C H A P T E R 3
Applications of Differentiation

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C H A P T E R 3
Applications of Differentiation

Section 3.1 Extrema on an Interval

1. $f(c)$ is the low point of the graph of f on the interval I. That is, $f(c) \le f(x)$ for all x in I.

2. The Extreme Value Theorem states that if f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

3. A relative maximum is a peak of the graph. An absolute maximum is the greatest value on the interval I.

4. Let f be defined at the point $x = c$. c is a critical number if $f'(c) = 0$ or f is not differentiable at c.

5. To find the critical numbers of f, determine all values of x for which $f'(x) = 0$ or f' does not exist.

6. Let f be continuous on $[a, b]$. To find the extrema of f, determine the critical number(s) of f in (a, b). Then evaluate f at each critical number and at the two endpoints a and b. The least of these values is the minimum, and the greatest is the maximum.

7. $f(x) = \dfrac{x^2}{x^2 + 4}$

$$f'(x) = \frac{(x^2 + 4)(2x) - (x^2)(2x)}{(x^2 + 4)^2} = \frac{8x}{(x^2 + 4)^2}$$

$f'(0) = 0$

8. $f(x) = \cos \dfrac{\pi x}{2}$

$f'(x) = -\dfrac{\pi}{2} \sin \dfrac{\pi x}{2}$

$f'(0) = 0$

$f'(2) = 0$

9. $f(x) = x + \dfrac{4}{x^2} = x + 4x^{-2}$

$f'(x) = 1 - 8x^{-3} = 1 - \dfrac{8}{x^3}$

$f'(2) = 0$

10. $f(x) = -3x\sqrt{x + 1}$

$$\begin{aligned} f'(x) &= -3x\left[\tfrac{1}{2}(x + 1)^{-1/2}\right] + \sqrt{x + 1}(-3) \\ &= -\tfrac{3}{2}(x + 1)^{-1/2}\left[x + 2(x + 1)\right] \\ &= -\tfrac{3}{2}(x + 1)^{-1/2}(3x + 2) \end{aligned}$$

$f'\left(-\tfrac{2}{3}\right) = 0$

11. $f(x) = (x + 2)^{2/3}$

$f'(x) = \tfrac{2}{3}(x + 2)^{-1/3}$

$f'(-2)$ is undefined.

12. Using the limit definition of the derivative,

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{(4 - |x|) - 4}{x} = 1$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{(4 - |x|) - 4}{x - 0} = -1$$

$f'(0)$ does not exist, because the one-sided derivatives are not equal.

13. Critical number: $x = 2$

$x = 2$: absolute maximum (and relative maximum)

14. Critical number: $x = 0$

$x = 0$: neither

15. Critical numbers: $x = 1, 2, 3$

$x = 1, 3$: absolute maxima (and relative maxima)

$x = 2$: absolute minimum (and relative minimum)

16. Critical numbers: $x = 2, 5$

$x = 2$: neither

$x = 5$: absolute maximum (and relative maximum)

17. $f(x) = 4x^2 - 6x$

$f'(x) = 8x - 6 = 2(4x - 3)$

Critical number: $\tfrac{3}{4}$

18. $g(x) = x - \sqrt{x}$

$g'(x) = 1 - \frac{1}{2\sqrt{x}}$

$g'(x) = 0 \Rightarrow 1 = \frac{1}{2\sqrt{x}} \Rightarrow 2\sqrt{x} = 1 \Rightarrow x = \frac{1}{4}$

Critical numbers: $x = 0, \frac{1}{4}$

19. $g(t) = t\sqrt{4 - t}, \quad t < 3$

$g'(t) = t\left[\frac{1}{2}(4 - t)^{-1/2}(-1)\right] + (4 - t)^{1/2}$

$= \frac{1}{2}(4 - t)^{-1/2}\left[-t + 2(4 - t)\right]$

$= \frac{8 - 3t}{2\sqrt{4 - t}}$

Critical number: $t = \frac{8}{3}$

20. $f(x) = \frac{4x}{x^2 + 1}$

$f'(x) = \frac{(x^2 + 1)(4) - (4x)(2x)}{(x^2 + 1)^2} = \frac{4(1 - x^2)}{(x^2 + 1)^2}$

