -0.708$$

71. $\Delta x = \dfrac{b-a}{n},\ c_i = a + i(\Delta x) = a + i\left(\dfrac{b-a}{n}\right)$

$$\int_0^b x\,dx = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(c_i)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[a + i\left(\frac{b-a}{n}\right)\right]\left(\frac{b-a}{n}\right)$$

$$= \lim_{n\to\infty}\left[\left(\frac{b-a}{n}\right)\sum_{i=1}^{n} a + \left(\frac{b-a}{n}\right)^2\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty}\left[\frac{b-a}{n}(an) + \left(\frac{b-a}{n}\right)^2\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[a(b-a) + \frac{(b-a)^2}{n}\frac{n+1}{2}\right]$$

$$= a(b-a) + \frac{(b-a)^2}{2}$$

$$= (b-a)\left[a + \frac{b-a}{2}\right]$$

$$= \frac{(b-a)(a+b)}{2} = \frac{b^2 - a^2}{2}$$

72. $\Delta x = \dfrac{b-a}{n},\ c_i = a + i(\Delta x) = a + i\left(\dfrac{b-a}{n}\right)$

$$\int_a^b x^2\,dx = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(c_i)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[a + i\left(\frac{b-a}{n}\right)\right]^2\left(\frac{b-a}{n}\right)$$

$$= \lim_{n\to\infty}\left[\left(\frac{b-a}{n}\right)\sum_{i=1}^{n}\left(a^2 + \frac{2ai(b-a)}{n} + i^2\left(\frac{b-a}{n}\right)^2\right)\right]$$

$$= \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\left[na^2 + \frac{2a(b-a)}{n}\frac{n(n+1)}{2} + \left(\frac{b-a}{n}\right)^2\frac{n(n+1)(2n+1)}{6}\right]$$

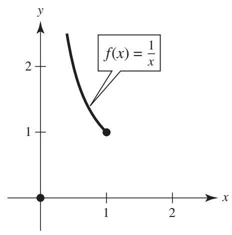
$$= \lim_{n\to\infty}\left[a^2(b-a) + \frac{a(b-a)^2(n+1)}{n} + \frac{(b-a)^3}{6}\frac{(n+1)(2n+1)}{n^2}\right]$$

$$= a^2(b-a) + a(b-a)^2 + \frac{1}{3}(b-a)^3 = \frac{1}{3}\left(b^3 - a^3\right)$$

73. $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$

is not integrable on the interval $[0, 1]$. As $\|\Delta\| \to 0$, $f(c_i) = 1$ or $f(c_i) = 0$ in each subinterval because there are an infinite number of both rational and irrational numbers in any interval, no matter how small.

74. $f(x) = \begin{cases} 0, & x = 0 \\ \dfrac{1}{x}, & 0 < x \le 1 \end{cases}$



The limit

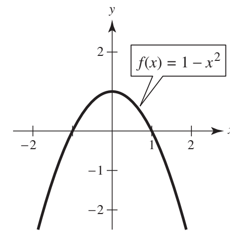
$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

does not exist.
This does not contradict Theorem 4.4 because f is not continuous on $[0, 1]$.

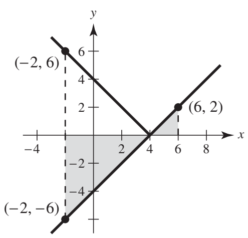
75. The function f is nonnegative between $x = -1$ and $x = 1$.

So, $\int_a^b (1 - x^2)\,dx$

is a maximum for $a = -1$ and $b = 1$.



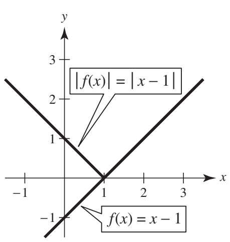
76.



$$\int_{-2}^{6} (x - 4)dx = \frac{-1}{2}6(6) + \frac{1}{2}2(2) = -18 + 2 = -16 \Rightarrow \left|\int_{-2}^{6} (x - 4)dx\right| = 16$$

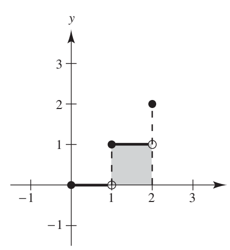
$$\int_{-2}^{6} |x - 4|\,dx = \frac{1}{2}6(6) + \frac{1}{2}2(2) = 18 + 2 = 20$$

77. Answers will vary. *Sample answer:*



The integrals are equal when f is always greater than or equal to 0 on $[a, b]$.

78. To find $\int_0^2 [\![x]\!]\,dx$, use a geometric approach.



So, $\int_0^2 [\![x]\!]\,dx = 1(2 - 1) = 1.$

79. Let $f(x) = x^2$, $0 \le x \le 1$, and $\Delta x_i = 1/n$. The appropriate Riemann Sum is

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2.$$

$$\lim_{n\to\infty} \frac{1}{n^3}\left[1^2 + 2^2 + 3^2 + \cdots + n^2\right] = \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n(2n + 1)(n + 1)}{6} = \lim_{n\to\infty} \frac{2n^2 + 3n + 1}{6n^2} = \lim_{n\to\infty}\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}$$

Section 4.4 The Fundamental Theorem of Calculus

1. Find an antiderivative F of the function and evaluate the difference of the antiderivative at the upper limit of integration and the lower limit of integration, $F(b) - F(a)$.

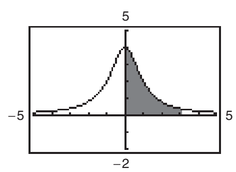
2. The Mean Value Theorem states that somewhere between the inscribed and circumscribed rectangles, there is a rectangle whose area is equal to the area of the region under the curve.

3. The average value of a function on an interval is the integral of the function on $[a, b]$ times $\dfrac{1}{b - a}$.

4. $F(x) = \int_0^x f(t)\,dt$ is considered an accumulation function because as x increases, $F(x)$ "accumulates" more area under the curve $f(t)$. Here assume $f(t) \ge 0$.

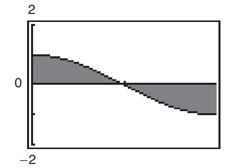
5. $f(x) = \dfrac{4}{x^2 + 1}$

$\int_0^{\pi} \dfrac{4}{x^2 + 1}\,dx$ is positive.



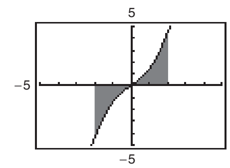
6. $f(x) = \cos x$

$\int_0^{\pi} \cos x\,dx = 0$



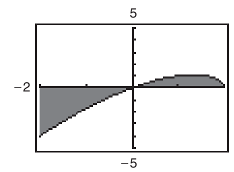
7. $f(x) = x\sqrt{x^2 + 1}$

$\int_{-2}^{2} x\sqrt{x^2 + 1}\,dx = 0$



8. $f(x) = x\sqrt{2 - x}$

$\int_{-2}^{2} x\sqrt{2 - x}\,dx$ is negative.



9. $\int_{-1}^{0} (2x - 1)\,dx = \left[x^2 - x\right]_{-1}^{0}$

$= 0 - \left((-1)^2 - (-1)\right) = -(1 + 1) = -2$

10. $\int_{-1}^{2} (7 - 3t)\,dt = \left[7t - \frac{3}{2}t^2\right]_{-1}^{2}$

$= \left[7(2) - \frac{3}{2}(4)\right] - \left[7(-1) - \frac{3}{2}(-1)^2\right]$

$= 14 - 6 + 7 + \frac{3}{2} = \frac{33}{2}$

11. $\int_{-1}^{1} \left(t^2 - 5\right) dt = \left[\dfrac{t^3}{3} - 5t\right]_{-1}^{1} = \left(\dfrac{1}{3} - 5\right) - \left(-\dfrac{1}{3} + 5\right) = \dfrac{-28}{3}$

12. $\int_1^2 \left(6x^2 - 3x\right) dx = \left[2x^3 - \frac{3}{2}x^2\right]_1^2 = \left[2(8) - \frac{3}{2}(4)\right] - \left[2(1) - \frac{3}{2}(1)\right] = (16 - 6) - \left(2 - \frac{3}{2}\right) = \frac{19}{2}$

13. $\int_0^1 (2t - 1)^2\,dt = \int_0^1 \left(4t^2 - 4t + 1\right) dt = \left[\frac{4}{3}t^3 - 2t^2 + t\right]_0^1 = \frac{4}{3} - 2 + 1 = \frac{1}{3}$

14. $\int_1^4 \left(8x^3 - x\right) dx = \left[2x^4 - \dfrac{x^2}{2}\right]_1^4$

$= (512 - 8) - \left(2 - \dfrac{1}{2}\right)$

$= \dfrac{1005}{2}$

15. $\int_1^2 \left(\dfrac{3}{x^2} - 1\right) dx = \left[-\dfrac{3}{x} - x\right]_1^2 = \left(-\dfrac{3}{2} - 2\right) - (-3 - 1) = \dfrac{1}{2}$

16. $\int_{-2}^{-1} \left(u - \dfrac{1}{u^2}\right) du = \left[\dfrac{u^2}{2} + \dfrac{1}{u}\right]_{-2}^{-1} = \left(\dfrac{1}{2} - 1\right) - \left(2 - \dfrac{1}{2}\right) = -2$

17. $\int_1^4 \frac{u-2}{\sqrt{u}}\,du = \int_1^4 \left(u^{1/2} - 2u^{-1/2}\right) du = \left[\frac{2}{3}u^{3/2} - 4u^{1/2}\right]_1^4 = \left[\frac{2}{3}\left(\sqrt{4}\right)^3 - 4\sqrt{4}\right] - \left[\frac{2}{3} - 4\right] = \frac{2}{3}$

18. $\int_{-8}^8 x^{1/3}\,dx = \left[\frac{3}{4}x^{4/3}\right]_{-8}^8 = \frac{3}{4}\left[8^{4/3} - (-8)^{4/3}\right] = \frac{3}{4}(16 - 16) = 0$

19. $\int_{-1}^1 \left(\sqrt[3]{t} - 2\right) dt = \left[\frac{3}{4}t^{4/3} - 2t\right]_{-1}^1 = \left(\frac{3}{4} - 2\right) - \left(\frac{3}{4} + 2\right) = -4$

20. $\int_1^8 \sqrt{\frac{2}{x}}\,dx = \sqrt{2}\int_1^8 x^{-1/2}\,dx = \left[\sqrt{2}(2)x^{1/2}\right]_1^8 = \left[2\sqrt{2x}\right]_1^8 = 8 - 2\sqrt{2}$

21. $\int_0^1 \frac{x - \sqrt{x}}{3}\,dx = \frac{1}{3}\int_0^1 \left(x - x^{1/2}\right) dx = \frac{1}{3}\left[\frac{x^2}{2} - \frac{2}{3}x^{3/2}\right]_0^1 = \frac{1}{3}\left(\frac{1}{2} - \frac{2}{3}\right) = -\frac{1}{18}$

22. $\int_0^2 (6 - t)\sqrt{t}\,dt = \int_0^2 \left(6t^{1/2} - t^{3/2}\right) dt$

$$= \left[\frac{6t^{3/2}}{(3/2)} - \frac{t^{5/2}}{(5/2)}\right]_0^2$$

$$= \left[4t^{3/2} - \frac{2}{5}t^{5/2}\right]_0^2$$

$$= 8(2)^{1/2} - \frac{8}{5}(2)^{1/2} = \frac{32}{5}\sqrt{2}$$

23. $\int_{-1}^0 \left(t^{1/3} - t^{2/3}\right) dt = \left[\frac{3}{4}t^{4/3} - \frac{3}{5}t^{5/3}\right]_{-1}^0 = 0 - \left(\frac{3}{4} + \frac{3}{5}\right) = -\frac{27}{20}$

24. $\int_{-8}^{-1} \frac{x - x^2}{2\sqrt[3]{x}}\,dx = \frac{1}{2}\int_{-8}^{-1} \left(x^{2/3} - x^{5/3}\right) dx$

$$= \frac{1}{2}\left[\frac{3}{5}x^{5/3} - \frac{3}{8}x^{8/3}\right]_{-8}^{-1} = \left[\frac{x^{5/3}}{80}(24 - 15x)\right]_{-8}^{-1} = -\frac{1}{80}(39) + \frac{32}{80}(144) = \frac{4569}{80}$$

25. $\int_0^5 |2x - 5|\,dx = \int_0^{5/2} (5 - 2x)\,dx + \int_{5/2}^5 (2x - 5)\,dx$ (split up the integral at the zero $x = \frac{5}{2}$)

$$= \left[5x - x^2\right]_0^{5/2} + \left[x^2 - 5x\right]_{5/2}^5 = \left(\frac{25}{2} - \frac{25}{4}\right) - 0 + (25 - 25) - \left(\frac{25}{4} - \frac{25}{2}\right) = 2\left(\frac{25}{2} - \frac{25}{4}\right) = \frac{25}{2}$$

Note: By Symmetry, $\int_0^5 |2x - 5|\,dx = 2\int_{5/2}^5 (2x - 5)\,dx$.

26. $\int_1^4 (3 - 1x - 31)\,dx = \int_1^3 \left[3 + (x - 3)\right] dx + \int_3^4 \left[3 - (x - 3)\right] dx$

$$= \int_1^3 x\,dx + \int_3^4 (6 - x)dx$$

$$= \left[\frac{x^2}{2}\right]_1^3 + \left[6x - \frac{x^2}{2}\right]_3^4$$

$$= \left(\frac{9}{2} - \frac{1}{2}\right) + \left[(24 - 8) - \left(18 - \frac{9}{2}\right)\right]$$

$$= 4 + 16 - 18 + \frac{9}{2} = \frac{13}{2}$$

27. $\int_0^4 \left|x^2 - 9\right| dx = \int_0^3 \left(9 - x^2\right) dx + \int_3^4 \left(x^2 - 9\right) dx$ (split up integral at the zero $x = 3$)

$$= \left[9x - \frac{x^3}{3}\right]_0^3 + \left[\frac{x^3}{3} - 9x\right]_3^4 = (27 - 9) + \left(\frac{64}{3} - 36\right) - (9 - 27) = \frac{64}{3}$$

28. $\int_0^4 \left|x^2 - 4x + 3\right| dx = \int_0^1 \left(x^2 - 4x + 3\right) dx - \int_1^3 \left(x^2 - 4x + 3\right) dx + \int_3^4 \left(x^2 - 4x + 3\right) dx$ (split up the integral at the zeros $x = 1, 3$)

$$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_0^1 - \left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3 + \left[\frac{x^3}{3} - 2x^2 + 3x\right]_3^4$$

$$= \left(\frac{1}{3} - 2 + 3\right) - (9 - 18 + 9) + \left(\frac{1}{3} - 2 + 3\right) + \left(\frac{64}{3} - 32 + 12\right) - (9 - 18 + 9)$$

$$= \frac{4}{3} - 0 + \frac{4}{3} + \frac{4}{3} - 0 = 4$$

29. $\int_0^\pi (\sin x - 7)\, dx = \left[-\cos x - 7x\right]_0^\pi$

$$= \left[-\cos\pi - 7(\pi)\right] - (-\cos 0 - 0)$$

$$= 1 - 7\pi + 1 = 2 - 7\pi$$

30. $\int_0^\pi (2 + \cos x)\, dx = \left[2x + \sin x\right]_0^\pi = (2\pi + 0) - 0 = 2\pi$

31. $\int_0^{\pi/4} \frac{1 - \sin^2\theta}{\cos^2\theta}\, d\theta = \int_0^{\pi/4} d\theta = \left[\theta\right]_0^{\pi/4} = \frac{\pi}{4}$

32. $\int_0^{\pi/4} \frac{\sec^2\theta}{\tan^2\theta + 1}\, d\theta = \int_0^{\pi/4} \frac{\sec^2\theta}{\sec^2\theta}\, d\theta = \int_0^{\pi/4} d\theta = \left[\theta\right]_0^{\pi/4} = \frac{\pi}{4}$

33. $\int_{-\pi/6}^{\pi/6} \sec^2 x\, dx = \left[\tan x\right]_{-\pi/6}^{\pi/6} = \frac{\sqrt{3}}{3} - \left(-\frac{\sqrt{3}}{3}\right) = \frac{2\sqrt{3}}{3}$

34. $\int_{\pi/4}^{\pi/2} \left(2 - \csc^2 x\right) dx = \left[2x + \cot x\right]_{\pi/4}^{\pi/2} = (\pi + 0) - \left(\frac{\pi}{2} + 1\right) = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$

35. $\int_{-\pi/3}^{\pi/3} 4\sec\theta\tan\theta\, d\theta = \left[4\sec\theta\right]_{-\pi/3}^{\pi/3} = 4(2) - 4(2) = 0$

36. $\int_{-\pi/2}^{\pi/2} (2t + \cos t)\, dt = \left[t^2 + \sin t\right]_{-\pi/2}^{\pi/2} = \left(\frac{\pi^2}{4} + 1\right) - \left(\frac{\pi^2}{4} - 1\right) = 2$

37. $A = \int_0^1 \left(x - x^2\right) dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$

38. $A = \int_1^2 \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_1^2 = \frac{1}{2}$

39. $A = \int_0^{\pi/2} \cos x\, dx = \left[\sin x\right]_0^{\pi/2} = 1$

40. $A = \int_0^\pi (x + \sin x)\, dx = \left[\frac{x^2}{2} - \cos x\right]_0^\pi = \frac{\pi^2}{2} + 2 = \frac{\pi^2 + 4}{2}$

41. Because $y > 0$ on $[0, 2]$,

$$\text{Area} = \int_0^2 \left(5x^2 + 2\right) dx = \left[\tfrac{5}{3}x^3 + 2x\right]_0^2 = \tfrac{40}{3} + 4 = \tfrac{52}{3}.$$

42. Because $y \geq 0$ on $[0, 2]$,

$$\begin{aligned}\text{Area} &= \int_0^2 \left(x^3 + 6x\right) dx\\ &= \left[\frac{x^4}{4} + 3x^2\right]_0^2\\ &= \frac{16}{4} + 3(4)\\ &= 16.\end{aligned}$$

43. Because $y > 0$ on $[0, 8]$,

$$\text{Area} = \int_0^8 \left(1 + x^{1/3}\right) dx = \left[x + \frac{3}{4}x^{4/3}\right]_0^8 = 8 + \frac{3}{4}(16) = 20.$$

44. Because $y \geq 0$ on $[0, 4]$,

$$\begin{aligned}\text{Area} &= \int_0^4 \left(2\sqrt{x} - x\right) dx\\ &= \int_0^4 \left(2x^{1/2} - x\right) dx = \left[\frac{4}{3}x^{3/2} - \frac{x^2}{2}\right]_0^4 = \frac{32}{3} - 8 = \frac{8}{3}.\end{aligned}$$

45. Because $y > 0$ on $[0, 4]$,

$$\text{Area} = \int_0^4 \left(-x^2 + 4x\right) dx = \left[-\frac{x^3}{3} + 2x^2\right]_0^4 = -\frac{64}{3} + 32 = \frac{32}{3}.$$

46. Because $y > 0$ on $[-1, 1]$,

$$\begin{aligned}\text{Area} &= \int_{-1}^1 \left(1 - x^4\right) dx\\ &= 2\int_0^1 \left(1 - x^4\right) dx\\ &= 2\left[x - \frac{x^5}{5}\right]_0^1 = 2\left(1 - \frac{1}{5}\right) = \frac{8}{5}.\end{aligned}$$