Critical numbers: $x = \pm 1$

21. $h(x) = \sin^2 x + \cos x, \quad 0 < x < 2\pi$

$h'(x) = 2 \sin x \cos x - \sin x = \sin x(2 \cos x - 1)$

Critical numbers in $(0, 2\pi)$: $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

22. $f(\theta) = 2 \sec \theta + \tan \theta, \quad 0 < \theta < 2\pi$

$f'(\theta) = 2 \sec \theta \tan \theta + \sec^2 \theta$

$= \sec \theta(2 \tan \theta + \sec \theta)$

$= \sec \theta\left[2\left(\frac{\sin \theta}{\cos \theta}\right) + \frac{1}{\cos \theta}\right]$

$= \sec^2 \theta(2 \sin \theta + 1)$

Critical numbers in $(0, 2\pi)$: $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

23. $f(x) = 3 - x, \quad [-1, 2]$

$f'(x) = -1 \Rightarrow$ no critical numbers

Left endpoint: $(-1, 4)$ Maximum

Right endpoint: $(2, 1)$ Minimum

24. $f(x) = \frac{3}{4}x + 2, [0, 4]$

$f'(x) = \frac{3}{4} \Rightarrow$ no critical numbers

Left endpoint: $(0, 2)$ Minimum

Right endpoint: $(4, 5)$ Maximum

25. $h(x) = 5 - 2x^2, [-3, 1]$

$h'(x) = -4x$

Critical number: $x = 0$

Left endpoint: $(-3, -13)$ Minimum

Critical number: $(0, 5)$ Maximum

Right endpoint: $(1, 3)$

26. $f(x) = 7x^2 + 1, [-1, 2]$

$f'(x) = 14x = 0 \Rightarrow x = 0$ critical number

Left endpoint: $(-1, 8)$

Right endpoint: $(2, 29)$ Maximum

Critical number: $(0, 1)$ Minimum

27. $f(x) = x^3 - \frac{3}{2}x^2, [-1, 2]$

$f'(x) = 3x^2 - 3x = 3x(x - 1)$

Left endpoint: $\left(-1, -\frac{5}{2}\right)$ Minimum

Right endpoint: $(2, 2)$ Maximum

Critical number: $(0, 0)$

Critical number: $\left(1, -\frac{1}{2}\right)$

28. $f(x) = 2x^3 - 6x, [0, 3]$

$f'(x) = 6x^2 - 6 = 6(x^2 - 1)$

Critical number: $x = 1$ ($x = -1$ not in interval.)

Left endpoint: $(0, 0)$

Critical number: $(1, -4)$ Minimum

Right endpoint: $(3, 36)$ Maximum

29. $f(x) = 3x^{2/3} - 2x, \quad [-1, 1]$

$f'(x) = 2x^{-1/3} - 2 = \frac{2(1 - \sqrt[3]{x})}{\sqrt[3]{x}}$

Left endpoint: $(-1, 5)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $(1, 1)$

30. $g(x) = \sqrt[3]{x} = x^{1/3}, [-8, 8]$

$g'(x) = \dfrac{1}{3x^{2/3}}$

Critical number: $x = 0$

Left endpoint: $(-8, -2)$ Minimum

Critical number: $(0, 0)$

Right endpoint: $(8, 2)$ Maximum

31. $g(x) = \dfrac{6x^2}{x - 2}, [-2, 1]$

$g'(x) = \dfrac{(x - 2)(12x) - 6x^2(1)}{(x - 2)^2} = \dfrac{6x^2 - 24x}{(x - 2)^2} = \dfrac{6x(x - 4)}{(x - 2)^2}$

Critical number: $x = 0$ $(x = 2$ and $x = 4$ are outside the interval.)

Left endpoint: $(-2, -6)$ Mimimum

Right endpoint: $(1, -6)$ Minimum

Critical number: $(0, 0)$ Maximum

32. $h(t) = \dfrac{t}{t + 3}, [-1, 6]$

$h'(t) = \dfrac{(t + 3)(1) - t(1)}{(t + 3)^2} = \dfrac{3}{(t + 3)^2}$

No critical numbers

Left endpoint: $\left(-1, -\dfrac{1}{2}\right)$ Minimum

Right endpoint: $\left(6, \dfrac{2}{3}\right)$ Maximum

33. $y = 3 - |t - 3|, \quad [-1, 5]$

For $x < 3, y = 3 + (t - 3) = t$

and $y' = 1 \neq 0$ on $[-1, 3)$

For $x > 3, y = 3 - (t - 3) = 6 - t$

and $y' = -1 \neq 0$ on $(3, 5]$

So, $x = 3$ is the only critical number.

Left endpoint: $(-1, -1)$ Minimum

Right endpoint: $(5, 1)$

Critical number: $(3, 3)$ Maximum

34. $g(x) = |x + 4|, \quad [-7, 1]$

g is the absolute value function shifted 4 units to the left. So, the critical number is $x = -4$.

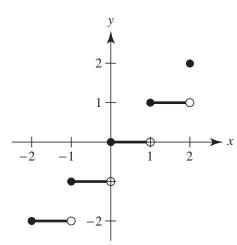
Left endpoint: $(-7, 3)$

Critical number: $(-4, 0)$ Minimum

Right endpoint: $(1, 5)$ Maximum

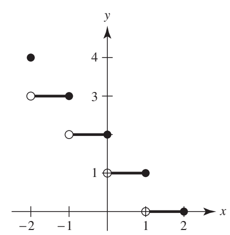
35. $f(x) = [\![x]\!], \quad [-2, 2]$

From the graph of f, you see that the maximum value of f is 2 for $x = 2$, and the minimum value is –2 for $-2 \leq x < -1$.



36. $h(x) = [\![2 - x]\!], \quad [-2, 2]$

From the graph you see that the maximum value of h is 4 at $x = -2$, and the minimum value is 0 for $1 < x \leq 2$.



37. $f(x) = \sin x, \left[\frac{5\pi}{6}, \frac{11\pi}{6}\right]$

$f'(x) = \cos x$

Critical number: $x = \frac{3\pi}{2}$

Left endpoint: $\left(\frac{5\pi}{6}, \frac{1}{2}\right)$ Maximum

Critical number: $\left(\frac{3\pi}{2}, -1\right)$ Minimum

Right endpoint: $\left(\frac{11\pi}{6}, -\frac{1}{2}\right)$

38. $g(x) = \sec x, \quad \left[-\frac{\pi}{6}, \frac{\pi}{3}\right]$

$g'(x) = \sec x \tan x$

Left endpoint: $\left(-\frac{\pi}{6}, \frac{2}{\sqrt{3}}\right) \approx \left(-\frac{\pi}{6}, 1.1547\right)$