47. $\displaystyle\int_0^3 x^3\, dx = \left[\frac{x^4}{4}\right]_0^3 = \frac{81}{4}$

$$\begin{aligned}f(c)(3 - 0) &= \frac{81}{4}\\ f(c) &= \frac{27}{4}\\ c^3 &= \frac{27}{4}\\ c &= \frac{3}{\sqrt[3]{4}} = \frac{3}{2}\sqrt[3]{2} \approx 1.8899\end{aligned}$$

48. $\displaystyle\int_4^9 \sqrt{x}\, dx = \left[\tfrac{2}{3}x^{3/2}\right]_4^9 = \tfrac{2}{3}(27 - 8) = \tfrac{38}{3}$

$$\begin{aligned}f(c)(9 - 4) &= \tfrac{38}{3}\\ f(c) &= \tfrac{38}{15}\\ \sqrt{c} &= \tfrac{38}{15}\\ c &= \tfrac{1444}{225} \approx 6.4178\end{aligned}$$

49. $\displaystyle\int_0^6 \frac{x^2}{4}\, dx = \left[\frac{x^3}{12}\right]_0^6 = \frac{216}{12} = 18$

$$\begin{aligned}f(c)(6 - 0) &= 18\\ f(c) &= 3\\ \frac{c^2}{4} &= 3\\ c &= \pm\sqrt{12}\\ c &= 2\sqrt{3}\end{aligned}$$

$\left(-2\sqrt{3}\text{ is not in interval.}\right)$

50. $\int_1^3 \frac{9}{x^3}\,dx = \left[-\frac{9}{2x^2}\right]_1^3 = -\frac{1}{2} + \frac{9}{2} = 4$

$$f(c)(3-1) = 4$$
$$\frac{9}{c^3} = 2$$
$$c^3 = \frac{9}{2}$$
$$c = \sqrt[3]{\frac{9}{2}} \approx 1.6510$$

51. $\int_{-\pi/4}^{\pi/4} 2\sec^2 x\,dx = \left[2\tan x\right]_{-\pi/4}^{\pi/4} = 2(1) - 2(-1) = 4$

$$f(c)\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 4$$
$$2\sec^2 c = \frac{8}{\pi}$$
$$\sec^2 c = \frac{4}{\pi}$$
$$\sec c = \pm\frac{2}{\sqrt{\pi}}$$
$$c = \pm\operatorname{arcsec}\left(\frac{2}{\sqrt{\pi}}\right)$$
$$= \pm\arccos\frac{\sqrt{\pi}}{2} \approx \pm 0.4817$$

52. $\int_{-\pi/3}^{\pi/3} \cos x\,dx = \left[\sin x\right]_{-\pi/3}^{\pi/3} = \sqrt{3}$

$$f(c)\left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right] = \sqrt{3}$$
$$\cos c = \frac{3\sqrt{3}}{2\pi}$$
$$c \approx \pm 0.5971$$

53. $f(x) = 4 - x^2, [-2, 2]$

$$\frac{1}{2-(-2)}\int_{-2}^{2} \left(4 - x^2\right)dx = \frac{1}{4}\left[4x - \frac{1}{3}x^3\right]_{-2}^{2}$$
$$= \frac{1}{4}\left[\left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)\right]$$
$$= \frac{8}{3}$$

Average value $= \frac{8}{3}$

$4 - x^2 = \frac{8}{3}$ when $x^2 = 4 - \frac{8}{3}$ or

$x = \pm\sqrt{\frac{4}{3}} = \pm\frac{2\sqrt{3}}{3} \approx \pm 1.1547$

54. $f(x) = \frac{4\left(x^2+1\right)}{x^2}, [1, 3]$

$$\frac{1}{3-1}\int_1^3 \frac{4\left(x^2+1\right)}{x^2}\,dx = 2\int_1^3 \left(1 + x^{-2}\right)dx$$
$$= 2\left[x - \frac{1}{x}\right]_1^3$$
$$= 2\left(3 - \frac{1}{3}\right) = \frac{16}{3}$$

Average value $= \frac{16}{3}$

$\frac{4\left(x^2+1\right)}{x^2} = \frac{16}{3} \Rightarrow x = \sqrt{3}$ (on $[1, 3]$)

55. $f(x) = x^4 + 7, [0, 2]$

$$\frac{1}{2-0}\int_0^2 \left(x^4 + 7\right)dx = \frac{1}{2}\left[\frac{x^5}{5} + 7x\right]_0^2$$
$$= \frac{1}{2}\left[\frac{32}{5} + 14\right] = \frac{51}{5}$$

Average value $= \frac{51}{5} = 10.2$

$$x^4 + 7 = \frac{51}{5}$$
$$x^4 = \frac{16}{5}$$
$$x = \frac{2}{5^{1/4}} \approx 1.3375$$

56. $f(x) = 4x^3 - 3x^2, [0, 1]$

$$\frac{1}{1-0}\int_0^1 \left(4x^3 - 3x^2\right)dx = \left[x^4 - x^3\right]_0^1 = 0$$

Average value $= 0$

$$4x^3 - 3x^2 = 0$$
$$x^2(4x - 3) = 0$$
$$x = 0, \tfrac{3}{4}$$

57. $f(x) = \sin x, [0, \pi]$

$$\frac{1}{\pi - 0}\int_0^{\pi} \sin x\,dx = \left[-\frac{1}{\pi}\cos x\right]_0^{\pi} = \frac{2}{\pi}$$

Average value $= \frac{2}{\pi}$

$$\sin x = \frac{2}{\pi}$$
$$x \approx 0.690,\ 2.451$$

58. $f(x) = \cos x, \left[0, \dfrac{\pi}{2}\right]$

$$\frac{1}{(\pi/2) - 0}\int_0^{\pi/2} \cos x\, dx = \left[\frac{2}{\pi}\sin x\right]_0^{\pi/2} = \frac{2}{\pi}$$

$$\text{Average value} = \frac{2}{\pi}$$

$$\cos x = \frac{2}{\pi}$$

$$x \approx 0.881$$

59. (a) $F(x) = k\sec^2 x$

$F(0) = k = 500$

$F(x) = 500\sec^2 x$

(b) $$\frac{1}{\pi/3 - 0}\int_0^{\pi/3} 500\sec^2 x\, dx = \frac{1500}{\pi}\left[\tan x\right]_0^{\pi/3}$$

$$= \frac{1500}{\pi}\left(\sqrt{3} - 0\right)$$

$$\approx 826.99 \text{ newtons}$$

$$\approx 827 \text{ newtons}$$

60. $$\frac{1}{5 - 0}\int_0^5 \left(0.1729t + 0.1522t^2 - 0.0374t^3\right) dt \approx \frac{1}{5}\left[0.08645t^2 + 0.05073t^3 - 0.00935t^4\right]_0^5 \approx 0.5318 \text{ L}$$

61. $$P = \frac{2}{\pi}\int_0^{\pi/2} \sin\theta\, d\theta = \left[-\frac{2}{\pi}\cos\theta\right]_0^{\pi/2} = -\frac{2}{\pi}(0 - 1) = \frac{2}{\pi} \approx 63.7\%$$

62. (a) Because $y < 0$ on [0, 2], $\int_0^2 f(x)\, dx = -(\text{area of region } A) = -1.5.$

(b) $\int_2^6 f(x)\, dx = (\text{area of region } B) = \int_0^6 f(x)\, dx - \int_0^2 f(x)\, dx = 3.5 - (-1.5) = 5.0$

(c) $\int_0^6 |f(x)|\, dx = -\int_0^2 f(x)\, dx + \int_2^6 f(x)\, dx = 1.5 + 5.0 = 6.5$

(d) $\int_0^2 -2f(x)\, dx = -2\int_0^2 f(x)\, dx = -2(-1.5) = 3.0$

(e) $\int_0^6 \left[2 + f(x)\right] dx = \int_0^6 2\, dx + \int_0^6 f(x)\, dx = 12 + 3.5 = 15.5$

(f) Average value $= \frac{1}{6}\int_0^6 f(x)\, dx = \frac{1}{6}(3.5) = 0.5833$

63. $$F(x) = \int_1^x \frac{20}{v^2}\, dv = \int_1^x 20v^{-2}\, dv = -\frac{20}{v}\Big]_1^x$$

$$= -\frac{20}{x} + 20 = 20\left(1 - \frac{1}{x}\right)$$

$F(2) = 20\left(\frac{1}{2}\right) = 10$

$F(5) = 20\left(\frac{4}{5}\right) = 16$

$F(8) = 20\left(\frac{7}{8}\right) = \frac{35}{2}$

64. $$F(x) = \int_2^x \left(t^3 + 2t - 2\right) dt = \left[\frac{t^4}{4} + t^2 - 2t\right]_2^x = \left(\frac{x^4}{4} + x^2 - 2x\right) - (4 + 4 - 4) = \frac{x^4}{4} + x^2 - 2x - 4$$

$F(2) = 4 + 4 - 4 - 4 = 0$ $\left[\textbf{Note: } F(2) = \int_2^2 \left(t^3 + 2t - 2\right) dt = 0\right]$

$F(5) = \dfrac{625}{4} + 25 - 10 - 4 = 167.25$

$F(8) = \dfrac{8^4}{4} + 64 - 16 - 4 = 1068$

65. $F(x) = \int_0^x \cos\theta\, d\theta = \left[\sin\theta\right]_0^x = \sin x - \sin 0 = \sin x$

$F(0) = 0$

$F\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$F\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$

66. $F(x) = \int_{-\pi}^x \sin\theta\, d\theta = \left[-\cos\theta\right]_{-\pi}^x = -\cos x + \cos(-\pi) = -\cos x - 1$

$F(0) = -\cos 0 - 1 = -2$

$F\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} - 1 = -\frac{\sqrt{2}}{2} - 1 = \frac{-2 - \sqrt{2}}{2}$

$F\left(\frac{\pi}{2}\right) = -\cos\frac{\pi}{2} - 1 = -1$

67. $g(x) = \int_0^x f(t)\, dt$

(a) $g(0) = \int_0^0 f(t)\, dt = 0$

$g(2) = \int_0^2 f(t)\, dt \approx 4 + 2 + 1 = 7$

$g(4) = \int_0^4 f(t)\, dt \approx 7 + 2 = 9$

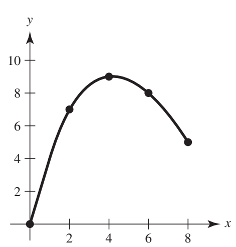
$g(6) = \int_0^6 f(t)\, dt \approx 9 + (-1) = 8$

$g(8) = \int_0^8 f(t)\, dt \approx 8 - 3 = 5$

(b) g increasing on (0, 4) and decreasing on (4, 8)

(c) g is a maximum of 9 at $x = 4$.

(d)



68. $g(x) = \int_0^x f(t)\, dt$

(a) $g(0) = \int_0^0 f(t)\, dt = 0$

$g(2) = \int_0^2 f(t)\, dt = -\frac{1}{2}(2)(4) = -4$

$g(4) = \int_0^4 f(t)\, dt = -\frac{1}{2}(4)(4) = -8$

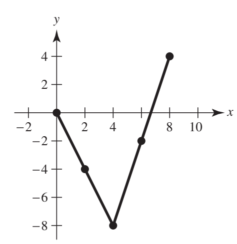
$g(6) = \int_0^6 f(t)\, dt = -8 + 2 + 4 = -2$

$g(8) = \int_0^8 f(t)\, dt = -2 + 6 = 4$

(b) g decreasing on (0, 4) and increasing on (4, 8)

(c) g is a minimum of -8 at $x = 4$.

(d)



69. (a) $\int_0^x (t + 2)\, dt = \left[\frac{t^2}{2} + 2t\right]_0^x = \frac{1}{2}x^2 + 2x$

(b) $\frac{d}{dx}\left[\frac{1}{2}x^2 + 2x\right] = x + 2$

70. (a) $\int_0^x t(t^2+1)\,dt = \int_0^x (t^3+t)\,dt$

$$= \left[\frac{1}{4}t^4 + \frac{1}{2}t^2\right]_0^x$$

$$= \frac{1}{4}x^4 + \frac{1}{2}x^2 = \frac{x^2}{4}(x^2+2)$$

(b) $\frac{d}{dx}\left[\frac{1}{4}x^4 + \frac{1}{2}x^2\right] = x^3 + x = x(x^2+1)$

71. (a) $\int_8^x \sqrt[3]{t}\,dt = \left[\frac{3}{4}t^{4/3}\right]_8^x = \frac{3}{4}(x^{4/3}-16) = \frac{3}{4}x^{4/3} - 12$

(b) $\frac{d}{dx}\left[\frac{3}{4}x^{4/3} - 12\right] = x^{1/3} = \sqrt[3]{x}$

72. (a) $F(x) = \int_4^x t^{3/2}\,dt = \left[\frac{2}{5}t^{5/2}\right]_4^x = \frac{2}{5}x^{5/2} - \frac{64}{5}$

(b) $\frac{d}{dx}\left[\frac{2}{5}x^{5/2} - \frac{64}{5}\right] = x^{3/2}$

73. (a) $\int_{\pi/4}^x \sec^2 t\,dt = [\tan t]_{\pi/4}^x = \tan x - 1$

(b) $\frac{d}{dx}[\tan x - 1] = \sec^2 x$

74. (a) $\int_{\pi/3}^x \sec t\tan t\,dt = [\sec t]_{\pi/3}^x = \sec x - 2$

(b) $\frac{d}{dx}[\sec x - 2] = \sec x\tan x$

75. $F(x) = \int_{-2}^x (t^2-2t)\,dt$

$F'(x) = x^2 - 2x$

76. $F(x) = \int_1^x \frac{t^2}{t^2+1}\,dt$

$F'(x) = \frac{x^2}{x^2+1}$

77. $F(x) = \int_{-1}^x \sqrt{t^4+1}\,dt$

$F'(x) = \sqrt{x^4+1}$

78. $F(x) = \int_1^x \sqrt[4]{t}\,dt$

$F'(x) = \sqrt[4]{x}$

79. $F(x) = \int_1^x \sqrt{t}\csc t\,dt$

$F'(x) = \sqrt{x}\csc x$

80. $F(x) = \int_0^x \sec^3 t\,dt$

$F'(x) = \sec^3 x$

81. $F(x) = \int_x^{x+2} (4t+1)\,dt$

$$= \left[2t^2 + t\right]_x^{x+2}$$

$$= \left[2(x+2)^2 + (x+2)\right] - \left[2x^2 + x\right]$$

$$= 8x + 10$$

$F'(x) = 8$

Alternate solution:

$F(x) = \int_x^{x+2} (4t+1)\,dt$

$$= \int_x^0 (4t+1)\,dt + \int_0^{x+2} (4t+1)\,dt$$

$$= -\int_0^x (4t+1)\,dt + \int_0^{x+2} (4t+1)\,dt$$

$F'(x) = -(4x+1) + 4(x+2) + 1 = 8$

82. $F(x) = \int_{-x}^x t^3\,dt = \left[\frac{t^4}{4}\right]_{-x}^x = 0$

$F'(x) = 0$

Alternate solution:

$F(x) = \int_{-x}^x t^3\,dt$

$$= \int_{-x}^0 t^3\,dt + \int_0^x t^3\,dt$$

$$= -\int_0^{-x} t^3\,dt + \int_0^x t^3\,dt$$

$F'(x) = -(-x)^3(-1) + (x^3) = 0$

83. $F(x) = \int_0^{\sin x} \sqrt{t}\, dt = \left[\frac{2}{3}t^{3/2}\right]_0^{\sin x} = \frac{2}{3}(\sin x)^{3/2}$

$F'(x) = (\sin x)^{1/2}\cos x = \cos x\sqrt{\sin x}$

Alternate solution:

$F(x) = \int_0^{\sin x} \sqrt{t}\, dt$

$F'(x) = \sqrt{\sin x}\,\frac{d}{dx}(\sin x) = \sqrt{\sin x}(\cos x)$

84. $F(x) = \int_2^{x^2} t^{-3}\, dt = \left[\frac{t^{-2}}{-2}\right]_2^{x^2} = \left[-\frac{1}{2t^2}\right]_2^{x^2} = \frac{-1}{2x^4} + \frac{1}{8}$

$F'(x) = 2x^{-5}$

Alternate solution:

$F(x) = \int_2^{x^2} t^{-3}\, dt$

$F'(x) = \left(x^2\right)^{-3}(2x) = 2x^{-5}$

85. $F(x) = \int_0^{x^3} \sin t^2\, dt$

$F'(x) = \sin\left(x^3\right)^2 \cdot 3x^2 = 3x^2 \sin x^6$

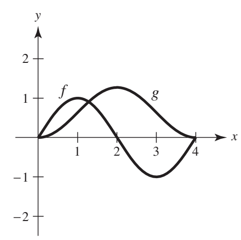
86. $F(x) = \int_0^{2x} \cos t^4\, dt$

$F'(x) = \cos(2x)^4(2) = 2\cos\left(16x^4\right)$

87. $g(x) = \int_0^x f(t)\, dt$

$g(0) = 0, g(1) \approx \frac{1}{2}, g(2) \approx 1, g(3) \approx \frac{1}{2}, g(4) = 0$

g has a relative maximum at $x = 2$.



88. (a) $g(t) = 4 - \frac{4}{t^2}$

$\lim_{t\to\infty} g(t) = 4$

Horizontal asymptote: $y = 4$

(b) $A(x) = \int_1^x \left(4 - \frac{4}{t^2}\right) dt = \left[4t + \frac{4}{t}\right]_1^x = 4x + \frac{4}{x} - 8 = \frac{4x^2 - 8x + 4}{x} = \frac{4(x-1)^2}{x}$

$\lim_{x\to\infty} A(x) = \lim_{x\to\infty}\left(4x + \frac{4}{x} - 8\right) = \infty + 0 - 8 = \infty$

The graph of $A(x)$ does not have a horizontal asymptote.

89. Let $c(t)$ be the amount of water that is flowing out of the tank. Then $c'(t) = 500 - 5t$ liters per minute is the rate of flow.

$\int_0^{18} c'(t)dt = \int_0^{18} (500 - 5t)\, dt = \left[500t - \frac{5t^2}{2}\right]_0^{18} = 9000 - 810 = 8190 \text{ L}$

90. Let $c(t)$ be the amount of oil leaking and $t = 0$ represent 1 p.m. Then $c'(t) = 4 + 0.75t$ gallons per minute is the rate of flow.

(a) From 1 p.m. to 4 p.m. (3 hr):

$\int_0^3 (4 + 0.75t)\, dt = \left[4t + \frac{0.75}{2}t^2\right]_0^3 = \frac{123}{8} = 15.375 \text{ gal}$

(b) From 4 p.m. to 7 p.m. (3 hr)

$\int_3^6 (4 + 0.75t)\, dt = \left[4t + \frac{0.75}{2}t^2\right]_3^6 = 22.125 \text{ gal}$

(c) The second answer is larger because the rate of flow is increasing.

91. The distance traveled is $\int_0^8 v(t)\, dt$. The area under the curve from $0 \le t \le 8$ is approximately

(18 squares) $(30) \approx 540$ feet.

92. The distance traveled is $\int_0^5 v(t)\,dt$. The area under the curve from $0 \le t \le 5$ is approximately (29 squares) $(5) = 145$ feet.