Right endpoint: $\left(\frac{\pi}{3}, 2\right)$ Maximum

Critical number: $(0, 1)$ Minimum

39. $y = 3\cos x, \quad [0, 2\pi]$

$y' = -3\sin x$

Critical number in $(0, 2\pi)$: $x = \pi$

Left endpoint: $(0, 3)$ Maximum

Critical number: $(\pi, -3)$ Minimum

Right endpoint: $(2\pi, 3)$ Maximum

40. $y = \tan\left(\frac{\pi x}{8}\right), \quad [0, 2]$

$y' = \frac{\pi}{8}\sec^2\left(\frac{\pi x}{8}\right) \neq 0$

Left endpoint: $(0, 0)$ Minimum

Right endpoint: $(2, 1)$ Maximum

41. $f(x) = 2x - 3$

(a) Minimum: $(0, -3)$

Maximum: $(2, 1)$

(b) Minimum: $(0, -3)$

(c) Maximum: $(2, 1)$

(d) No extrema

42. $f(x) = 5 - x$

(a) Minimum: $(4, 1)$

Maximum: $(1, 4)$

(b) Maximum: $(1, 4)$

(c) Minimum: $(4, 1)$

(d) No extrema

43. $f(x) = x^2 - 2x$

(a) Minimum: $(1, -1)$

Maximum: $(-1, 3)$

(b) Maximum: $(3, 3)$

(c) Minimum: $(1, -1)$

(d) Minimum: $(1, -1)$

44. $f(x) = \sqrt{4 - x^2}$

(a) Minima: $(-2, 0)$ and $(2, 0)$

Maximum: $(0, 2)$

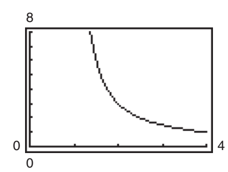
(b) Minimum: $(-2, 0)$

(c) Maximum: $(0, 2)$

(d) Maximum: $(1, \sqrt{3})$

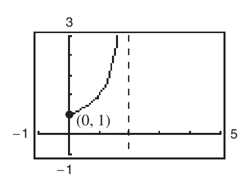
45. $f(x) = \frac{3}{x - 1}, \quad (1, 4]$

Right endpoint: $(4, 1)$ Minimum

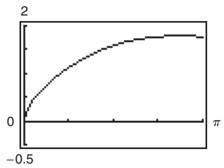


46. $f(x) = \frac{2}{2 - x}, \quad [0, 2)$

Left endpoint: $(0, 1)$ Minimum



47.



$f(x) = \sqrt{x} + \dfrac{\sin x}{3}, [0, \pi]$

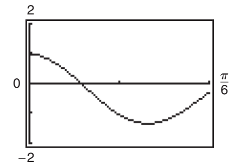
$f'(x) = \dfrac{1}{2\sqrt{x}} + \dfrac{1}{3}\cos x = 0 \Rightarrow x \approx 2.715$

Left endpoint: $(0, 0)$ Minimum

Right endpoint: $(\pi, 1.772)$

Critical number: $(2.715, 1.786)$ Maximum

48.



$f(x) = -x + \cos 3\pi x, \left[0, \dfrac{\pi}{6}\right]$

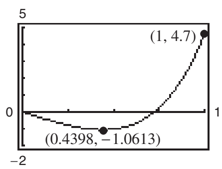
$f'(x) = -1 - 3\pi \sin 3\pi x = 0 \Rightarrow x \approx 0.345$

Left endpoint: $(0, 1)$ Maximum

Right endpoint: $\left(\dfrac{\pi}{6}, -0.303\right)$

Critical number: $(0.345, -1.339)$ Minimum

49. (a)



Minimum: $(0.4398, -1.0613)$

(b) $f(x) = 3.2x^5 + 5x^3 - 3.5x, \quad [0, 1]$

$f'(x) = 16x^4 + 15x^2 - 3.5$

$16x^4 + 15x^2 - 3.5 = 0$

$$x^2 = \frac{-15 \pm \sqrt{(15)^2 - 4(16)(-3.5)}}{2(16)} = \frac{-15 \pm \sqrt{449}}{32}$$

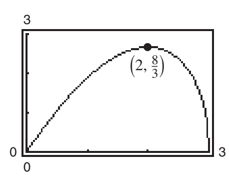
$$x = \sqrt{\frac{-15 + \sqrt{449}}{32}} \approx 0.4398$$

Left endpoint: $(0, 0)$

Critical point: $(0.4398, -1.0613)$ Minimum

Right endpoint: $(1, 4.7)$ Maximum

50. (a)



Maximum: $\left(2, \dfrac{8}{3}\right)$

(b) $f(x) = \dfrac{4}{3}x\sqrt{3 - x}, \quad [0, 3]$

$$f'(x) = \frac{4}{3}\left[x\left(\frac{1}{2}\right)(3 - x)^{-1/2}(-1) + (3 - x)^{1/2}(1)\right] = \frac{4}{3}(3 - x)^{-1/2}\left(\frac{1}{2}\right)\left[-x + 2(3 - x)\right] = \frac{2(6 - 3x)}{3\sqrt{3 - x}} = \frac{6(2 - x)}{3\sqrt{3 - x}} = \frac{2(2 - x)}{\sqrt{3 - x}}$$