93. (a) $v(t) = 5t - 7,\ 0 \le t \le 3$

$$\text{Displacement} = \int_0^3 (5t - 7)\,dt = \left[\frac{5t^2}{2} - 7t\right]_0^3 = \frac{45}{2} - 21 = \frac{3}{2} \text{ ft to the right}$$

(b)
$$\begin{aligned}\text{Total distance traveled} &= \int_0^3 |5t - 7|\,dt\\ &= \int_0^{7/5} (7 - 5t)\,dt + \int_{7/5}^3 (5t - 7)\,dt\\ &= \left[7t - \frac{5t^2}{2}\right]_0^{7/5} + \left[\frac{5t^2}{2} - 7t\right]_{7/5}^3\\ &= 7\left(\frac{7}{5}\right) - \frac{5}{2}\left(\frac{7}{2}\right)^2 + \left(\frac{5}{2}(9) - 21\right) - \left(\frac{5}{2}\left(\frac{7}{5}\right)^2 - 7\left(\frac{7}{5}\right)\right)\\ &= \frac{49}{5} - \frac{49}{10} + \frac{45}{2} - 21 - \frac{49}{10} + \frac{49}{5} = \frac{113}{10} \text{ ft}\end{aligned}$$

94. (a) $v(t) = t^2 - t - 12 = (t - 4)(t + 3),\ 1 \le t \le 5$

$$\begin{aligned}\text{Displacement} &= \int_1^5 \left(t^2 - t - 12\right) dt\\ &= \left[\frac{t^3}{3} - \frac{t^2}{2} - 12t\right]_1^5 = \left(\frac{125}{3} - \frac{25}{2} - 60\right) - \left(\frac{1}{3} - \frac{1}{2} - 12\right) = -\frac{56}{3}\ \left(\frac{56}{3} \text{ ft to the left}\right)\end{aligned}$$

(b)
$$\begin{aligned}\text{Total distance traveled} &= \int_1^4 \left(-t^2 + t + 12\right) dt + \int_4^5 \left(t^2 - t - 12\right) dt\\ &= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 12t\right]_1^4 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 12t\right]_4^5\\ &= \left(-\frac{64}{3} + 8 + 48\right) - \left(-\frac{1}{3} + \frac{1}{2} + 12\right) + \left(\frac{125}{3} - \frac{25}{2} - 60\right) - \left(\frac{64}{3} - 8 - 48\right)\\ &= \frac{104}{3} - \frac{73}{6} + \left(-\frac{185}{6}\right) - \left(-\frac{104}{3}\right) = \frac{79}{3} \text{ ft}\end{aligned}$$

95. (a) $v(t) = t^3 - 10t^2 + 27t - 18 = (t - 1)(t - 3)(t - 6),\ 1 \le t \le 7$

$$\begin{aligned}\text{Displacement} &= \int_1^7 \left(t^3 - 10t^2 + 27t - 18\right) dt\\ &= \left[\frac{t^4}{4} - \frac{10t^3}{3} + \frac{27t^2}{2} - 18t\right]_1^7\\ &= \left[\frac{7^4}{4} - \frac{10(7^3)}{3} + \frac{27(7^2)}{2} - 18(7)\right] - \left[\frac{1}{4} - \frac{10}{3} + \frac{27}{2} - 18\right]\\ &= -\frac{91}{12} - \left(-\frac{91}{12}\right) = 0\end{aligned}$$

(b) Total distance traveled $= \int_1^7 |v(t)|\,dt$

$$= \int_1^3 (t^3 - 10t^2 + 27t - 18)\,dt - \int_3^6 (t^3 - 10t^2 + 27t - 18)\,dt + \int_6^7 (t^3 - 10t^2 + 27t - 18)\,dt$$

Evaluating each of these integrals, you obtain

Total distance $= \frac{16}{3} - \left(-\frac{63}{4}\right) + \frac{125}{12} = \frac{63}{2}$ ft

96. (a) $v(t) = t^3 - 8t^2 + 15t = t(t-3)(t-5),\ 0 \le t \le 5$

Displacement $= \int_0^5 (t^3 - 8t^2 + 15t)\,dt$

$$= \left[\frac{t^4}{4} - \frac{8t^3}{3} + \frac{15t^2}{2}\right]_0^5$$

$$= \frac{625}{4} - \frac{8(125)}{3} + \frac{375}{2} = \frac{125}{12} \text{ ft to the right}$$

(b) Total distance traveled $= \int_0^5 |v(t)|\,dt$

$$= \int_0^3 (t^3 - 8t^2 + 15t)\,dt - \int_3^5 (t^3 - 8t^2 + 15t)\,dt$$

Evaluating each of these integrals, you obtain

Total distance $= \frac{63}{4} - \left(-\frac{16}{3}\right) = \frac{253}{12} \approx 21.08$ ft

97. (a) $v(t) = \frac{1}{\sqrt{t}},\ 1 \le t \le 4$

Because $v(t) > 0$,

Displacement = Total Distance

Displacement $= \int_1^4 t^{-1/2}\,dt = \left[2t^{1/2}\right]_1^4 = 4 - 2 = 2$ ft to the right

(b) Total distance = 2 ft

98. (a) $v(t) = \cos t,\ 0 \le t \le 3\pi$

Displacement $= \int_0^{3\pi} \cos t\,dt = \left[\sin t\right]_0^{3\pi} = 0$ ft

(b) Total distance $= \int_0^{\pi/2} \cos t\,dt - \int_{\pi/2}^{3\pi/2} \cos t\,dt + \int_{3\pi/2}^{5\pi/2} \cos t\,dt - \int_{5\pi/2}^{3\pi} \cos t\,dt$

$$= \left[\sin t\right]_0^{\pi/2} - \left[\sin t\right]_{\pi/2}^{3\pi/2} + \left[\sin t\right]_{3\pi/2}^{5\pi/2} - \left[\sin t\right]_{5\pi/2}^{3\pi} = 1 - (-2) + 2 - (-1) = 6$$

99. The displacement and total distance traveled are equal when the particle is always moving in the same direction on an interval.

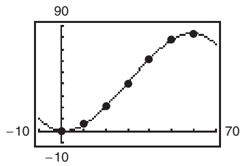
100. $r(t)$ represents the weight in pounds of the dog at time t.

$\int_2^6 r'(t)\,dt$ represents the net change in the weight of the dog from year 2 to year 6.

101. The Fundamental Theorem of Calculus requires that f be continuous on $[a, b]$ and that F be an antiderivative for f on the entire interval. On an interval containing c, the function $f(x) = \frac{1}{x - c}$ is not continuous at c.

102. (a) $v = -0.00086t^3 + 0.0782t^2 - 0.208t + 0.10$

(b)



(c) $\int_0^{60} v(t)\,dt = \left[\frac{-0.00086t^4}{4} + \frac{0.0782t^3}{3} - \frac{0.208t^2}{2} + 0.10t\right]_0^{60} \approx 2476 \text{ m}$

103. $x(t) = t^3 - 6t^2 + 9t - 2$

$x'(t) = 3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t-3)(t-1)$

$$\text{Total distance} = \int_0^5 |x'(t)|\,dt$$
$$= \int_0^5 3|(t-3)(t-1)|\,dt$$
$$= 3\int_0^1 (t^2 - 4t + 3)\,dt - 3\int_1^3 (t^2 - 4t + 3)\,dt + 3\int_3^5 (t^2 - 4t + 3)\,dt = 4 + 4 + 20 = 28 \text{ units}$$

104. $x(t) = (t-1)(t-3)^2 = t^3 - 7t^2 + 15t - 9$

$x'(t) = 3t^2 - 14t + 15$

Using a graphing utility,

$$\text{Total distance} = \int_0^5 |x'(t)|\,dt \approx 27.37 \text{ units.}$$

105. The function $f(x) = x^{-2}$ is not continuous on $[-1, 1]$.

$\int_{-1}^{1} x^{-2}\,dx = \int_{-1}^{0} x^{-2}\,dx + \int_0^1 x^{-2}\,dx$

Each of these integrals is infinite. $f(x) = x^{-2}$ has a nonremovable discontinuity at $x = 0$.

106. The function $f(x) = \frac{2}{x^3}$ is not continuous on $[-2, 1]$.

$\int_{-2}^{1} \frac{2}{x^3}\,dx = \int_{-2}^{0} \frac{2}{x^3}\,dx + \int_0^1 \frac{2}{x^3}\,dx$

Each of these integrals is infinite. $f(x) = \frac{2}{x^3}$ has a nonremovable discontinuity at $x = 0$.

107. The function $f(x) = \sec^2 x$ is not continuous on $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

$\int_{\pi/4}^{3\pi/4} \sec^2 x\,dx = \int_{\pi/4}^{\pi/2} \sec^2 x\,dx + \int_{\pi/2}^{3\pi/4} \sec^2 x\,dx$

Each of these integrals is infinite. $f(x) = \sec^2 x$ has a nonremovable discontinuity at $x = \frac{\pi}{2}$

108. The function $f(x) = \csc x \cot x$ is not continuous on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

$\int_{\pi/2}^{3\pi/2} \csc x \cot x\,dx = \int_{\pi/2}^{\pi} \csc x \cot x\,dx + \int_{\pi}^{3\pi/2} \csc x \cot x\,dx$

Each of these integrals is infinite $f(x) = \csc x \cot x$ has a nonremovable discontinuity at $x = \pi$.

109. True

110. False. For example, let $F(x) = x$ and $G(x) = x^3$ on $[0, 1]$. Then $F(1) - F(0) = 1 - 0 = 1$ and $G(1) - G(0) = 1 - 0 = 1$. But, $F'(x) \neq G'(x)$ on $[0, 1]$.

111. $f(x) = \int_0^{1/x} \frac{1}{t^2 + 1}\, dt + \int_0^x \frac{1}{t^2 + 1}\, dt$

By the Second Fundamental Theorem of Calculus, you have $f'(x) = \frac{1}{(1/x)^2 + 1}\left(-\frac{1}{x^2}\right) + \frac{1}{x^2 + 1} = -\frac{1}{1 + x^2} + \frac{1}{x^2 + 1} = 0.$

Because $f'(x) = 0$, $f(x)$ must be constant.

112. $\int_c^x f(t)\, dt = x^2 + x - 2$

Let $f(t) = 2t + 1$. Then

$$\int_c^x f(t)dt = \int_c^x (2t + 1)dt = \left[t^2 + t\right]_c^x =$$

$$x^2 + x - c^2 - c = x^2 + x - 2$$

$$-c^2 - c = -2$$

$$c^2 + c - 2 = 0$$

$$(c + 2)(c - 1) = 0 \Rightarrow c = 1, -2.$$

So, $f(x) = 2x + 1$, and $c = 1$ or $c = -2$.

113. $G(x) = \int_0^x \left[s\int_0^s f(t)\, dt\right] ds$

(a) $G(0) = \int_0^0 \left[s\int_0^s f(t)\, dt\right] ds = 0$

(b) Let $F(s) = s\int_0^s f(t)\, dt$.

$$G(x) = \int_0^x F(s)\, ds$$

$$G'(x) = F(x) = x\int_0^x f(t)\, dt$$

$$G'(0) = 0\int_0^0 f(t)\, dt = 0$$

(c) $G''(x) = x \cdot f(x) + \int_0^x f(t)\, dt$

(d) $G''(0) = 0 \cdot f(0) + \int_0^0 f(t)\, dt = 0$

114. Let $F(t)$ be an antiderivative of $f(t)$. Then,

$$\int_{u(x)}^{v(x)} f(t)\, dt = \left[F(t)\right]_{u(x)}^{v(x)} = F(v(x)) - F(u(x))$$

$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\, dt\right] = \frac{d}{dx}\left[F(v(x)) - F(u(x))\right] = F'(v(x))v'(x) - F'(u(x))u'(x) = f(v(x))v'(x) - f(u(x))u'(x).$$

115. $I(f) - J(f) = \int_0^1 x^2 f(x)\, dx - \int_0^1 xf(x)^2\, dx.$

Observe that

$$\frac{x^3}{4} - x\left(f(x) - \frac{x}{2}\right)^2 = \frac{x^3}{4} - x\left(f(x)^2 - xf(x) + \frac{x^2}{4}\right) = \frac{x^3}{4} - xf(x)^2 + x^2 f(x) - \frac{x^3}{4} = x^2 f(x) - xf(x)^2$$

So, $I(f) - J(f) = \int_0^1 \left[x^2 f(x) - xf(x)^2\right] dx = \int_0^1 \left[\frac{x^3}{4} - x\left(f(x) - \frac{x}{2}\right)^2\right] dx \leq \int_0^1 \frac{x^3}{4}\, dx = \frac{1}{16}$

Furthermore, $6 + f(x) = \frac{x}{2}$. Then $I(f) = \int_0^1 x^2\left(\frac{x}{2}\right) dx = \frac{1}{8}$ and $J(f) = \int_0^1 x\left(\frac{x^2}{4}\right) = \frac{1}{16}$

So $I(f) - J(f) = \frac{1}{8} - \frac{1}{16} = \frac{1}{16}$

The maximum value is $\frac{1}{16}$.

Section 4.5 Integration by Substitution

1. Multiply and divide by a constant, if necessary.

2. To make a change of variables when finding an indefinite integral, first choose a substitution $u = g(x)$. It is best to choose the inner part of a composite function. Then compute $du = g'(x)\,dx$. Rewrite the integral in terms of the variable u. Find the resulting integral in terms of u. Replace u by $g(x)$ to obtain an antiderivative in terms of x. Finally, check your answer by differentiating.

3. The integral of $\left[g(x)\right]^n g'(x)$ is $\dfrac{\left[g(x)\right]^{n+1}}{n+1} + C,\ n \neq -1$. Recall the power rule for polynomials.

4. $f(x) = x(x^2 + 1)^2$ is odd. So, $\int_{-2}^{2} x(x^2 + 1)^2\,dx = 0$.

	$\int f(g(x))g'(x)\,dx$	$u = g(x)$	$du = g'(x)\,dx$
5.	$\int (5x^2 + 1)^2(10x)dx$	$5x^2 + 1$	$10x\ dx$
6.	$\int x^2\sqrt{x^3 + 1}\,dx$	$x^3 + 1$	$3x^2\ dx$
7.	$\int \tan^2 x \sec^2 x\,dx$	$\tan x$	$\sec^2 x\,dx$
8.	$\int \dfrac{\cos x}{\sin^2 x}\,dx$	$\sin x$	$\cos x\,dx$

9. $\int (1 + 6x)^4(6)\,dx = \dfrac{(1 + 6x)^5}{5} + C$

Check: $\dfrac{d}{dx}\left[\dfrac{(1 + 6x)^5}{5} + C\right] = 6(1 + 6x)^4$

10. $\int (x^2 - 9)^3(2x)\,dx = \dfrac{(x^2 - 9)^4}{4} + C$

Check: $\dfrac{d}{dx}\left[\dfrac{(x^2 - 9)^4}{4} + C\right] = \dfrac{4(x^2 - 9)^3}{4}(2x) = (x^2 - 9)^3(2x)$

11. $\int \sqrt{25 - x^2}(-2x)\,dx = \dfrac{(25 - x^2)^{3/2}}{3/2} + C = \dfrac{2}{3}(25 - x^2)^{3/2} + C$

Check: $\dfrac{d}{dx}\left[\dfrac{2}{3}(25 - x^2)^{3/2} + C\right] = \dfrac{2}{3}\left(\dfrac{3}{2}\right)(25 - x^2)^{1/2}(-2x) = \sqrt{25 - x^2}(-2x)$

12. $\int \sqrt[3]{3 - 4x^2}(-8x)\,dx = \int (3 - 4x^2)^{1/3}(-8x)\,dx = \dfrac{(3 - 4x^2)^{4/3}}{4/3} + C = \dfrac{3}{4}(3 - 4x^2)^{4/3} + C$

Check: $\dfrac{d}{dx}\left[\dfrac{3}{4}(3 - 4x^2)^{4/3} + C\right] = \dfrac{3}{4}\left(\dfrac{4}{3}\right)(3 - 4x^2)^{1/3}(-8x) = (3 - 4x^2)^{1/3}(-8x)$

13. $\int x^3(x^4+3)^2\,dx = \frac{1}{4}\int (x^4+3)^2(4x^3)\,dx = \frac{1}{4}\frac{(x^4+3)^3}{3} + C = \frac{(x^4+3)^3}{12} + C$

Check: $\frac{d}{dx}\left[\frac{(x^4+3)^3}{12} + C\right] = \frac{3(x^4+3)^2}{12}(4x^3) = (x^4+3)^2(x^3)$

14. $\int x^2(6-x^3)\,dx = -\frac{1}{3}\int (6-x^3)^5(-3x^2)\,dx = -\frac{1}{3}\cdot\frac{(6-x^3)^6}{6} + C = -\frac{(6-x^3)^6}{18} + C$

Check: $\frac{d}{dx}\left[-\frac{(6-x^3)^6}{18} + C\right] = \frac{-6(6-x^3)^5(-3x^2)}{18} = x^2(6-x^3)^5$

15. $\int x^2(2x^3-1)^4\,dx = \frac{1}{6}\int (2x^3-1)^4(6x^2)\,dx = \frac{1}{6}\left[\frac{1}{5}(2x^3-1)^5\right] + C = \frac{(2x^3-1)^5}{30} + C$

Check: $\frac{d}{dx}\left[\frac{(2x^3-1)^5}{30} + C\right] = 5\,\frac{(2x^3-1)^4(6x^2)}{30} = x^2(2x^3-1)^4$

16. $\int x(5x^2+4)^3\,dx = \frac{1}{10}\int (5x^2+4)^3(10x)\,dx = \frac{1}{10}\left[\frac{(5x^2+4)^4}{4}\right] + C = \frac{(5x^2+4)^4}{40} + C$

Check: $\frac{d}{dx}\left[\frac{(5x^2+4)^4}{40} + C\right] = \frac{4(5x^2+4)^3(10x)}{40} = x(5x^2+4)^3$

17. $\int t\sqrt{t^2+2}\,dt = \frac{1}{2}\int (t^2+2)^{1/2}(2t)\,dt = \frac{1}{2}\frac{(t^2+2)^{3/2}}{3/2} + C = \frac{(t^2+2)^{3/2}}{3} + C$

Check: $\frac{d}{dt}\left[\frac{(t^2+2)^{3/2}}{3} + C\right] = \frac{3/2(t^2+2)^{1/2}(2t)}{3} = (t^2+2)^{1/2}t$

18. $\int t^3\sqrt{2t^4+3}\,dt = \frac{1}{8}\int (2t^4+3)^{1/2}(8t^3)\,dt = \frac{1}{8}\cdot\frac{(2t^4+3)^{3/2}}{(3/2)} + C = \frac{(2t^4+3)^{3/2}}{12} + C$

Check: $\frac{d}{dt}\left[\frac{(2t^4+3)^{3/2}}{12} + C\right] = \frac{\frac{3}{2}(2t^4+3)^{1/2}(8t^3)}{12} = t^3\sqrt{2t^4+3}$

19. $\int 5x(1-x^2)^{1/3}\,dx = -\frac{5}{2}\int (1-x^2)^{1/3}(-2x)\,dx = -\frac{5}{2}\cdot\frac{(1-x^2)^{4/3}}{4/3} + C = -\frac{15}{8}(1-x^2)^{4/3} + C$

Check: $\frac{d}{dx}\left[-\frac{15}{8}(1-x^2)^{4/3} + C\right] = -\frac{15}{8}\cdot\frac{4}{3}(1-x^2)^{1/3}(-2x) = 5x(1-x^2)^{1/3} = 5x\sqrt[3]{1-x^2}$

20. $\int 6u^6\sqrt{u^7+8}\,du = \frac{6}{7}\int (u^7+8)^{1/2}(7u^6)\,du$

$$= \frac{6}{7}\cdot\frac{(u^7+8)^{3/2}}{(3/2)} + C = \frac{4}{7}(u^7+8)^{3/2} + C$$

Check: $\frac{d}{du}\left[\frac{4}{7}(u^7+8)^{3/2}\right] = \frac{4}{7}\left(\frac{3}{2}\right)(u^7+8)^{1/2}(7u^6)$

$$= 6u^6\sqrt{u^7+8}$$

21. $\int \frac{7x}{(1-x^2)^3}\,dx = \frac{7}{-2}\int (1-x^2)^{-3}(-2x)\,dx$

$$= \frac{7}{-2}\frac{(1-x^2)^{-2}}{(-2)} + C = \frac{7}{4(1-x^2)^2} + C$$

Check: $\frac{d}{dx}\left[\frac{7}{4(1-x^2)^2} + C\right] = \frac{d}{dx}\left[\frac{7}{4}(1-x^2)^{-2} + C\right]$

$$= \frac{7}{4}(-2)(1-x^2)^{-3}(-2x) = 7x(1-x^2)^{-3} = \frac{7x}{(1-x^2)^3}$$

22. $\int \frac{x^3}{(1+x^4)^2}\,dx = \frac{1}{4}\int (1+x^4)^{-2}(4x^3)\,dx = -\frac{1}{4}(1+x^4)^{-1} + C = \frac{-1}{4(1+x^4)} + C$

Check: $\frac{d}{dx}\left[\frac{-1}{4(1+x^4)} + C\right] = \frac{1}{4}(1+x^4)^{-2}(4x^3) = \frac{x^3}{(1+x^4)^2}$

23. $\int \frac{x^2}{(1+x^3)^2}\,dx = \frac{1}{3}\int (1+x^3)^{-2}(3x^2)\,dx = \frac{1}{3}\left[\frac{(1+x^3)^{-1}}{-1}\right] + C = -\frac{1}{3(1+x^3)} + C$

Check: $\frac{d}{dx}\left[-\frac{1}{3(1+x^3)} + C\right] = -\frac{1}{3}(-1)(1+x^3)^{-2}(3x^2) = \frac{x^2}{(1+x^3)^2}$

24. $\int \frac{6x^2}{(4x^3-9)^3}\,dx = \frac{1}{2}\int (4x^3-9)^{-3}(12x^2)\,dx = \frac{1}{2}\cdot\frac{(4x^3-9)^{-2}}{-2} + C = -\frac{1}{4(4x^3-9)^2} + C$

Check: $\frac{d}{dx}\left[\frac{-1}{4(4x^3-9)^2} + C\right] = \frac{d}{dx}\left[-\frac{1}{4}(4x^3-9)^{-2} + C\right] = -\frac{1}{4}(-2)(4x^3-9)^{-3}(12x^2) = \frac{6x^2}{(4x^3-9)^3}$

25. $\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\,dx = -\frac{1}{2}\frac{(1-x^2)^{1/2}}{1/2} + C = -\sqrt{1-x^2} + C$

Check: $\frac{d}{dx}\left[-(1-x^2)^{1/2} + C\right] = -\frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{1-x^2}}$

26. $\displaystyle\int \frac{x^3}{\sqrt{1+x^4}}\,dx = \frac{1}{4}\int \left(1+x^4\right)^{-1/2}\left(4x^3\right)dx = \frac{1}{4}\frac{\left(1+x^4\right)^{1/2}}{1/2} + C = \frac{\sqrt{1+x^4}}{2} + C$

Check: $\displaystyle\frac{d}{dx}\left[\frac{\sqrt{1+x^4}}{2} + C\right] = \frac{1}{2}\cdot\frac{1}{2}\left(1+x^4\right)^{-1/2}\left(4x^3\right) = \frac{x^3}{\sqrt{1+x^4}}$

27. $\displaystyle\int\left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt = -\int\left(1+\frac{1}{t}\right)^3\left(-\frac{1}{t^2}\right)dt = -\frac{\left[1+\left(\frac{1}{t}\right)\right]^4}{4} + C$

Check: $\displaystyle\frac{d}{dt}\left[-\frac{\left[1+(1/t)\right]^4}{4} + C\right] = -\frac{1}{4}(4)\left(1+\frac{1}{t}\right)^3\left(-\frac{1}{t^2}\right) = \frac{1}{t^2}\left(1+\frac{1}{t}\right)^3$

28. $\displaystyle\int\left(8-\frac{1}{t^4}\right)^2\left(\frac{1}{t^5}\right)dt = \frac{1}{4}\int\left(8-t^{-4}\right)^2\left(4t^{-5}\right)dt$

$$= \frac{1}{4}\frac{\left(8-t^{-4}\right)^3}{3} + C = \frac{1}{12}\left(8-t^{-4}\right)^3 + C$$

Check: $\displaystyle\frac{d}{dt}\left[\frac{1}{12}\left(8-t^{-4}\right)^3 + C\right] = \frac{1}{12}(3)\left(8-t^{-4}\right)^2\left(4t^{-5}\right)$

$$= \left(8-t^{-4}\right)^2\frac{1}{t^5}$$

29. $\displaystyle\int\frac{1}{\sqrt{2x}}\,dx = \frac{1}{2}\int(2x)^{-1/2}2\,dx = \frac{1}{2}\left[\frac{(2x)^{1/2}}{1/2}\right] + C = \sqrt{2x} + C$

Alternate Solution: $\displaystyle\int\frac{1}{\sqrt{2x}}\,dx = \frac{1}{\sqrt{2}}\int x^{-1/2}\,dx = \frac{1}{\sqrt{2}}\frac{x^{1/2}}{(1/2)} + C = \sqrt{2x} + C$

Check: $\displaystyle\frac{d}{dx}\left[\sqrt{2x} + C\right] = \frac{1}{2}(2x)^{-1/2}(2) = \frac{1}{\sqrt{2x}}$

30. $\displaystyle\int\frac{x}{\sqrt[3]{5x^2}}\,dx = \int\frac{1}{\sqrt[3]{5}}x^{1/3}\,dx = \frac{1}{\sqrt[3]{5}}\cdot\frac{3}{4}x^{4/3} + C = \frac{3}{20}\sqrt[3]{25x^4} + C$

Alternate Solution:

$\displaystyle\int\frac{x}{\sqrt[3]{5x^2}}\,dx = \int\left(5x^2\right)^{-1/3}x\,dx = \frac{1}{10}\int\left(5x^2\right)^{-1/3}(10x)dx = \frac{1}{10}\cdot\frac{\left(5x^2\right)^{2/3}}{(2/3)} + C = \frac{3}{20}\left(5x^2\right)^{2/3} + C = \frac{3}{4}\cdot\frac{1}{\sqrt[3]{5}}x^{4/3} + C$

Check: $\displaystyle\frac{d}{dx}\left[\frac{1}{\sqrt[3]{5}}\cdot\frac{3}{4}x^{4/3} + C\right] = \frac{1}{\sqrt[3]{5}}\cdot\frac{3}{4}\cdot\frac{4}{3}x^{1/3} = \frac{x}{\sqrt[3]{5x^2}}$

31. $\displaystyle y = \int\left[4x + \frac{4x}{\sqrt{16-x^2}}\right]dx = 4\int x\,dx - 2\int\left(16-x^2\right)^{-1/2}(-2x)\,dx = 4\left(\frac{x^2}{2}\right) - 2\left[\frac{\left(16-x^2\right)^{1/2}}{1/2}\right] + C = 2x^2 - 4\sqrt{16-x^2} + C$

32. $y = \int \frac{10x^2}{\sqrt{1 + x^3}}\,dx$

$$= \frac{10}{3}\int \left(1 + x^3\right)^{-1/2}\left(3x^2\right) dx$$

$$= \frac{10}{3}\left[\frac{\left(1 + x^3\right)^{1/2}}{1/2}\right] + C$$

$$= \frac{20}{3}\sqrt{1 + x^3} + C$$

33. $y = \int \frac{x + 1}{\left(x^2 + 2x - 3\right)^2}\,dx$

$$= \frac{1}{2}\int \left(x^2 + 2x - 3\right)^{-2}(2x + 2)\,dx$$

$$= \frac{1}{2}\left[\frac{\left(x^2 + 2x - 3\right)^{-1}}{-1}\right] + C$$

$$= -\frac{1}{2\left(x^2 + 2x - 3\right)} + C$$

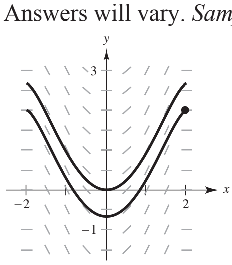
34. $y = \int \frac{18 - 6x^2}{\sqrt{x^3 - 9x + 7}}\,dx$

$$= -2\int\left(x^3 - 9x + 7\right)^{-1/2}\left(3x^2 - 9\right) dx$$

$$= -2\frac{\left(x^3 - 9x + 7\right)^{1/2}}{(1/2)} + C$$

$$= -4\sqrt{x^3 - 9x + 7} + C$$

35. (a) Answers will vary. *Sample answer:*



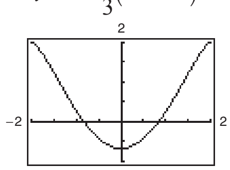
(b) $\frac{dy}{dx} = x\sqrt{4 - x^2}$, $(2, 2)$

$$y = \int x\sqrt{4 - x^2}\,dx = -\frac{1}{2}\int \left(4 - x^2\right)^{1/2}(-2x\,dx)$$

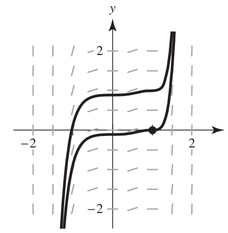
$$= -\frac{1}{2}\cdot\frac{2}{3}\left(4 - x^2\right)^{3/2} + C = -\frac{1}{3}\left(4 - x^2\right)^{3/2} + C$$

$$(2, 2)\!: 2 = -\frac{1}{3}\left(4 - 2^2\right)^{3/2} + C \Rightarrow C = 2$$

$$y = -\frac{1}{3}\left(4 - x^2\right)^{3/2} + 2$$



36. (a) Answers will vary. *Sample answer:*



(b)

$$\frac{dy}{dx} = x^2\left(x^3 - 1\right)^2,\ (1, 0)$$

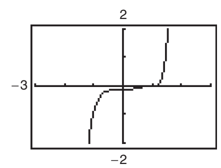
$$y = \int x^2\left(x^3 - 1\right)^2 dx$$

$$\left(u = x^3 - 1\right) = \frac{1}{3}\int \left(x^3 - 1\right)^2\left(3x^2\,dx\right)$$

$$= \frac{1}{3}\frac{\left(x^3 - 1\right)^3}{3} + C = \frac{1}{9}\left(x^3 - 1\right)^3 + C$$

$$0 = C$$

$$y = \frac{1}{9}\left(x^3 - 1\right)^3$$



37. $\frac{dy}{dx} = 18x^2\left(2x^3 + 1\right)^2, (0, 4)$

$$y = 3\int \left(2x^3 + 1\right)^2\left(6x^2\right) dx \quad \left(u = 2x^3 + 1\right)$$

$$y = 3\frac{\left(2x^3 + 1\right)^3}{3} + C = \left(2x^3 + 1\right)^3 + C$$

$$4 = 1^3 + C \Rightarrow C = 3$$

$$y = \left(2x^3 + 1\right)^3 + 3$$

38. $\frac{dy}{dx} = \frac{-48}{(3x + 5)^3}$, $(-1, 3)$

$$y = -48\int (3x + 5)^{-3}dx$$

$$= (-48)\frac{1}{3}\int (3x + 5)^{-3}3\,dx$$

$$= \frac{-16(3x + 5)^{-2}}{-2} + C$$

$$= \frac{8}{(3x + 5)^2} + C$$

$$3 = \frac{8}{\left(3(-1) + 5\right)^2} + C = \frac{8}{4} + C \Rightarrow C = 1$$

$$y = \frac{8}{(3x + 5)^2} + 1$$

39. $\int \pi \sin \pi x\, dx = -\cos \pi x + C$

40. $\int \sin 4x\, dx = \frac{1}{4}\int (\sin 4x)(4)\, dx = -\frac{1}{4}\cos 4x + C$

41. $\int \cos 6x\, dx = \frac{1}{6}\int \cos 6x(6)\, dx$

$= \frac{1}{6}\sin 6x + C$

42. $\int \csc^2\left(\frac{x}{2}\right) dx = 2\int \csc^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right) dx = -2\cot\left(\frac{x}{2}\right) + C$

43. $\int \frac{1}{\theta^2}\cos\frac{1}{\theta}\, d\theta = -\int \cos\frac{1}{\theta}\left(-\frac{1}{\theta^2}\right) d\theta = -\sin\frac{1}{\theta} + C$

44. $\int x \sin x^2\, dx = \frac{1}{2}\int (\sin x^2)(2x)\, dx = -\frac{1}{2}\cos x^2 + C$

45. $\int \sin 2x \cos 2x\, dx = \frac{1}{2}\int (\sin 2x)(2\cos 2x)\, dx = \frac{1}{2}\frac{(\sin 2x)^2}{2} + C = \frac{1}{4}\sin^2 2x + C$ OR

$\int \sin 2x \cos 2x\, dx = -\frac{1}{2}\int (\cos 2x)(-2\sin 2x)\, dx = -\frac{1}{2}\frac{(\cos 2x)^2}{2} + C_1 = -\frac{1}{4}\cos^2 2x + C_1$ OR

$\int \sin 2x \cos 2x\, dx = \frac{1}{2}\int 2\sin 2x \cos 2x\, dx = \frac{1}{2}\int \sin 4x\, dx = -\frac{1}{8}\cos 4x + C_2$

46. $\int \sqrt[3]{\tan x}\sec^2 x\, dx = \int (\tan x)^{1/3}\sec^2 x\, dx$

$= \frac{(\tan x)^{4/3}}{(4/3)} + C = \frac{3}{4}(\tan x)^{4/3}$

47. $\int \frac{\csc^2 x}{\cot^3 x}\, dx = -\int (\cot x)^{-3}(-\csc^2 x)\, dx$

$= -\frac{(\cot x)^{-2}}{-2} + C = \frac{1}{2\cot^2 x} + C = \frac{1}{2}\tan^2 x + C = \frac{1}{2}(\sec^2 x - 1) + C = \frac{1}{2}\sec^2 x + C_1$

48. $\int \frac{\sin x}{\cos^3 x}\, dx = -\int (\cos x)^{-3}(-\sin x)\, dx$

$= -\frac{(\cos x)^{-2}}{-2} + C$

$= \frac{1}{2\cos^2 x} + C = \frac{1}{2}\sec^2 x + C$

49. $f(x) = \int -\sin\frac{x}{2}\, dx = 2\cos\frac{x}{2} + C$

Because $f(0) = 6 = 2\cos\left(\frac{0}{2}\right) + C$, $C = 4$. So,

$f(x) = 2\cos\frac{x}{2} + 4.$

50. $f'(x) = \sec^2(2x), \left(\frac{\pi}{2}, 2\right)$

$f(x) = \frac{1}{2}\tan(2x) + C$

$f\left(\frac{\pi}{2}\right) = \frac{1}{2}\tan\left(2\left(\frac{\pi}{2}\right)\right) + C = 2$

$\frac{1}{2}(0) + C = 2$

$C = 2$

$f(x) = \frac{1}{2}\tan(2x) + 2$

51. $f'(x) = 2x(4x^2 - 10)^2, (2, 10)$

$f(x) = \frac{(4x^2 - 10)^3}{12} + C = \frac{2(2x^2 - 5)^3}{3} + C$

$f(2) = \frac{2(8 - 5)^3}{3} + C = 18 + C = 10 \Rightarrow C = -8$

$f(x) = \frac{2}{3}(2x^2 - 5)^3 - 8$

52. $f'(x) = -2x\sqrt{8 - x^2}, (2, 7)$

$$f(x) = \frac{2(8 - x^2)^{3/2}}{3} + C$$

$$f(2) = \frac{2(4)^{3/2}}{3} + C = \frac{16}{3} + C = 7 \Rightarrow C = \frac{5}{3}$$

$$f(x) = \frac{2(8 - x^2)^{3/2}}{3} + \frac{5}{3}$$

53. $u = x + 6, x = u - 6, dx = du$

$$\int x\sqrt{x + 6}\,dx = \int (u - 6)\sqrt{u}\,du$$
$$= \int \left(u^{3/2} - 6u^{1/2}\right) du$$
$$= \frac{2}{5}u^{5/2} - 4u^{3/2} + C$$
$$= \frac{2u^{3/2}}{5}(u - 10) + C$$
$$= \frac{2}{5}(x + 6)^{3/2}\left[(x + 6) - 10\right] + C$$
$$= \frac{2}{5}(x + 6)^{3/2}(x - 4) + C$$

54. $u = 3x - 4, x = \frac{u + 4}{3}, dx = \frac{1}{3}du$

$$\int x\sqrt{3x - 4}\,dx = \int \frac{u + 4}{3} \cdot \sqrt{u} \cdot \frac{1}{3}du$$
$$= \frac{1}{9}\int \left(u^{3/2} + 4u^{1/2}\right) du$$
$$= \frac{1}{9}\left(\frac{2}{5}u^{5/2} + \frac{8}{3}u^{3/2}\right) + C$$
$$= \frac{2}{45}(3x - 4)^{5/2} + \frac{8}{27}(3x - 4)^{3/2} + C$$
$$= \frac{2}{135}(3x - 4)^{3/2}\left[3(3x - 4) + 20\right] + C$$
$$= \frac{2}{135}(3x - 4)^{3/2}(9x + 8) + C$$

55. $u = 1 - x, x = 1 - u, dx = -du$

$$\int x^2\sqrt{1 - x}\,dx = -\int (1 - u)^2\sqrt{u}\,du$$
$$= -\int \left(u^{1/2} - 2u^{3/2} + u^{5/2}\right) du$$
$$= -\left(\frac{2}{3}u^{3/2} - \frac{4}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right) + C$$
$$= -\frac{2u^{3/2}}{105}\left(35 - 42u + 15u^2\right) + C$$
$$= -\frac{2}{105}(1 - x)^{3/2}\left[35 - 42(1 - x) + 15(1 - x)^2\right] + C$$
$$= -\frac{2}{105}(1 - x)^{3/2}\left(15x^2 + 12x + 8\right) + C$$

56. $u = 2 - x, x = 2 - u, dx = -du$

$$\int (x + 1)\sqrt{2 - x}\,dx = -\int (3 - u)\sqrt{u}\,du$$
$$= -\int \left(3u^{1/2} - u^{3/2}\right) du$$
$$= -\left(2u^{3/2} - \frac{2}{5}u^{5/2}\right) + C$$
$$= -\frac{2u^{3/2}}{5}(5 - u) + C$$
$$= -\frac{2}{5}(2 - x)^{3/2}\left[5 - (2 - x)\right] + C$$
$$= -\frac{2}{5}(2 - x)^{3/2}(x + 3) + C$$