Left endpoint: $(0, 0)$ Minimum

Critical point: $\left(2, \dfrac{8}{3}\right)$ Maximum

Right endpoint: $(3, 0)$ Minimum

51. $f(x) = (1 + x^3)^{1/2}, \quad [0, 2]$

$f'(x) = \frac{3}{2}x^2(1 + x^3)^{-1/2}$

$f''(x) = \frac{3}{4}(x^4 + 4x)(1 + x^3)^{-3/2}$

$f'''(x) = -\frac{3}{8}(x^6 + 20x^3 - 8)(1 + x^3)^{-5/2}$

Setting $f''' = 0$, you have $x^6 + 20x^3 - 8 = 0$.

$x^3 = \frac{-20 \pm \sqrt{400 - 4(1)(-8)}}{2}$

$x = \sqrt[3]{-10 \pm \sqrt{108}} = \sqrt{3} - 1$

In the interval $[0, 2]$, choose

$x = \sqrt[3]{-10 \pm \sqrt{108}} = \sqrt{3} - 1 \approx 0.732.$

$\left|f''\left(\sqrt[3]{-10 + \sqrt{108}}\right)\right| \approx 1.47$ is the maximum value.

52. $f(x) = \frac{1}{x^2 + 1}, \quad \left[\frac{1}{2}, 3\right]$

$f'(x) = \frac{-2x}{(x^2 + 1)^2}$

$f''(x) = \frac{-2(1 - 3x^2)}{(x^2 + 1)^3}$

$f'''(x) = \frac{24x - 24x^3}{(x^2 + 1)^4}$

Setting $f''' = 0$, you have $x = 0, \pm 1$.

$\left|f''(1)\right| = \frac{1}{2}$ is the maximum value.

53. $f(x) = (x + 1)^{2/3}, \quad [0, 2]$

$f'(x) = \frac{2}{3}(x + 1)^{-1/3}$

$f''(x) = -\frac{2}{9}(x + 1)^{-4/3}$

$f'''(x) = \frac{8}{27}(x + 1)^{-7/3}$

$f^{(4)}(x) = -\frac{56}{81}(x + 1)^{-10/3}$

$f^{(5)}(x) = \frac{560}{243}(x + 1)^{-13/3}$

$\left|f^{(4)}(0)\right| = \frac{56}{81}$ is the maximum value.

54. $f(x) = \frac{1}{x^2 + 1}, \quad [-1, 1]$

$f'''(x) = \frac{24x - 24x^3}{(x^2 + 1)^4}$

$f^{(4)}(x) = \frac{24(5x^4 - 10x^2 + 1)}{(x^2 + 1)^5}$

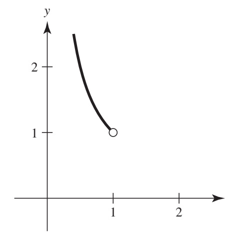
$f^{(5)}(x) = \frac{-240x(3x^4 - 10x^2 + 3)}{(x^2 + 1)^6}$

$\left|f^{(4)}(0)\right| = 24$ is the maximum value.

55. Answers will vary.
Sample answer:

$y = \frac{1}{x}$ on the interval $(0, 1)$

There is no maximum or minimum value.



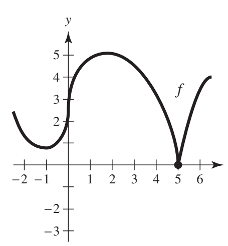
56. A: absolute minimum
B: relative maximum
C: neither
D: relative minimum
E: relative maximum
F: relative minimum
G: neither

57. (a) Yes
(b) No

58. (a) No
(b) Yes

59. No. The function is not defined at $x = -2$.

60.



61. $P = VI - RI^2 = 12I - 0.5I^2, 0 \le I \le 15$

$P = 0$ when $I = 0$.

$P = 67.5$ when $I = 15$.