57. $u = 2x - 1,\ x = \frac{1}{2}(u + 1),\ dx = \frac{1}{2}\,du$

$$\begin{aligned}\int \frac{x^2 - 1}{\sqrt{2x-1}}\,dx &= \int \frac{\left[(1/2)(u+1)\right]^2 - 1}{\sqrt{u}}\frac{1}{2}\,du\\ &= \frac{1}{8}\int u^{-1/2}\left[\left(u^2 + 2u + 1\right) - 4\right]du\\ &= \frac{1}{8}\int \left(u^{3/2} + 2u^{1/2} - 3u^{-1/2}\right)du\\ &= \frac{1}{8}\left(\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2} - 6u^{1/2}\right) + C\\ &= \frac{u^{1/2}}{60}\left(3u^2 + 10u - 45\right) + C\\ &= \frac{\sqrt{2x-1}}{60}\left[3(2x-1)^2 + 10(2x-1) - 45\right] + C\\ &= \frac{1}{60}\sqrt{2x-1}\left(12x^2 + 8x - 52\right) + C\\ &= \frac{1}{15}\sqrt{2x-1}\left(3x^2 + 2x - 13\right) + C\end{aligned}$$

58. $u = x + 4,\ x = u - 4,\ du = dx$

$$\begin{aligned}\int \frac{2x+1}{\sqrt{x+4}}\,dx &= \int \frac{2(u-4)+1}{\sqrt{u}}\,du\\ &= \int \left(2u^{1/2} - 7u^{-1/2}\right)du\\ &= \frac{4}{3}u^{3/2} - 14u^{1/2} + C\\ &= \frac{2}{3}u^{1/2}(2u - 21) + C\\ &= \frac{2}{3}\sqrt{x+4}\left[2(x+4) - 21\right] + C\\ &= \frac{2}{3}\sqrt{x+4}(2x - 13) + C\end{aligned}$$

59. $u = 2x, du = 2\,dx \Rightarrow dx = \frac{du}{2}$

$$\begin{aligned}\int \cos^3 2x \sin 2x\,dx &= \frac{1}{2}\int \cos^3 u \sin u\,du\\ &= \frac{1}{2}\left(-\frac{\cos^4 u}{4}\right) + C = -\frac{1}{8}\cos^4 2x + C\end{aligned}$$

60. $u = 7x, du = 7\,dx \Rightarrow dx = \frac{1}{7}\,du$

$$\begin{aligned}\int \sec^5 7x \tan 7x\,dx &= \int \sec^5 u \tan u\left(\frac{1}{7}\,du\right).\\ &= \frac{1}{7}\int \sec^4 u(\sec u \tan u)\,du\\ &= \frac{1}{7}\cdot\frac{\sec^5 u}{5} + C = \frac{1}{35}\sec^5(7x) + C\end{aligned}$$

61. Let $u = x^2 + 1$, $du = 2x\,dx$.

$$\int_{-1}^{1} x(x^2+1)^3\,dx = \tfrac{1}{2}\int_{-1}^{1}(x^2+1)^3(2x)\,dx = \left[\tfrac{1}{8}(x^2+1)^4\right]_{-1}^{1} = 0$$

62. Let $u = 2x^4 + 1$, $du = 8x^3\,dx$.

$$\int_{0}^{1} x^3(2x^4+1)^2\,dx = \tfrac{1}{8}\int_{0}^{1}(2x^4+1)^2(8x^3)\,dx = \left[\tfrac{1}{8}\cdot\frac{(2x^4+1)^3}{3}\right]_{0}^{1} = \frac{1}{24}(3^3-1^3) = \frac{13}{12}$$

63. Let $u = x^3 + 1$, $du = 3x^2\,dx$.

$$\int_{1}^{2} 2x^2\sqrt{x^3+1}\,dx = 2\cdot\frac{1}{3}\int_{1}^{2}(x^3+1)^{1/2}(3x^2)\,dx = \left[\frac{(x^3+1)^{3/2}}{3/2}\right]_{1}^{2} = \frac{4}{9}\left[(x^3+1)^{3/2}\right]_{1}^{2} = \frac{4}{9}\left[27-2\sqrt{2}\right] = 12-\frac{8}{9}\sqrt{2}$$

64.
$$\int_{-1}^{0} x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int_{-1}^{0}(1-x^2)^{1/2}(-2x)\,dx$$
$$= \left[-\frac{1}{2}\frac{(1-x^2)^{3/2}}{(3/2)}\right]_{-1}^{0}$$
$$= \left[-\frac{1}{3}(1-x^2)^{3/2}\right]_{-1}^{0}$$
$$= -\frac{1}{3}$$

65. Let $u = 2x + 1$, $du = 2\,dx$.

$$\int_{0}^{4}\frac{1}{\sqrt{2x+1}}\,dx = \frac{1}{2}\int_{0}^{4}(2x+1)^{-1/2}(2)\,dx = \left[\sqrt{2x+1}\right]_{0}^{4} = \sqrt{9}-\sqrt{1} = 2$$

66. Let $u = 1 + 2x^2$, $du = 4x\,dx$.

$$\int_{0}^{2}\frac{x}{\sqrt{1+2x^2}}\,dx = \frac{1}{4}\int_{0}^{2}(1+2x^2)^{-1/2}(4x)\,dx = \left[\frac{1}{2}\sqrt{1+2x^2}\right]_{0}^{2} = \frac{3}{2}-\frac{1}{2} = 1$$

67. Let $u = 1 + \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$.

$$\int_{1}^{9}\frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx = 2\int_{1}^{9}(1+\sqrt{x})^{-2}\left(\frac{1}{2\sqrt{x}}\right)dx = \left[-\frac{2}{1+\sqrt{x}}\right]_{1}^{9} = -\frac{1}{2}+1 = \frac{1}{2}$$

68. Let $u = 2x - 6, du = 2\,dx, x = \dfrac{u + 6}{2}$.

$$\int_3^5 \frac{x}{\sqrt{2x-6}}\,dx = \int_0^4 u^{-1/2}\left(\frac{u+6}{2}\right)\left(\frac{1}{2}\,du\right)$$
$$= \frac{1}{4}\int_2^4 \left(u^{1/2} + 6u^{-1/2}\right)du$$
$$= \frac{1}{4}\left[\frac{2}{3}u^{3/2} + 12u^{1/2}\right]_0^4$$
$$= \frac{1}{4}\left[\left(\frac{16}{3} + 24\right) - (0 + 0)\right]$$
$$= \frac{22}{3}$$

69. $u = x + 1, x = u - 1, dx = du$

When $x = 0, u = 1$. When $x = 7, u = 8$.

$$\text{Area} = \int_0^7 x\sqrt[3]{x+1}\,dx = \int_1^8 (u-1)\sqrt[3]{u}\,du$$
$$= \int_1^8 \left(u^{4/3} - u^{1/3}\right)du$$
$$= \left[\frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3}\right]_1^8$$
$$= \left(\frac{384}{7} - 12\right) - \left(\frac{3}{7} - \frac{3}{4}\right)$$
$$= \frac{1209}{28}$$

70. $u = x + 2, x = u - 2, dx = du$

When $x = -2, u = 0$. When $x = 6, u = 8$.

$$\text{Area} = \int_{-2}^6 x^2\sqrt[3]{x+2}\,dx$$
$$= \int_0^8 (u-2)^2\sqrt[3]{u}\,du$$
$$= \int_0^8 \left(u^{7/3} - 4u^{4/3} + 4u^{1/3}\right)du$$
$$= \left[\frac{3}{10}u^{10/3} - \frac{12}{7}u^{7/3} + 3u^{4/3}\right]_0^8 = \frac{4752}{35}$$

71. $$\text{Area} = \int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right)dx$$
$$= 2\int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)dx$$
$$= \left[2\tan\left(\frac{x}{2}\right)\right]_{\pi/2}^{2\pi/3} = 2\left(\sqrt{3} - 1\right)$$

72. Let $u = 2x,\ du = 2\,dx$.

$$\text{Area} = \int_{\pi/12}^{\pi/4} \csc 2x \cot 2x\,dx$$
$$= \tfrac{1}{2}\int_{\pi/12}^{\pi/4} \csc 2x \cot 2x(2)\,dx$$
$$= \left[-\tfrac{1}{2}\csc 2x\right]_{\pi/12}^{\pi/4} = \tfrac{1}{2}$$

73. $f(x) = x^2(x^2 + 1)$ is even.

$$\int_{-2}^2 x^2(x^2+1)\,dx = 2\int_0^2 \left(x^4 + x^2\right)dx = 2\left[\frac{x^5}{5} + \frac{x^3}{3}\right]_0^2$$
$$= 2\left[\frac{32}{5} + \frac{8}{3}\right] = \frac{272}{15}$$

74. $f(x) = x(x^2 + 1)^3$ is odd.

$$\int_{-2}^2 x(x^2+1)^3\,dx = 0$$

75. $f(x) = \sin x \cos x$ is odd.

$$\int_{-\pi/2}^{\pi/2} \sin x \cos x\,dx = 0$$

76. $f(x) = \sin^2 x \cos x$ is even.

$$\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\,dx = 2\int_0^{\pi/2} \sin^2 x(\cos x)\,dx$$
$$= 2\left[\frac{\sin^3 x}{3}\right]_0^{\pi/2}$$
$$= \frac{2}{3}$$

77. $\int_0^6 x^2\,dx = 72$ and $y = x^2$ is even.

(a) $\int_{-6}^6 x^2\,dx = 2\int_0^6 x^2\,dx = 2(72) = 144$

(b) $\int_{-6}^0 x^2\,dx = \int_0^6 x^2\,dx = 72$

(c) $\int_0^6 -2x^2\,dx = -2\int_0^6 x^2\,dx = -2(72) = -144$

(d) $\int_{-6}^6 3x^2\,dx = 3\int_{-6}^6 x^2\,dx = 3(144) = 432$

78. (a) $\int_{-\pi/4}^{\pi/4} \sin x\,dx = 0$ because $\sin x$ is symmetric to the origin.

(b) $\int_{-\pi/4}^{\pi/4} \cos x\,dx = 2\int_0^{\pi/4} \cos x\,dx = \left[2\sin x\right]_0^{\pi/4} = \sqrt{2}$ because $\cos x$ is symmetric to the y-axis.

(c) $\int_{-\pi/2}^{\pi/2} \cos x\,dx = 2\int_0^{\pi/2} \cos x\,dx = \left[2\sin x\right]_0^{\pi/2} = 2$

(d) $\int_{-\pi/2}^{\pi/2} \sin x \cos x\,dx = 0$ because $\sin(-x)\cos(-x) = -\sin x \cos x$ and so, is symmetric to the origin.

79. $\int_{-3}^{3} \left(x^3 + 4x^2 - 3x - 6\right) dx = \int_{-3}^{3} \left(x^3 - 3x\right) dx + \int_{-3}^{3} \left(4x^2 - 6\right) dx = 0 + 2\int_0^3 \left(4x^2 - 6\right) dx = 2\left[\frac{4}{3}x^3 - 6x\right]_0^3 = 36$

80. $\int_{-\pi/2}^{\pi/2} (\sin 4x + \cos 4x)\,dx = \int_{-\pi/2}^{\pi/2} \sin 4x\,dx + \int_{-\pi/2}^{\pi/2} \cos 4x\,dx = 0 + 2\int_0^{\pi/2} \cos 4x\,dx = \left[\frac{2}{4}\sin 4x\right]_0^{\pi/2} = 0$

81. (a) $\int x^2\sqrt{x^3 + 1}\,dx$ is easier to compute. Use substitution with $u = x^3 + 1$.

(b) $\int \cot^3 2x \csc^2 2x\,dx$ is easier to compute. Use substitution with $u = \cot 2x$.

82. (a)
$$\int (2x - 1)^2\,dx = \int \left(4x^2 - 4x + 1\right) dx = \frac{4x^3}{3} - 2x^2 + x + C_1$$

OR

$$\int (2x - 1)^2\,dx = \frac{1}{2}\int (2x - 1)^2\,2\,dx = \frac{1}{2}\cdot\frac{(2x - 1)^3}{3} + C = \frac{1}{6}\left(8x^3 - 12x^2 + 6x - 1\right) + C = \frac{4}{3}x^3 - 2x^2 + x + C_1 \qquad \left(C_1 = C - \frac{1}{6}\right)$$

(b) $\int \sin x \cos x\,dx = \frac{\sin^2 x}{2} + C \qquad (u = \sin x)$

OR

$\int \sin x \cos x\,dx = \frac{-\cos^2 x}{2} + C_1 \qquad (u = \cos x)$

These answers differ by a constant because $\sin^2 x + \cos^2 x = 1$.

83. $\frac{dV}{dt} = \frac{k}{(t + 1)^2}$

$V(t) = \int \frac{k}{(t + 1)^2}\,dt = -\frac{k}{t + 1} + C$

$V(0) = -k + C = 500{,}000$

$V(1) = -\frac{1}{2}k + C = 400{,}000$

Solving this system yields $k = -200{,}000$ and $C = 300{,}000$. So, $V(t) = \frac{200{,}000}{t + 1} + 300{,}000$.

When $t = 4, V(4) = \$340{,}000$.

84. (a) The maximum flow is approximately $R \approx 62$ thousand gallons at 9:00 A.M. $(t \approx 9)$.

(b) The volume of water used during the day is the area under the curve for $0 \le t \le 24$. That is, $V = \int_0^{24} R(t)\,dt$.

(c) The least amount of water is used approximately from 1 A.M. to 3 A.M. $(1 \le t \le 3)$.

85. $$\frac{1}{b-a}\int_a^b \left[74.50 + 43.75 \sin\frac{\pi t}{6}\right] dt = \frac{1}{b-a}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_a^b$$

(a) $$\frac{1}{3}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_0^3 = \frac{1}{3}\left(223.5 + \frac{262.5}{\pi}\right) \approx 102.352 \text{ thousand units}$$

(b) $$\frac{1}{3}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_3^6 = \frac{1}{3}\left(447 + \frac{262.5}{\pi} - 223.5\right) \approx 102.352 \text{ thousand units}$$

(c) $$\frac{1}{12}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_0^{12} = \frac{1}{12}\left(894 - \frac{262.5}{\pi} + \frac{262.5}{\pi}\right) = 74.5 \text{ thousand units}$$

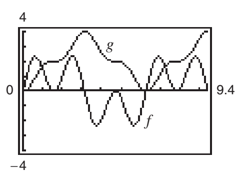
86. $$\frac{1}{b-a}\int_a^b \left[2\sin(60\pi t) + \cos(120\pi t)\right] dt = \frac{1}{b-a}\left[-\frac{1}{30\pi}\cos(60\pi t) + \frac{1}{120\pi}\sin(120\pi t)\right]_a^b$$

(a) $$\frac{1}{(1/60) - 0}\left[-\frac{1}{30\pi}\cos(60\pi t) + \frac{1}{120\pi}\sin(120\pi t)\right]_0^{1/60} = 60\left[\left(\frac{1}{30\pi} + 0\right) - \left(-\frac{1}{30\pi}\right)\right] = \frac{4}{\pi} \approx 1.273 \text{ amps}$$

(b) $$\frac{1}{(1/240) - 0}\left[-\frac{1}{30\pi}\cos(60\pi t) + \frac{1}{120\pi}\sin(120\pi t)\right]_0^{1/240} = 240\left[\left(-\frac{1}{30\sqrt{2}\pi} + \frac{1}{120\pi}\right) - \left(-\frac{1}{30\pi}\right)\right]$$
$$= \frac{2}{\pi}\left(5 - 2\sqrt{2}\right) \approx 1.382 \text{ amps}$$

(c) $$\frac{1}{(1/30) - 0}\left[-\frac{1}{30\pi}\cos(60\pi t) + \frac{1}{120\pi}\sin(120\pi t)\right]_0^{1/30} = 30\left[\left(-\frac{1}{30\pi}\right) - \left(-\frac{1}{30\pi}\right)\right] = 0 \text{ amp}$$

87. (a)

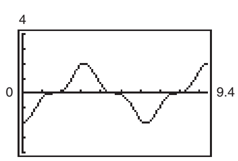


(b) g is nonnegative because the graph of f is positive at the beginning, and generally has more positive sections than negative ones.

(c) The points on g that correspond to the extrema of f are points of inflection of g.

(d) No, some zeros of f, like $x = \pi/2$, do not correspond to an extrema of g. The graph of g continues to increase after $x = \pi/2$ because f remains above the x-axis.

(e) The graph of h is that of g shifted 2 units downward.



$$g(t) = \int_0^t f(x)\,dx = \int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^t f(x)\,dx = 2 + h(t).$$

88. Let $f(x) = \sin \pi x$, $0 \le x \le 1$.

Let $\Delta x = \dfrac{1}{n}$ and use righthand endpoints

$c_i = \dfrac{i}{n}$, $i = 1, 2, \ldots, n$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sin(i\pi/n)}{n} = \lim_{\|\Delta x\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x$$
$$= \int_0^1 \sin \pi x\, dx$$
$$= -\frac{1}{\pi}\cos \pi x \Big]_0^1$$
$$= -\frac{1}{\pi}(-1 - 1) = \frac{2}{\pi}$$

89. (a) Let $u = 1 - x$ and $du = -dx$.

$$\int_0^1 x^3(1-x)^8\, dx = \int_1^0 (1-u)^3(u)^8(-du)$$
$$= \int_0^1 (1-x)^3 x^8\, dx$$

(b) Let $u = 1 - x$ and $du = -dx$.

$$\int_0^1 x^a(1-x)^b\, dx = \int_1^0 (1-u)^a u^b(-du)$$
$$= \int_0^1 (1-x)^a x^b\, dx$$

90. (a) $\sin x = \cos\left(\dfrac{\pi}{2} - x\right)$ and $\cos x = \sin\left(\dfrac{\pi}{2} - x\right)$

Let $u = \dfrac{\pi}{2} - x$, $du = -dx$, $x = \dfrac{\pi}{2} - u$:

$$\int_0^{\pi/2} \sin^2 x\, dx = \int_0^{\pi/2} \cos^2\left(\frac{\pi}{2} - x\right) dx$$
$$= \int_{\pi/2}^{0} \cos^2 u(-du)$$
$$= \int_0^{\pi/2} \cos^2 u\, du = \int_0^{\pi/2} \cos^2 x\, dx$$

(b) Let $u = \dfrac{\pi}{2} - x$ as in part (a):

$$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n\left(\frac{\pi}{2} - x\right) dx$$
$$= \int_{\pi/2}^{0} \cos^n u(-du)$$
$$= \int_0^{\pi/2} \cos^n u\, du = \int_0^{\pi/2} \cos^n x\, dx$$

91. $u = 1 - x$, $x = 1 - u$, $dx = -du$

When $x = a$, $u = 1 - a$. When $x = b$, $u = 1 - b$.

$$P_{a,\,b} = \int_a^b \frac{15}{4} x\sqrt{1-x}\, dx = \frac{15}{4}\int_{1-a}^{1-b} -(1-u)\sqrt{u}\, du$$
$$= \frac{15}{4}\int_{1-a}^{1-b} \left(u^{3/2} - u^{1/2}\right) du = \frac{15}{4}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_{1-a}^{1-b} = \frac{15}{4}\left[\frac{2u^{3/2}}{15}(3u - 5)\right]_{1-a}^{1-b} = \left[-\frac{(1-x)^{3/2}}{2}(3x + 2)\right]_a^b$$

(a) $$P_{0.50,\,0.75} = \left[-\frac{(1-x)^{3/2}}{2}(3x + 2)\right]_{0.50}^{0.75} = 0.353 = 35.3\%$$

(b) $$P_{0,\,b} = \left[-\frac{(1-x)^{3/2}}{2}(3x + 2)\right]_0^b = -\frac{(1-b)^{3/2}}{2}(3b + 2) + 1 = 0.5$$
$$(1-b)^{3/2}(3b + 2) = 1$$
$$b \approx 0.586 = 58.6\%$$

92. $u = 1 - x,\ x = 1 - u,\ dx = -du$

When $x = a,\ u = 1 - a$. When $x = b,\ u = 1 - b$.

$$P_{a,b} = \int_a^b \frac{1155}{32}x^3(1-x)^{3/2}\,dx = \frac{1155}{32}\int_{1-a}^{1-b} -(1-u)^3u^{3/2}\,du$$

$$= \frac{1155}{32}\int_{1-a}^{1-b}\left(u^{9/2} - 3u^{7/2} + 3u^{5/2} - u^{3/2}\right)du = \frac{1155}{32}\left[\frac{2}{11}u^{11/2} - \frac{2}{3}u^{9/2} + \frac{6}{7}u^{7/2} - \frac{2}{5}u^{5/2}\right]_{1-a}^{1-b}$$

$$= \frac{1155}{32}\left[\frac{2u^{5/2}}{1155}\left(105u^3 - 385u^2 + 495u - 231\right)\right]_{1-a}^{1-b} = \left[\frac{u^{5/2}}{16}\left(105u^3 - 385u^2 + 495u - 231\right)\right]_{1-a}^{1-b}$$

(a) $P_{0,\,0.25} = \left[\frac{u^{5/2}}{16}\left(105u^3 - 385u^2 + 495u - 231\right)\right]_1^{0.75} \approx 0.025 = 2.5\%$

(b) $P_{0.5,\,1} = \left[\frac{u^{5/2}}{16}\left(105u^3 - 385u^2 + 495u - 231\right)\right]_{0.5}^{0} \approx 0.736 = 73.6\%$

93. True. Let $u = x^3 + 5$.

94. False

$$\int x\left(x^2+1\right)dx = \tfrac{1}{2}\int\left(x^2+1\right)(2x)\,dx = \tfrac{1}{4}\left(x^2+1\right)^2 + C$$

95. True

$$\int_{-10}^{10}\left(ax^3 + bx^2 + cx + d\right)dx = \underbrace{\int_{-10}^{10}\left(ax^3 + cx\right)dx}_{\text{Odd}} + \underbrace{\int_{-10}^{10}\left(bx^2 + d\right)dx}_{\text{Even}} = 0 + 2\int_0^{10}\left(bx^2 + d\right)dx$$

96. True

$$\int_a^b \sin x\,dx = \left[-\cos x\right]_a^b = -\cos b + \cos a = -\cos(b + 2\pi) + \cos a = \int_a^{b+2\pi}\sin x\,dx$$

97. True

$$4\int \sin x\cos x\,dx = 2\int \sin 2x\,dx = -\cos 2x + C$$

98. False

$$\int \sin^2 2x\cos 2x\,dx = \frac{1}{2}\int(\sin 2x)^2(2\cos 2x)\,dx$$
$$= \frac{1}{2}\frac{(\sin 2x)^3}{3} + C$$
$$= \frac{1}{6}\sin^3 2x + C$$

99. Let $u = cx,\ du = c\,dx$:

$$c\int_a^b f(cx)\,dx = c\int_{ca}^{cb} f(u)\frac{du}{c}$$
$$= \int_{ca}^{cb} f(u)\,du$$
$$= \int_{ca}^{cb} f(x)\,dx$$

100. (a) $\frac{d}{du}\left[\sin u - u\cos u + C\right] = \cos u - \cos u + u\sin u$
$= u\sin u$

So, $\int u\sin u\,du = \sin u - u\cos u + C$.

(b) Let $u = \sqrt{x},\ u^2 = x,\ 2u\,du = dx$.

$$\int_0^{\pi^2}\sin\sqrt{x}\,dx = \int_0^{\pi}\sin u(2u\,du)$$
$$= 2\int_0^{\pi} u\sin u\,du$$
$$= 2\left[\sin u - u\cos u\right]_0^{\pi} \quad \text{(part (a))}$$
$$= 2\left[-\pi\cos(\pi)\right]$$
$$= 2\pi$$

101. Because f is odd, $f(-x) = -f(x)$. Then

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$
$$= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx.$$

Let $x = -u$, $dx = -du$ in the first integral.

When $x = 0$, $u = 0$. When $x = -a$, $u = a$.

$$\int_{-a}^{1} f(x)\,dx = -\int_{0}^{a} f(-u)(-du) + \int_{0}^{a} f(x)\,dx$$
$$= -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 0$$

102. Let $u = x + h$, then $du = dx$.

When $x = a$, $u = a + h$.

When $x = b$, $u = b + h$. So,

$$\int_{a}^{b} f(x + h)\,dx = \int_{a+h}^{b+h} f(u)\,du = \int_{a+h}^{b+h} f(x)\,dx.$$

103. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$.

$$\int_{0}^{1} f(x)\,dx = \left[a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + \cdots + a_n\frac{x^{n+1}}{n+1}\right]_0^1$$
$$= a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0 \text{ (Given)}$$

By the Mean Value Theorem for Integrals, there exists c in [0, 1] such that

$$\int_{0}^{1} f(x)\,dx = f(c)(1 - 0)$$
$$0 = f(c).$$

So the equation has at least one real zero.

104. $\alpha^2\int_{0}^{1} f(x)\,dx = \alpha^2(1) = \alpha^2$

$$-2\alpha\int_{0}^{1} f(x)x\,dx = -2\alpha(\alpha) = -2\alpha^2$$

$$\int_{0}^{1} f(x)x^2\,dx = \alpha^2$$

Adding,

$$\int_{0}^{1}\left[\alpha^2 f(x) - 2\alpha\,xf(x) + x^2 f(x)\right]dx = 0$$

$$\int_{0}^{1} f(x)(\alpha - x)^2\,dx = 0.$$

Because $(\alpha - x)^2 \ge 0$, $f = 0$. So, there are no such functions.

Review Exercises for Chapter 4

1. $\int\left(x^3 + 4\right)dx = \dfrac{x^4}{4} + 4x + C$

2. $\int\left(x^4 + 3\right)dx = \dfrac{x^5}{5} + 3x + C$

3. $\int\left(4x^2 + x + 3\right)dx = \frac{4}{3}x^3 + \frac{1}{2}x^2 + 3x + C$

4. $\int \dfrac{6}{\sqrt[3]{x}}\,dx = \int 6x^{-1/3}\,dx = 6\cdot\dfrac{x^{2/3}}{(2/3)} + C = 9x^{2/3} + C$

5. $\int \dfrac{x^4 + 8}{x^3}\,dx = \int\left(x + 8x^{-3}\right)dx = \dfrac{1}{2}x^2 - \dfrac{4}{x^2} + C$

6. $\int \dfrac{x^2 + 2x - 6}{x^4}\,dx = \int\left(x^{-2} + 2x^{-3} - 6x^{-4}\right)dx$

$$= \frac{x^{-1}}{(-1)} + 2\cdot\frac{x^{-2}}{(-2)} - 6\cdot\frac{x^{-3}}{-3} + C$$
$$= -\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} + C$$

7. $\int\left(2\csc^2 x - 9\sin x\right)dx = -2\cot x + 9\cos x + C$

8. $\int\left(5\cos x - 2\sec^2 x\right)dx = 5\sin x - 2\tan x + C$

9. $f'(x) = -6x$, $f(1) = -2$

$f(x) = -3x^2 + C$

$f(1) = -2 = -3(1)^2 + C \Rightarrow C = 1$

$f(x) = -3x^2 + 1$

10. $f'(x) = 9x^2 + 1$, $f(0) = 7$

$f(x) = 3x^3 + x + C$

$f(0) = 7 = 3(0)^2 + 0 + C \Rightarrow C = 7$

$f(x) = 3x^3 + x + 7$

11. $f''(x) = 24x,\ f'(-1) = 7,\ f(1) = -4$

$$f'(x) = 12x^2 + C_1$$
$$f'(-1) = 7 = 12(-1)^2 + C_1 \Rightarrow C_1 = -5$$
$$f'(x) = 12x^2 - 5$$
$$f(x) = 4x^3 - 5x + C_2$$
$$f(1) = -4 = 4(1)^3 - 5(1) + C_2 \Rightarrow C_2 = -3$$
$$f(x) = 4x^3 - 5x - 3$$

12. $f''(x) = 2\cos x,\ f'(0) = 4,\ f(0) = -5$

$$f'(x) = 2\sin x + C_1$$
$$f'(0) = 4 = 2\sin 0 + C_1 \Rightarrow C_1 = 4$$
$$f'(x) = 2\sin x + 4$$
$$f(x) = -2\cos x + 4x + C_2$$
$$f(0) = -5 = -2\cos 0 + 4(0) + C_2$$
$$= -2 + C_2$$
$$\Rightarrow C_2 = -3$$
$$f(x) = -2\cos x + 4x - 3$$

13. $a(t) = -32$

$$v(t) = -32t + 96$$
$$s(t) = -16t^2 + 96t$$

(a) $v(t) = -32t + 96 = 0$ when $t = 3$ sec.

$s(3) = -144 + 288 = 144$ ft

(b) $v(t) = -32t + 96 = \frac{96}{2}$ when $t = \frac{3}{2}$ sec.

(c) $s\left(\frac{3}{2}\right) = -16\left(\frac{9}{4}\right) + 96\left(\frac{3}{2}\right) = 108$ ft

14. $s(t) = -\dfrac{1}{2}gt^2 + v_0t + s_0, \qquad g = 9.8$

$$150 = -\frac{1}{2}(9.8)t^2 + v_0t + 3 \Rightarrow 147 = -4.9t^2 + v_0t$$
$$v(t) = s'(t) = -gt + v_0$$
$$0 = -9.8t + v_0 \Rightarrow t = \frac{v_0}{9.8}$$
$$147 = -4.9\left(\frac{v_0}{9.8}\right)^2 + v_0\left(\frac{v_0}{9.8}\right)$$
$$147 = -\frac{5}{98}v_0^{\,2} + \frac{5}{49}v_o^{\,2}$$
$$147 = \frac{5}{98}v_0^{\,2}$$
$$v_0 \approx 53.68$$

The initial velocity is about 53.68 meters per second.

15. $\displaystyle\sum_{i=1}^{5}(5i - 3) = 2 + 7 + 12 + 17 + 22 = 60$

16. $\displaystyle\sum_{k=0}^{3}\left(k^2 + 1\right) = 1 + 2 + 5 + 10 = 18$

17. $\dfrac{1}{5(3)} + \dfrac{2}{5(4)} + \dfrac{3}{5(5)} + \ldots + \dfrac{10}{5(12)} = \displaystyle\sum_{i=1}^{10}\frac{i}{5(i + 2)}$

18. $\displaystyle\sum_{i=1}^{n}\left(\frac{3}{n}\right)\left(\frac{i + 1}{n}\right)^2 = \frac{3}{n}\left(\frac{1 + 1}{n}\right)^2 + \frac{3}{n}\left(\frac{2 + 1}{n}\right)^2 + \cdots + \frac{3}{n}\left(\frac{n + 1}{n}\right)^2$

19. $\sum_{i=1}^{24} 8 = 8(24) = 192$

20. $\sum_{i=1}^{75} 5i = 5\sum_{i=1}^{75} i = 5 \cdot \frac{75(76)}{2} = 14{,}250$

21. $\sum_{i=1}^{20} 2i = 2\left(\frac{20(21)}{2}\right) = 420$

22. $\sum_{i=1}^{30} (3i - 4) = 3\sum_{i=1}^{30} i - 4\sum_{i=1}^{30} 1$

$= 3 \cdot \frac{(30)(31)}{2} - 4(30)$

$= 1395 - 120 - 1275$

23. $\sum_{i=1}^{20} (i+1)^2 = \sum_{i=1}^{20} (i^2 + 2i + 1)$

$= \frac{20(21)(41)}{6} = 2\frac{20(21)}{2} + 20$

$= 2870 + 420 + 20 = 3310$

24. $\sum_{i=1}^{12} i(i^2 - 1) = \sum_{i=1}^{12} (i^3 - i)$

$= \frac{(12^2)(13^2)}{4} - \frac{12(13)}{2}$

$= 6084 - 78 = 6006$

25. $y = \frac{10}{x^2+1}$, $\Delta x = \frac{1}{2}$, $n = 4$

$$S(n) = S(4) = \frac{1}{2}\left[\frac{10}{1} + \frac{10}{(1/2)^2 + 1} + \frac{10}{(1)^2 + 1} + \frac{10}{(3/2)^2 + 1}\right] \approx 13.0385$$

$$s(n) = s(4) = \frac{1}{2}\left[\frac{10}{(1/2)^2 + 1} + \frac{10}{1+1} + \frac{10}{(3/2)^2 + 1} + \frac{10}{2^2 + 1}\right] \approx 9.0385$$

$9.0385 <$ Area of Region < 13.0385

26. $y = 9 - \frac{1}{4}x^2$, $\Delta x = 1$, $n = 4$

$$S(4) = 1\left[\left(9 - \tfrac{1}{4}(4)\right) + \left(9 - \tfrac{1}{4}(9)\right) + \left(9 - \tfrac{1}{4}(16)\right) + 9 - \tfrac{1}{4}(25)\right] \approx 22.5$$

$$s(4) = 1\left[\left(9 - \tfrac{1}{4}(9)\right) + \left(9 - \tfrac{1}{4}(16)\right) + \left(9 - \tfrac{1}{4}(25)\right) + (9 - 9)\right] \approx 14.5$$

$14.5 <$ Area of Region < 22.5

27. $f(x) = 4x + 1$, $[2, 3]$

$\Delta x = \frac{3-2}{n} = \frac{1}{n}$

Right endpoints for upper sum: $M_i = 2 + i\Delta x = 2 + i\left(\frac{1}{n}\right)$

Left endpoints for lower sum: $m_i = 2 + (i-1)\Delta x = 2 + (i-1)\left(\frac{1}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f(Mi)\Delta x = \sum_{i=1}^{n}\left[4\left(2 + i\left(\frac{1}{n}\right)\right) + 1\right]\left(\frac{1}{n}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} 9 + \frac{4}{n^2}\sum_{i=1}^{n} i = \frac{1}{n}(9n) + \frac{4}{n^2}\left[\frac{n(n+1)}{2}\right] = 11 + \frac{2}{n}$$

$$s(n) = \sum_{i=1}^{n} f(m_i)\Delta x = \sum_{i=1}^{n}\left[4\left(2 + (i-1)\left(\frac{1}{n}\right)\right) + 1\right]\left(\frac{1}{n}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} 9 + \frac{4}{n^2}\sum_{i=1}^{4} (i-1) = 9 + \frac{4}{n^2}\left[\frac{n(n-1)}{2}\right] = 11 - \frac{2}{n}$$

28. $f(x) = 7x^2, \; [0, 3]$

$\Delta x = \dfrac{3-0}{n} = \dfrac{3}{n}$

Right endpoints for upper sum: $M_i = i\Delta x = i\left(\dfrac{3}{n}\right)$

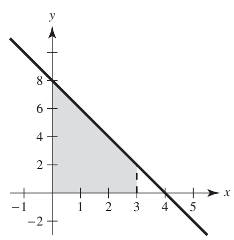
Left endpoints for lower sum: $m_i = (i-1)\Delta x = (i-1)\left(\dfrac{3}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f(M_i)\Delta x = \sum_{i=1}^{n} 7\left(\frac{3i}{n}\right)^2\left(\frac{3}{n}\right)$$
$$= \frac{189}{n^3}\sum_{i=1}^{n} i^2 = \frac{189}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$
$$= 63 + \frac{189}{2n} + \frac{63}{2n^2}$$

$$s(n) = \sum_{i=1}^{n} f(m_i)\Delta x = \sum_{i=1}^{n} 7\left[\frac{3(i-1)}{n}\right]^2\left(\frac{3}{n}\right)$$
$$= \frac{189}{n^3}\sum_{i=1}^{n}(i-1)^2 = \frac{189}{n^3}\cdot\frac{(n-1)n(2n-1)}{6}$$
$$= 63 - \frac{189}{2n} + \frac{63}{2n^2}$$

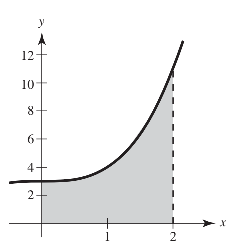
29. $y = 8 - 2x, \; \Delta x = \dfrac{3}{n}$, right endpoints

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(ci)\Delta x$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(8 - 2\left(\frac{3i}{n}\right)\right)\frac{3}{n}$$
$$= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left(8 - \frac{6i}{n}\right)$$
$$= \lim_{n\to\infty}\frac{3}{n}\left[8n - \frac{6}{n}\frac{n(n+1)}{2}\right]$$
$$= \lim_{n\to\infty}\left[24 - 9\frac{n+1}{n}\right] = 24 - 9 = 15$$



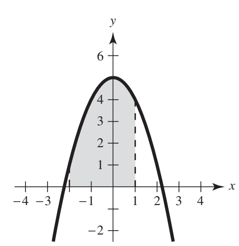
30. $y = x^2 + 3, \; \Delta x = \dfrac{2}{n}$, right endpoints

$$\text{Area} \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^2 + 3\right]\left(\frac{2}{n}\right)$$
$$= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[\frac{4i^2}{n^2} + 3\right]$$
$$= \lim_{n\to\infty}\frac{2}{n}\left[\frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} + 3n\right]$$
$$= \lim_{n\to\infty}\left[\frac{4}{3}\frac{(n+1)(2n+1)}{n^2} + 6\right] = \frac{8}{3} + 6 = \frac{26}{3}$$

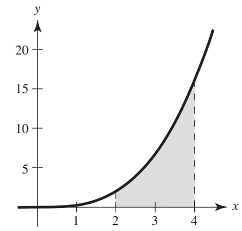


31. $y = 5 - x^2, \; \Delta x = \dfrac{3}{n}$

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[5 - \left(-2 + \frac{3i}{n}\right)^2\right]\left(\frac{3}{n}\right)$$
$$= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[1 + \frac{12i}{n} - \frac{9i^2}{n^2}\right]$$
$$= \lim_{n\to\infty}\frac{3}{n}\left[n + \frac{12}{n}\frac{n(n+1)}{2} - \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$
$$= \lim_{n\to\infty}\left[3 + 18\frac{n+1}{n} - \frac{9}{2}\frac{(n+1)(2n+1)}{n^2}\right]$$
$$= 3 + 18 - 9 = 12$$