$P' = 12 - I = 0$

Critical number: $I = 12$ amps

When $I = 12$ amps, $P = 72$, the maximum output.

No, a 20-amp fuse would not increase the power output. P is decreasing for $I > 12$.

62. $x = \dfrac{v^2 \sin 2\theta}{32}, 45° \le \theta \le 135°$

$\dfrac{d\theta}{dt}$ is constant.

$\dfrac{dx}{dt} = \dfrac{dx}{d\theta}\dfrac{d\theta}{dt}$ (by the Chain Rule) $= \dfrac{v^2 \cos 2\theta}{16}\dfrac{d\theta}{dt}$

In the interval $[45°, 135°]$, $\theta = 45°$ and $\theta = 135°$ indicate minimums for dx/dt and $\theta = 90°$ indicates a maximum for dx/dt. This implies that the sprinkler waters longest when $\theta = 45°$ and $\theta = 135°$. So, the lawn farthest from the sprinkler gets the most water.

63. $S = 6hs + \dfrac{3s^2}{2}\left(\dfrac{\sqrt{3} - \cos\theta}{\sin\theta}\right), \dfrac{\pi}{6} \le \theta \le \dfrac{\pi}{2}$

$$\frac{dS}{d\theta} = \frac{3s^2}{2}\left(-\sqrt{3}\csc\theta\cot\theta + \csc^2\theta\right)$$

$$= \frac{3s^2}{2}\csc\theta\left(-\sqrt{3}\cot\theta + \csc\theta\right) = 0$$

$$\csc\theta = \sqrt{3}\cot\theta$$

$$\sec\theta = \sqrt{3}$$

$$\theta = \operatorname{arcsec}\sqrt{3} \approx 0.9553 \text{ radians}$$

$$S\left(\frac{\pi}{6}\right) = 6hs + \frac{3s^2}{2}\left(\sqrt{3}\right)$$

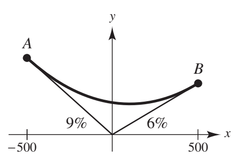
$$S\left(\frac{\pi}{6}\right) = 6hs + \frac{3s^2}{2}\left(\sqrt{3}\right)$$

$$S\left(\operatorname{arcsec}\sqrt{3}\right) = 6hs + \frac{3s^2}{2}\left(\sqrt{2}\right)$$

S is minimum when $\theta = \operatorname{arcsec}\sqrt{3} \approx 0.9553$ radian.

64. (a) Because the grade at A is 9%, $A(-500, 45)$

The grade at B is 6%, $B(500, 30)$.



(b) $y = ax^2 + bx + c$

$y' = 2ax + b$

At A: $2a(-500) + b = -0.09$

At B: $2a(500) + b = 0.06$

Solving these two equations, you obtain

$a = \dfrac{3}{40{,}000}$ and $b = -\dfrac{3}{200}$.

Using the points $A(-500, 45)$ and $B(500, 30)$, you obtain

$$45 = \frac{3}{40{,}000}(-500)^2 + \left(-\frac{3}{200}\right)(-500) + C$$

$$30 = \frac{3}{40{,}000}(500)^2 + \left(-\frac{3}{200}\right)(500) + C.$$

In both cases, $C = 18.75 = \dfrac{75}{4}$. So, $y = \dfrac{3}{40{,}000}x^2 - \dfrac{3}{200}x + \dfrac{75}{4}$

(c)

x	–500	–400	–300	–200	–100	0	100	200	300	400	500
d	0	0.75	3	6.75	12	18.75	12	6.75	3	0.75	0

For $-500 \le x \le 0$, $d = \left(ax^2 + bx + c\right) - (-0.09x)$.

For $0 \le x \le 500$, $d = \left(ax^2 + bx + c\right) - (0.06x)$.

(d) $y' = \frac{3}{20{,}000}x - \frac{3}{200} = 0$

$x = \frac{3}{200} \cdot \frac{20{,}000}{3} = 100$

The lowest point on the highway is $(100, 18)$, is not directly over the origin.

65. False. The function does not have a maximum on the open interval $(-3, 3)$. The maximum would be 9 if the interval were closed, $[-3, 3]$.

66. True. This is stated in the Extreme Value Theorem.

67. True

68. False. Let $f(x) = x^2$. $x = 0$ is a critical number of f.

$g(x) = f(x - k) = (x - k)^2$

$x = k$ is a critical number of g.

69. If f has a maximum value at $x = c$, then $f(c) \ge f(x)$ for all x in I. So, $-f(c) \le -f(x)$ for all x in I. So, $-f$ has a minimum value at $x = c$.

70. $f(x) = ax^3 + bx^2 + cx + d, \quad a \ne 0$

$f'(x) = 3ax^2 + 2bx + c$

The quadratic polynomial can have zero, one, or two zeros.

$x = \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$

Zero critical numbers: $b^2 < 3ac$.

Example: $(a = b = c = 1, d = 0)$ $f(x) = x^3 + x^2 + x$ has no critical numbers.

One critical number: $b^2 = 3ac$.

Example: $(a = 1, b = c = d = 0)$ $f(x) = x^3$ has one critical number, $x = 0$.

Two critical numbers: $b^2 > 3ac$.

Example:

$(a = c = 1, b = 2, d = 0)$ $f(x) = x^3 + 2x^2 + x$ has two critical numbers: $x = -1, -\frac{1}{3}$.

71. First do an example: Let $a = 4$ and $f(x) = 4$. Then R is the square $0 \le x \le 4, 0 \le y \le 4$. Its area and perimeter are both $k = 16$.

Claim that all real numbers $a > 2$ work. On the one hand, if $a > 2$ is given, then let $f(x) = 2a/(a - 2)$. Then the rectangle

$$R = \left\{(x, y): 0 \le x \le a, 0 \le y \le \frac{2a}{a - 2}\right\}$$

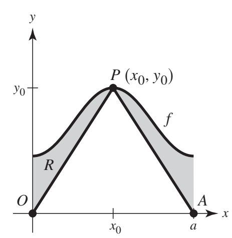
has $k = \frac{2a^2}{a - 2}$:

$$\text{Area} = a\left(\frac{2a}{a - 2}\right) = \frac{2a^2}{a - 2}$$

$$\text{Perimeter} = 2a + 2\left(\frac{2a}{a - 2}\right) = \frac{2a(a - 2) + 2(2a)}{a - 2} = \frac{2a^2}{a - 2}.$$

To see that a must be greater than 2, consider $R = \{(x, y): 0 \le x \le a, 0 \le y \le f(x)\}$.

f attains its maximum value on $[0, a]$ at some point $P(x_0, y_0)$, as indicated in the figure.



Draw segments $\overline{OP}$ and $\overline{PA}$. The region R is bounded by the rectangle $0 \le x \le a, 0 \le y \le y_0$, so $\text{area}(R) = k \le ay_0$. Furthermore, from the figure, $y_0 < \overline{OP}$ and $y_0 < \overline{PA}$. So, $k = \text{Perimeter}(R) > \overline{OP} + \overline{PA} > 2y_0$. Combining, $2y_0 < k \le ay_0 \Rightarrow a > 2$.

Section 3.2 Rolle's Theorem and the Mean Value Theorem

1. Let f be continuous on $[a, b]$ and differentiable on (a, b). Rolle's Theorem says that if $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.

2. Let f be continuous on $[a, b]$ and differentiable on (a, b). The Mean Value Theorem says that there is at least one number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

3. $f(x) = \left|\frac{1}{x}\right|$

$f(-1) = f(1) = 1$. But, f is not continuous on $[-1, 1]$.

4. Rolle's Theorem does not apply to $f(x) = \cot(x/2)$ over $[\pi, 3\pi]$ because f is not continuous at $x = 2\pi$.

5. Rolle's Theorem does not apply to $f(x) = 1 - |x - 1|$ over $[0, 2]$ because f is not differentiable at $x = 1$.

6. $f(x) = \sqrt{\left(2 - x^{2/3}\right)^3}$

$f(-1) = f(1) = 1$

$$f'(x) = \frac{-\sqrt{\left(2 - x^{2/3}\right)}}{x^{1/3}}$$

f is not differentiable at $x = 0$.