32. $y = \frac{1}{4}x^3,\ \Delta x = \frac{2}{n}$



$$\begin{aligned}\text{Area} &= \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{4}\left(2+\frac{2i}{n}\right)^3\left(\frac{2}{n}\right)\\ &= \lim_{n\to\infty}\frac{1}{2n}\sum_{i=1}^{n}\left[8+\frac{24i}{n}+\frac{24i^2}{n^2}+\frac{8i^3}{n^3}\right]\\ &= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left[1+\frac{3i}{n}+\frac{3i^2}{n^2}+\frac{i^3}{n^3}\right]\\ &= \lim_{n\to\infty}\frac{4}{n}\left[n+\frac{3}{n}\frac{n(n+1)}{2}+\frac{3}{n^2}\frac{n(n+1)(2n+1)}{6}+\frac{1}{n^3}\frac{n^2(n+1)^2}{4}\right] = 4+6+4+1 = 15\end{aligned}$$

33. $f(x) = 16 - x^2,\ [0, 4],\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$\Delta x = 1,\ c_1 = \frac{1}{2},\ c_2 = \frac{3}{2},\ c_3 = \frac{5}{2},\ c_4 = \frac{7}{2}$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{4}\left[16 - c_i^2\right](1) = \left[\left(16-\frac{1}{4}\right)+\left(16-\frac{9}{4}\right)+\left(16-\frac{25}{4}\right)+\left(16-\frac{49}{4}\right)\right] = 43$$

34. $f(x) = \sin \pi x,\ [0, 1],\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$\Delta x = \frac{1}{4},\ c_1 = \frac{1}{8},\ c_2 = \frac{3}{8},\ c_3 = \frac{5}{8},\ c_4 = \frac{7}{8}$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{4}(\sin \pi c_i)\left(\frac{1}{4}\right) = \frac{1}{4}\left(\sin\frac{\pi}{8}+\sin\frac{3\pi}{8}+\sin\frac{5\pi}{8}+\sin\frac{7\pi}{8}\right) \approx 0.6533$$

35. $y = 6x$ on $[-3, 5]$. $\left(\textbf{Note: } \Delta x = \dfrac{5-(-3)}{n} = \dfrac{8}{n}\right)$

$$\begin{aligned}S(n) &= \sum_{i=1}^{n} f\left(-3+\frac{8i}{n}\right)\left(\frac{8}{n}\right)\\ &= \frac{8}{n}\sum_{i=1}^{n} 6\left(-3+\frac{8i}{n}\right)\\ &= \frac{8}{n}\sum_{i=1}^{n} -18 + \left(\frac{8}{n}\right)^2\sum_{i=1}^{n} 6i\\ &= \frac{8}{n}(-18n) + \frac{384}{n^2}\cdot\left(\frac{n(n+1)}{2}\right)\end{aligned}$$

$$\lim_{n\to\infty} S(n) = -144 + 192 = 48$$

36. $f(x) = 1 - 2x^2, [0, 3]$ $\left(\textbf{Note: } \Delta x = \dfrac{3-0}{n} = \dfrac{3}{n}\right)$

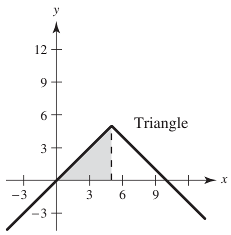
$$S(n) = \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \frac{3}{n}\sum_{i=1}^{n}\left[1 - 2\left(\frac{3i}{n}\right)^2\right]$$

$$= \frac{3}{n}\sum_{i=1}^{n} 1 - \frac{54}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{3}{n}(n) - \frac{54}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$\lim_{n\to\infty} S(n) = 3 - 54\left(\frac{1}{3}\right) = -15$$

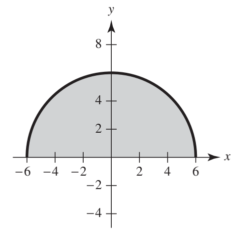
37.



$$\int_0^5 (5 - |x - 5|)\,dx = \tfrac{1}{2}(5)(5) = \tfrac{25}{2}$$

(triangle)

38.



$$\int_{-6}^{6} \sqrt{36 - x^2}\,dx = \tfrac{1}{2}\pi(6)^2 = 18\pi$$

(semicircle)

39. $\int_4^8 f(x)\,dx = 12, \int_4^8 g(x)\,dx = 5$

(a) $\int_4^8 [f(x) + g(x)]\,dx = \int_4^8 f(x)\,dx + \int_4^8 g(x)\,dx = 12 + 5 = 17$

(b) $\int_4^8 [f(x) - g(x)]\,dx = \int_4^8 f(x)\,dx - \int_4^8 g(x)\,dx = 12 - 5 = 7$

(c) $\int_4^8 [2f(x) - 3g(x)]\,dx = 2\int_4^8 f(x)\,dx - 3\int_4^8 g(x)\,dx = 2(12) - 3(5) = 9$

(d) $\int_4^8 7\,f(x)\,dx = 7\int_4^8 f(x)\,dx = 7(12) = 84$

40. $\int_0^2 f(x)\,dx = 2, \int_2^5 f(x)\,dx = -5$

(a) $\int_0^5 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^5 f(x)\,dx = 2 - 5 = -3$

(b) $\int_5^2 f(x)\,dx = -\int_2^5 f(x)\,dx = -(-5) = 5$

(c) $\int_3^3 f(x)\,dx = 0$

(d) $\int_2^5 -8f(x)\,dx = -8\int_2^5 f(x)\,dx = (-8)(-5) = 40$

41. $\int_0^6 (x-1)\,dx = \left[\frac{x^2}{2} - x\right]_0^6 = \left(\frac{36}{2} - 6\right) - 0 = 12$

42. $\int_{-2}^{1} \left(4x^4 - x\right) dx = \left[\frac{4x^5}{5} - \frac{x^2}{2}\right]_{-2}^{1}$

$= \left(\frac{4}{5} - \frac{1}{2}\right) - \left(\frac{-128}{5} - \frac{4}{2}\right)$

$= \frac{279}{10}$

43. $\int_4^9 x\sqrt{x}\,dx = \int_4^9 x^{3/2}\,dx = \left[\frac{2}{5}x^{5/2}\right]_4^9 = \frac{2}{5}\left[\left(\sqrt{9}\right)^5 - \left(\sqrt{4}\right)^5\right] = \frac{2}{5}(243 - 32) = \frac{422}{5}$

44. $\int_1^4 \left(\frac{1}{x^3} + x\right)dx = \int_1^4 \left(x^{-3} + x\right)dx = \left[\frac{x^{-2}}{-2} + \frac{x^2}{2}\right]_1^4 = \frac{1}{2}\left[x^2 - \frac{1}{x^2}\right]_1^4 = \frac{1}{2}\left[\left(16 - \frac{1}{16}\right) - (1 - 1)\right] = \frac{255}{32}$

45. $\int_0^{3\pi/4} \sin\theta\,d\theta = \left[-\cos\theta\right]_0^{3\pi/4} = -\left(-\frac{\sqrt{2}}{2}\right) + 1 = 1 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2} + 2}{2}$

46. $\int_{-\pi/4}^{\pi/4} \sec^2 t\,dt = \left[\tan t\right]_{-\pi/4}^{\pi/4} = 1 - (-1) = 2$

47. Area $= \int_0^{\pi/2} \sin x\,dx = \left[-\cos x\right]_0^{\pi/2} = 0 - (-1) = 1$

48. Area $= \int_0^{\pi/2} (x + \cos x)\,dx = \left[\frac{x^2}{2} + \sin x\right]_0^{\pi/2} = \frac{\pi^2}{8} + 1$

49. $A = \int_0^6 (8 - x)\,dx$

$= \left[8x - \frac{x^2}{2}\right]_0^6$

$= (48 - 18) - 0$

$= 30$

50. $y = -\left(x^2 - x - 6\right) = -(x - 3)(x + 2)$

$A = \int_{-2}^{3}\left(-x^2 + x + 6\right)dx$

$= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 6x\right]_{-2}^{3}$

$= \left(-9 + \frac{9}{2} + 18\right) - \left(\frac{8}{3} + 2 - 12\right)$

$= \frac{125}{6}$

51. $A = \int_0^1 \left(x - x^3\right) dx$

$= \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$

$= \left(\frac{1}{2} - \frac{1}{4}\right) - 0$

$= \frac{1}{4}$

52. $A = \int_0^1 \sqrt{x}\,(1 - x)\,dx = \int_0^1 \left(x^{1/2} - x^{3/2}\right)dx$

$= \left[\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^1$

$= \left(\frac{2}{3} - \frac{2}{5}\right) - (0)$

$= \frac{4}{15}$

53. $\frac{1}{3 - 1}\int_1^3 3x^2\,dx = \frac{1}{2}\left[x^3\right]_1^3 = \frac{1}{2}(27 - 1) = 13$

$3x^2 = 13$

$x^2 = \frac{13}{3}$

$x = \sqrt{\frac{13}{3}}$

54. $\dfrac{1}{\pi - 0}\int_0^{\pi} \sin x\,dx = \dfrac{1}{\pi}\Big[-\cos x\Big]_0^{\pi} = \dfrac{1}{\pi}(1 + 1) = \dfrac{2}{\pi}$

$\dfrac{2}{\pi} = \sin x$

Using a calculator, $x \approx 0.6901$.

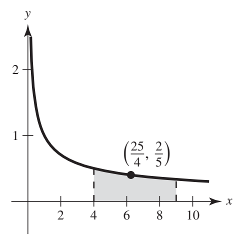
55. Average value: $\dfrac{1}{9 - 4}\int_4^9 \dfrac{1}{\sqrt{x}}\,dx = \left[\dfrac{1}{5}2\sqrt{x}\right]_4^9$

$= \dfrac{2}{5}(3 - 2) = \dfrac{2}{5}$

$\dfrac{2}{5} = \dfrac{1}{\sqrt{x}}$

$\sqrt{x} = \dfrac{5}{2}$

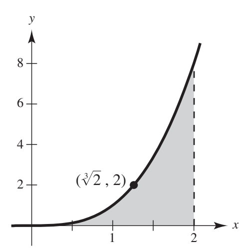
$x = \dfrac{25}{4}$



56. Average value: $\dfrac{1}{2 - 0}\int_0^2 x^3\,dx = \left[\dfrac{x^4}{8}\right]_0^2 = 2$

$x^3 = 2$

$x = \sqrt[3]{2}$



57. $F'(x) = x^2\sqrt{1 + x^3}$

58. $F'(x) = \dfrac{1}{x^2}$

59. $u = 1 - 3x^2,\ du = -6x\,dx$

$\int x(1 - 3x^2)^4\,dx = -\frac{1}{6}\int (1 - 3x^2)^4(-6x\,dx)$

$= -\frac{1}{30}(1 - 3x^2)^5 + C$

$= \frac{1}{30}(3x^2 - 1)^5 + C$

60. $u = 3x^4 + 2,\ du = 12x^3\,dx$

$\int 6x^3\sqrt{3x^4 + 2}\,dx = \dfrac{1}{2}\int (3x^4 + 2)^{1/2}(12x^3)\,dx$

$= \dfrac{1}{2}\cdot\dfrac{(3x^4 + 2)^{3/2}}{(3/2)} + C$

$= \dfrac{1}{3}(3x^4 + 2)^{3/2} + C$

61. $u = \sin x,\ du = \cos x\,dx$

$\int \sin^3 x\cos x\,dx = \frac{1}{4}\sin^4 x + C$

62. $u = 3x^2,\ du = 6x\,dx$

$\int x\sin 3x^2\,dx = \frac{1}{6}\int (\sin 3x^2)(6x)\,dx = -\frac{1}{6}\cos 3x^2 + C$

63. $u = 1 - \sin\theta,\ du = -\cos\theta\,d\theta$

$\int \dfrac{\cos\theta}{\sqrt{1 - \sin\theta}}\,d\theta = -\int (1 - \sin\theta)^{-1/2}(-\cos\theta)\,d\theta$

$= -2(1 - \sin\theta)^{1/2} + C$

$= -2\sqrt{1 - \sin\theta} + C$

64. $u = \cos x,\ du = -\sin x\,dx$

$\int \dfrac{\sin x}{\sqrt{\cos x}}\,dx = -\int (\cos x)^{-1/2}(-\sin x)\,dx$

$= -2(\cos x)^{1/2} + C$

$= -2\sqrt{\cos x} + C$

65. $u = 8 - x,\ du = -dx$

$\int x\sqrt{8 - x}\,dx = \int (8 - u)u^{1/2}(-du)$

$= \int \left(u^{3/2} - 8u^{1/2}\right)du$

$= \dfrac{2}{5}u^{5/2} - \dfrac{16}{3}u^{3/2} + C$

$= \dfrac{2}{5}(8 - x)^{5/2} - \dfrac{16}{3}(8 - x)^{3/2} + C$

66. $u = 1 + \sqrt{x},\ du = \dfrac{1}{2\sqrt{x}}\,dx$

Hence, $2\sqrt{x}\,du = 2(u - 1)\,du = dx$.

$\int \sqrt{1 + \sqrt{x}}\,dx = \int u^{1/2}\left[2(u - 1)\right]du$

$= 2\int \left(u^{3/2} - u^{1/2}\right)du$

$= 2\left[\dfrac{u^{5/2}}{(5/2)} - \dfrac{u^{3/2}}{(3/2)}\right] + C$

$= \dfrac{4}{5}\left(1 + \sqrt{x}\right)^{5/2} - \dfrac{4}{3}\left(1 + \sqrt{x}\right)^{3/2} + C$

67. $\int_0^1 (3x+1)^5\,dx = \frac{1}{3}\int_0^1 (3x+1)^5(3dx) = \frac{1}{3}\left[\frac{(3x+1)^6}{6}\right]_0^1 = \frac{1}{3}\left[\frac{4^6}{6} - \frac{1}{6}\right] = \frac{455}{2}$

68. $\int_0^1 x^2(x^3-2)^3\,dx$

$u = x^3 - 2,\ du = 3x^2\,dx,\ x^2\,dx = \frac{1}{3}\,du$

When $x = 0, u = -2$. When $x = 1, u = -1$

$\int_{-2}^{-1} u^3 \frac{1}{3}\,du = \left.\frac{u^4}{12}\right]_{-2}^{-1} = \frac{1}{12} - \frac{16}{12} = -\frac{15}{12} = -\frac{5}{4}$

69. $\int_0^3 \frac{1}{\sqrt{1+x}}\,dx = \int_0^3 (1+x)^{-1/2}\,dx = \left[2(1+x)^{1/2}\right]_0^3 = 4 - 2 = 2$

70. $\int_3^6 \frac{x}{3\sqrt{x^2-8}}\,dx = \frac{1}{6}\int_3^6 (x^2-8)^{-1/2}(2x)\,dx = \left[\frac{1}{3}(x^2-8)^{1/2}\right]_3^6 = \frac{1}{3}\left(2\sqrt{7}-1\right)$

71. $u = 1 - y,\ y = 1 - u,\ dy = -du$

When $y = 0, u = 1$. When $y = 1, u = 0$.

$2\pi\int_0^1 (y+1)\sqrt{1-y}\,dy = 2\pi\int_1^0 -\left[(1-u)+1\right]\sqrt{u}\,du = 2\pi\int_1^0 \left(u^{3/2} - 2u^{1/2}\right)du = 2\pi\left[\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2}\right]_1^0 = \frac{28\pi}{15}$

72. $u = x + 1, x = u - 1, dx = du$

When $x = -1, u = 0$. When $x = 0, u = 1$.

$2\pi\int_{-1}^0 x^2\sqrt{x+1}\,dx = 2\pi\int_0^1 (u-1)^2\sqrt{u}\,du = 2\pi\int_0^1 \left(u^{5/2} - 2u^{3/2} + u^{1/2}\right)du = 2\pi\left[\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right]_0^1 = \frac{32\pi}{105}$

73. $\int_1^9 x(x-1)^{1/3}\,dx$. Let $u = x - 1,\ du = dx$.

$$\begin{aligned} A &= \int_0^8 (u+1)u^{1/3}\,du \\ &= \int_0^8 \left(u^{4/3} + u^{1/3}\right)du \\ &= \left[\frac{3u^{7/3}}{7} + \frac{3u^{4/3}}{4}\right]_0^8 \\ &= \frac{3}{7}(128) + \frac{3}{4}(16) = \frac{468}{7} \end{aligned}$$

74. $\int_0^{\pi/2} (\cos x + \sin 2x)\,dx = \left[\sin x - \frac{1}{2}\cos 2x\right]_0^{\pi/2}$

$= \left(1 + \frac{1}{2}\right) - \left(0 - \frac{1}{2}\right) = 2$

75. $f(x) = x^3 - 2x$ is an odd function.

$\int_{-2}^2 \left(x^3 - 2x\right)dx = 0$

76. $f(x) = \cos x + x^2$ is an even function.

$$\begin{aligned} \int_{-\pi}^{\pi} \left(\cos x + x^2\right)dx &= 2\int_0^{\pi} \left(\cos x + x^2\right)dx \\ &= 2\left[\sin x + \frac{x^3}{3}\right]_0^{\pi} \\ &= 2\left[\frac{\pi^3}{3}\right] = \frac{2}{3}\pi^3 \end{aligned}$$

Problem Solving for Chapter 4

1. (a) $L(1) = \int_1^1 \frac{1}{t}\,dt = 0$

(b) $L'(x) = \frac{1}{x}$ by the Second Fundamental Theorem of Calculus.

$L'(1) = 1$

(c) $L(x) = 1 = \int_1^x \frac{1}{t}\,dt$ for $x \approx 2.718$

$\int_1^{2.718} \frac{1}{t}\,dt = 0.999896$

(**Note:** The exact value of x is e, the base of the natural logarithm function.)

(d) First show that $\int_1^{x_1} \frac{1}{t}\,dt = \int_{1/x_1}^{1} \frac{1}{t}\,dt$.

To see this, let $u = \frac{t}{x_1}$ and $du = \frac{1}{x_1}\,dt$.

Then $\int_1^{x_1} \frac{1}{t}\,dt = \int_{1/x_1}^{1} \frac{1}{ux_1}(x_1\,du) = \int_{1/x_1}^{1} \frac{1}{u}\,du = \int_{1/x_1}^{1} \frac{1}{t}\,dt.$

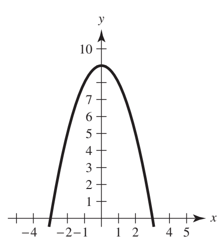
Now, $L(x_1 x_2) = \int_1^{x_1 x_2} \frac{1}{t}\,dt = \int_{1/x_1}^{x_2} \frac{1}{u}\,du \quad \left(\text{using } u = \frac{t}{x_1}\right)$

$$= \int_{1/x_1}^{1} \frac{1}{u}\,du + \int_1^{x_2} \frac{1}{u}\,du$$

$$= \int_1^{x_1} \frac{1}{u}\,du + \int_1^{x_2} \frac{1}{u}\,du$$

$$= L(x_1) + L(x_2).$$

2. (a)



Area $= \int_{-3}^{3} (9 - x^2)\,dx = 2\int_0^3 (9 - x^2)\,dx = 2\left[9x - \frac{x^3}{3}\right]_0^3 = 2[27 - 9] = 36$

(b) Base $= 6$, height $= 9$, Area $= \frac{2}{3}bh = \frac{2}{3}(6)(9) = 36$

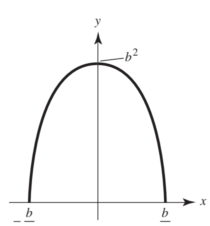
(c) Let the parabola be given by $y = b^2 - a^2x^2, a, b > 0$.

$$\text{Area} = 2\int_0^{b/a} \left(b^2 - a^2x^2\right) dx$$

$$= 2\left[b^2x - a^2\frac{x^3}{3}\right]_0^{b/a}$$

$$= 2\left[b^2\left(\frac{b}{a}\right) - \frac{a^2}{3}\left(\frac{b}{a}\right)^3\right]$$

$$= 2\left[\frac{b^3}{a} - \frac{1}{3}\frac{b^3}{a}\right] = \frac{4}{3}\frac{b^3}{a}$$



Base $= \frac{2b}{a}$, height $= b^2$

Archimedes' Formula: Area $= \frac{2}{3}\left(\frac{2b}{a}\right)\left(b^2\right) = \frac{4}{3}\frac{b^3}{a}$

3. $y = x^4 - 4x^3 + 4x^2$, $[0, 2]$, $c_i = \dfrac{2i}{n}$

(a) $\Delta x = \dfrac{2}{n}$, $f(x) = x^4 - 4x^3 + 4x^2$

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^4 - 4\left(\frac{2i}{n}\right)^3 + 4\left(\frac{2i}{n}\right)^2\right]\frac{2}{n} = \lim_{n\to\infty}\left[\frac{32}{n^s}\sum_{i=1}^{n} i^4 - \frac{64}{n^4}\sum_{i=1}^{n} i^3 + \frac{32}{n^3}\sum_{i=1}^{n} i^2\right]$$

(b) $$\left[\frac{32}{n^5}\sum_{i=1}^{n} i^4 - \frac{64}{n^4}\sum_{i=1}^{n} i^3 + \frac{32}{n^3}\sum_{i=1}^{n} i^2\right] = \left[\frac{32\cdot n(n+1)(2n+1)(3n^2+3n-1)}{n^5\cdot 30} - \frac{64\cdot n^2(n+1)^2}{n^4\cdot 4} + \frac{32\cdot n(n+1)(2n+1)}{n^3\cdot 6}\right]$$

$$= \left[\frac{16(n+1)(6n^3+9n^2+n-1)}{15n^4} - \frac{16(n+1)^2}{n^2} + \frac{16(n+1)(2n+1)}{3n^2}\right]$$

$$= \left[\frac{16(n+1)(n^3-n^2+n-1)}{15n^4}\right] = \frac{16n^4-16}{15n^4}$$

(c) $$A = \lim_{n\to\infty}\left[\frac{16n^4-16}{15n^4}\right] = \frac{16}{15}$$

4. $y = \dfrac{1}{2}x^5 + 2x^3$, $[0, 2]$, $c_i = \dfrac{2i}{n}$, $\Delta x = \dfrac{2}{n}$

(a) $$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{1}{2}\left(\frac{2i}{n}\right)^5 + 2\left(\frac{2i}{n}\right)^3\right]\frac{2}{n} = \lim_{n\to\infty}$$

(b) $$\left[\frac{32}{n^6}\sum_{i=1}^{n} i^5 + \frac{32}{n^4}\sum_{i=1}^{n} i^3\right]$$

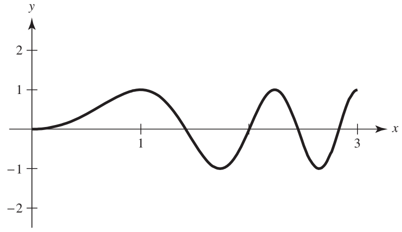
$$\left[\frac{32\cdot n^2(n+1)^2(2n^2+2n-1)}{n^6\cdot 12} + \frac{32\cdot n^2(n+1)^2}{n^4\cdot 4}\right] = \left[\frac{8(n+1)^2(2n^2+2n-1)}{3n^4} + \frac{8(n+1)^2}{n^2}\right]$$

$$= \left[\frac{8(n+1)^2(5n^2+2n-1)}{3n^4}\right]$$

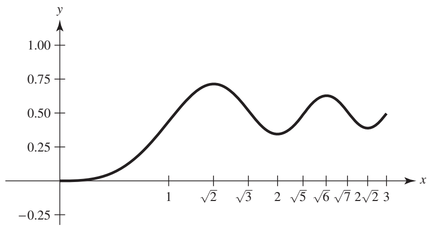
(c) $$A = \lim_{n\to\infty}\left[\frac{40n^4+96n^3+64n^2-8}{3n^4}\right] = \frac{40}{3}$$

5. $S(x) = \int_0^x \sin\left(\dfrac{\pi t^2}{2}\right)dt$

(a)



(b)



The zeros of $y = \sin\dfrac{\pi x^2}{2}$ correspond to the relative extrema of $S(x)$.

(c) $S'(x) = \sin\dfrac{\pi x^2}{2} = 0 \Rightarrow \dfrac{\pi x^2}{2} = n\pi \Rightarrow x^2 = 2n \Rightarrow x = \sqrt{2n},\ n$ integer

Relative maxima at $x = \sqrt{2} \approx 1.4142$ and $x = \sqrt{6} \approx 2.4495$

Relative minima at $x = 2$ and $x = 2\sqrt{2} \approx 2.8284$

(d) $S''(x) = \cos\left(\dfrac{\pi x^2}{2}\right)(\pi x) = 0 \Rightarrow \dfrac{\pi x^2}{2} = \dfrac{\pi}{2} + n\pi \Rightarrow x^2 = 1 + 2n \Rightarrow x = \sqrt{1 + 2n},\ n$ integer

Points of inflection at $x = 1, \sqrt{3}, \sqrt{5}$, and $\sqrt{7}$.

6. (a) $\displaystyle\int_{-1}^{1} \cos x\,dx \approx \cos\left(-\frac{1}{\sqrt{3}}\right) + \cos\left(\frac{1}{\sqrt{3}}\right) = 2\cos\left(\frac{1}{\sqrt{3}}\right) \approx 1.6758$

$\displaystyle\int_{-1}^{1} \cos x\,dx = \sin x\Big]_{-1}^{1} = 2\sin(1) \approx 1.6829$

Error: $|1.6829 - 1.6758| = 0.0071$

(b) $\displaystyle\int_{-1}^{1} \frac{1}{1 + x^2}\,dx \approx \frac{1}{1 + (1/3)} + \frac{1}{1 + (1/3)} = \frac{3}{2}$

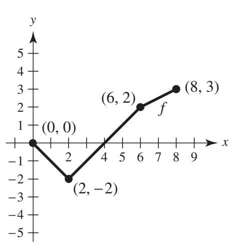
(**Note:** exact answer is $\pi/2 \approx 1.5708$)

(c) Let $p(x) = ax^3 + bx^2 + cx + d$.

$\displaystyle\int_{-1}^{1} p(x)\,dx = \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_{-1}^{1} = \frac{2b}{3} + 2d$

$\displaystyle p\left(-\frac{1}{\sqrt{3}}\right) + p\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{b}{3} + d\right) + \left(\frac{b}{3} + d\right) = \frac{2b}{3} + 2d$

7. (a)



(b)

x	0	1	2	3	4	5	6	7	8
$F(x)$	0	$-\frac{1}{2}$	-2	$-\frac{7}{2}$	-4	$-\frac{7}{2}$	-2	$\frac{1}{4}$	3

(c) $f(x) = \begin{cases} -x, & 0 \le x < 2 \\ x - 4, & 2 \le x < 6 \\ \frac{1}{2}x - 1, & 6 \le x \le 8 \end{cases}$

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} (-x^2/2), & 0 \le x < 2 \\ (x^2/2) - 4x + 4, & 2 \le x < 6 \\ (1/4)x^2 - x - 5, & 6 \le x \le 8 \end{cases}$$

$F'(x) = f(x)$. F is decreasing on (0, 4) and increasing on (4, 8). Therefore, the minimum is -4 at $x = 4$, and the maximum is 3 at $x = 8$.

(d) $F''(x) = f'(x) = \begin{cases} -1, & 0 < x < 2 \\ 1, & 2 < x < 6 \\ \frac{1}{2}, & 6 < x < 8 \end{cases}$

$x = 2$ is a point of inflection, whereas $x = 6$ is not.

8. Let d be the distance traversed and a be the uniform acceleration. You can assume that $v(0) = 0$ and $s(0) = 0$. Then

$a(t) = a$

$v(t) = at$

$s(t) = \frac{1}{2}at^2.$

$s(t) = d$ when $t = \sqrt{\frac{2d}{a}}$.

The highest speed is $v = a\sqrt{\frac{2d}{a}} = \sqrt{2ad}$.

The lowest speed is $v = 0$.

The mean speed is $\frac{1}{2}\left(\sqrt{2ad} + 0\right) = \sqrt{\frac{ad}{2}}$.

The time necessary to traverse the distance d at the mean speed is $t = \frac{d}{\sqrt{ad/2}} = \sqrt{\frac{2d}{a}}$

which is the same as the time calculated above.

9. $\int_0^x f(t)(x - t)\,dt = \int_0^x xf(t)\,dt - \int_0^x tf(t)\,dt = x\int_0^x f(t)\,dt - \int_0^x tf(t)\,dt$

So, $\frac{d}{dx}\int_0^x f(t)(x - t)\,dt = xf(x) + \int_0^x f(t)\,dt - xf(x) = \int_0^x f(t)\,dt$

Differentiating the other integral,

$\frac{d}{dx}\int_0^x \left(\int_0^x f(v)\,dv\right) dt = \int_0^x f(v)\,dv.$

So, the two original integrals have equal derivatives,

$\int_0^x f(t)(x - t)\,dt = \int_0^x \left(\int_0^t f(v)\,dv\right) dt + C.$

Letting $x = 0$, you see that $C = 0$.

10. Consider $F(x) = \left[f(x)\right]^2 \Rightarrow F'(x) = 2f(x)f'(x)$. So,

$$\begin{aligned}\int_a^b f(x)\,f'(x)\,dx &= \int_a^b \tfrac{1}{2}F'(x)\,dx\\ &= \left[\tfrac{1}{2}F(x)\right]_a^b\\ &= \tfrac{1}{2}\left[F(b) - F(a)\right]\\ &= \tfrac{1}{2}\left[f(b)^2 - f(a)^2\right].\end{aligned}$$

11. Consider $\int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big]_0^1 = \frac{2}{3}$. The corresponding Riemann Sum using right-hand endpoints is

$$S(n) = \frac{1}{n}\left[\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right] = \frac{1}{n^{3/2}}\left[\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}\right]. \text{ So, } \lim_{n\to\infty}\frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^{3/2}} = \frac{2}{3}.$$

12. Consider $\int_0^1 x^5\,dx = \frac{x^6}{6}\Big]_0^1 = \frac{1}{6}$.

The corresponding Riemann Sum using right endpoints is

$$S(n) = \frac{1}{n}\left[\left(\frac{1}{n}\right)^5 + \left(\frac{2}{n}\right)^5 + \cdots + \left(\frac{n}{n}\right)^5\right] = \frac{1}{n^6}\left[1^5 + 2^5 + \cdots + n^5\right]. \text{ So, } \lim_{n\to\infty} S(n) = \lim_{n\to\infty}\frac{1^5 + 2^5 + \cdots + n^5}{n^6} = \frac{1}{6}.$$

13. By Theorem 4.8, $0 < f(x) \le M \Rightarrow \int_a^b f(x)\,dx \le \int_a^b M\,dx = M(b - a)$.

Similarly, $m \le f(x) \Rightarrow m(b - a) = \int_a^b m\,dx \le \int_a^b f(x)\,dx$.

So, $m(b - a) \le \int_a^b f(x)\,dx \le M(b - a)$. On the interval [0, 1], $1 \le \sqrt{1 + x^4} \le \sqrt{2}$ and $b - a = 1$.

So, $1 \le \int_0^1 \sqrt{1 + x^4}\,dx \le \sqrt{2}$. (**Note:** $\int_0^1 \sqrt{1 + x^4}\,dx \approx 1.0894$)

14. (a) Let $A = \int_0^b \frac{f(x)}{f(x) + f(b - x)}\,dx$.

Let $u = b - x$, $du = -dx$.

$$A = \int_b^0 \frac{f(b - u)}{f(b - u) + f(u)}(-du) = \int_0^b \frac{f(b - u)}{f(b - u) + f(u)}\,du = \int_0^b \frac{f(b - x)}{f(b - x) + f(x)}\,dx$$

Then, $2A = \int_0^b \frac{f(x)}{f(x) + f(b - x)}\,dx + \int_0^b \frac{f(b - x)}{f(b - x) + f(x)}\,dx = \int_0^b 1\,dx = b$.

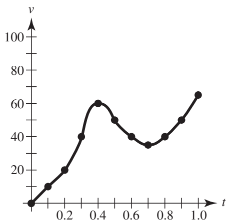
So, $A = \frac{b}{2}$.

(b) $b = 1 \Rightarrow \int_0^1 \frac{\sin x}{\sin(1 - x) + \sin x}\,dx = \frac{1}{2}$

(c) $b = 3$, $f(x) = \sqrt{x}$

$$\int_0^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{3 - x}}\,dx = \frac{3}{2}$$

15. (a)



(b) v is increasing (positive acceleration) on (0, 0.4) and (0.7, 1.0).

(c) Average acceleration $= \dfrac{v(0.4) - v(0)}{0.4 - 0} = \dfrac{60 - 0}{0.4} = 150 \text{ mi/h}^2$

(d) This integral is the total distance traveled in miles.

$$\int_0^1 v(t)\,dt \approx \tfrac{1}{5}[10 + 40 + 50 + 35 + 50] = 37$$

(e) One approximation is

$$a(0.8) \approx \frac{v(0.9) - v(0.8)}{0.9 - 0.8} = \frac{50 - 40}{0.1} = 100 \text{ mi/h}^2$$

(other answers possible)

16. Because $-\left|f(x)\right| \le f(x) \le \left|f(x)\right|$, $-\int_a^b \left|f(x)\right| dx \le \int_a^b f(x)\,dx \le \int_a^b \left|f(x)\right| dx \Rightarrow \left|\int_a^b f(x)\,dx\right| \le \int_a^b \left|f(x)\right| dx$.

17. (a) $(1 + i)^3 = 1 + 3i + 3i^2 + i^3 \Rightarrow (1 + i)^3 - i^3 = 3i^2 + 3i + 1$

(b)

$$3i^2 + 3i + 1 = (i + 1)^3 - i^3$$

$$\sum_{i=1}^{n} \left(3i^2 + 3i + 1\right) = \sum_{i=1}^{n} \left[(i + 1)^3 - i^3\right]$$

$$= \left(2^3 - 1^3\right) + \left(3^3 - 2^3\right) + \cdots + \left[\left((n + 1)^3 - n^3\right)\right] = (n + 1)^3 - 1$$

So, $(n + 1)^3 = \sum_{i=1}^{n} \left(3i^2 + 3i + 1\right) + 1$.

(c) $(n + 1)^3 - 1 = \sum_{i=1}^{n} \left(3i^2 + 3i + 1\right) = \sum_{i=1}^{n} 3i^2 + \dfrac{3(n)(n + 1)}{2} + n$

$$\Rightarrow \sum_{i=1}^{n} 3i^2 = n^3 + 3n^2 + 3n - \frac{3n(n + 1)}{2} - n$$

$$= \frac{2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n}{2}$$

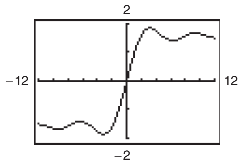
$$= \frac{2n^3 + 3n^2 + n}{2}$$

$$= \frac{n(n + 1)(2n + 1)}{2}$$

$$\Rightarrow \sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}$$

18. $\text{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$

(a)



(b) $\text{Si}'(x) = \frac{\sin x}{x} \quad \text{Si}'(x) = 0 \text{ for } x = 2n\pi$

For positive x, $x = (2n - 1)\pi$

For negative x, $x = 2n\pi$

Maxima at $\pi, 3\pi, 5\pi, \ldots$ and $-2\pi, -4\pi, -6\pi, \ldots$

(c) $\text{Si}''(x) = \frac{x\cos x - \sin x}{x^2} = 0$

$x\cos x = \sin x$ for $x \approx 4.4934$

$\text{Si}(4.4934) \approx 1.6556$

(d) Horizontal asymptotes at $y = \pm\frac{\pi}{2}$

$\lim_{x\to\infty} \text{Si}(x) = \frac{\pi}{2}$

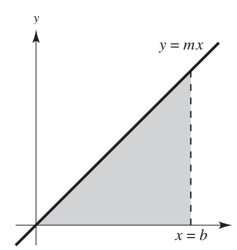
$\lim_{x\to-\infty} \text{Si}(x) = \frac{-\pi}{2}$

19. (a) $S = m\left(\frac{b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{2b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{3b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{4b}{4}\right)\left(\frac{b}{4}\right) = \frac{mb^2}{16}(1 + 2 + 3 + 4) = \frac{5mb^2}{8}$

$s = m(0)\left(\frac{b}{4}\right) + m\left(\frac{b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{2b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{3b}{4}\right)\left(\frac{b}{4}\right) = \frac{mb^2}{16}(1 + 2 + 3) = \frac{3mb^2}{8}$

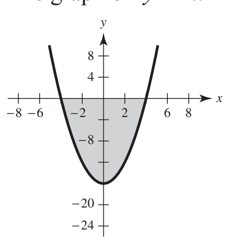
(b) $S(n) = \sum_{i=1}^{n} f\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = \sum_{i=1}^{n}\left(\frac{mbi}{n}\right)\left(\frac{b}{n}\right) = m\left(\frac{b}{n}\right)^2 \sum_{i=1}^{n} i = \frac{mb^2}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{mb^2(n+1)}{2n}$

$s(n) = \sum_{i=0}^{n-1} f\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = \sum_{i=0}^{n-1} m\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = m\left(\frac{b}{n}\right)^2 \sum_{i=0}^{n-1} i = \frac{mb^2}{n^2}\left(\frac{(n-1)n}{2}\right) = \frac{mb^2(n-1)}{2n}$



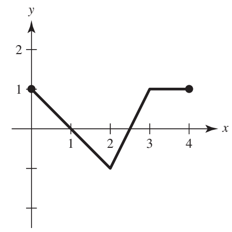
(c) $\text{Area} = \lim_{n\to\infty} \frac{mb^2(n+1)}{2n} = \lim_{n\to\infty} \frac{mb^2(n-1)}{2n} = \frac{1}{2}mb^2 = \frac{1}{2}(b)(mb) = \frac{1}{2}(\text{base})(\text{height})$

20. The graph of $y = x^2 - 16$ is a parabola:



The integral $\int_a^b (x^2 - 16)\,dx$ will be a minimum when $a = -4$ and $b = 4$, as indicated in the figure.

21. $f'(x) = \begin{cases} -1, & 0 \le x < 2 \\ 2, & 2 < x < 3 \\ 0, & 3 < x \le 4 \end{cases}$



$f(x) = \begin{cases} -x + C_1, & 0 \le x < 2 \\ 2x + C_2, & 2 < x < 3 \\ C_3, & 3 < x \le 4 \end{cases}$

$f(0) = 1 \Rightarrow C_1 = 1$

f continuous at $x = 2 \Rightarrow -2 + 1 = 4 + C_2 \Rightarrow C_2 = -5$

f continuous at $x = 3 \Rightarrow 6 - 5 = C_3 = 1$

$f(x) = \begin{cases} -x + 1, & 0 \le x < 2 \\ 2x - 5, & 2 \le x < 3 \\ 1, & 3 \le x \le 4 \end{cases}$