7. $f(x) = x^2 - x - 2 = (x - 2)(x + 1)$

x-intercepts: $(-1, 0), (2, 0)$

$f'(x) = 2x - 1 = 0$ at $x = \frac{1}{2}$.

8. $f(x) = x^2 + 6x = x(x + 6)$

x-intercepts: $(0, 0), (-6, 0)$

$f'(x) = 2x + 6 = 0$ at $x = -3$.

9. $f(x) = x\sqrt{x + 4}$

x-intercepts: $(-4, 0), (0, 0)$

$$f'(x) = x\frac{1}{2}(x + 4)^{-1/2} + (x + 4)^{1/2}$$

$$= (x + 4)^{-1/2}\left(\frac{x}{2} + (x + 4)\right)$$

$$f'(x) = \left(\frac{3}{2}x + 4\right)(x + 4)^{-1/2} = 0 \text{ at } x = -\frac{8}{3}$$

10. $f(x) = -3x\sqrt{x + 1}$

x-intercepts: $(-1, 0), (0, 0)$

$$f'(x) = -3x\frac{1}{2}(x + 1)^{-1/2} - 3(x + 1)^{1/2}$$

$$= -3(x + 1)^{-1/2}\left(\frac{x}{2} + (x + 1)\right)$$

$$f'(x) = -3(x + 1)^{-1/2}\left(\frac{3}{2}x + 1\right) = 0 \text{ at } x = -\frac{2}{3}$$

11. $f(x) = -x^2 + 3x, \quad [0, 3]$

$f(0) = -(0)^2 + 3(0)$

$f(3) = -(3)^2 + 3(3) = 0$

f is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Rolle's Theorem applies.

$f'(x) = -2x + 3 = 0$

$-2x = -3 \Rightarrow x = \frac{3}{2}$

c-value: $\frac{3}{2}$

12. $f(x) = x^2 - 8x + 5, [2, 6]$

$f(2) = 4 - 16 + 5 = -7$

$f(6) = 36 - 48 + 5 = -7$

f is continuous on $[2, 6]$ and differentiable on $(2, 6)$. Rolle's Theorem applies.

$f'(x) = 2x - 8 = 0$

$2x = 8 \Rightarrow x = 4$

c-value: 4

13. $f(x) = (x - 1)(x - 2)(x - 3), [1, 3]$

$f(1) = (1 - 1)(1 - 2)(1 - 3) = 0$

$f(3) = (3 - 1)(3 - 2)(3 - 3) = 0$

f is continuous on $[1, 3]$. f is differentiable on $(1, 3)$. Rolle's Theorem applies.

$f(x) = x^3 - 6x^2 + 11x - 6$

$f'(x) = 3x^2 - 12x + 11 = 0$

$$x = \frac{6 \pm \sqrt{3}}{3}$$

c-values: $\frac{6 - \sqrt{3}}{3}, \frac{6 + \sqrt{3}}{3}$

14. $f(x) = (x - 4)(x + 2)^2, [-2, 4]$

$f(-2) = (-2 - 4)(-2 + 2)^2 = 0$

$f(4) = (4 - 4)(4 + 2)^2 = 0$

f is continuous on $[-2, 4]$. f is differentiable on $(-2, 4)$. Rolle's Theorem applies.

$f(x) = (x - 4)(x^2 + 4x + 4) = x^3 - 12x - 16$

$f'(x) = 3x^2 - 12 = 0$

$3x^2 = 12$

$x^2 = 4 \Rightarrow x = \pm 2$

(Note: $x = -2$ is not in the interval.)

c-value: 2

15. $f(x) = x^{2/3} - 1, [-8, 8]$

$f(-8) = (-8)^{2/3} - 1 = 3$

$f(8) = (8)^{2/3} - 1 = 3$

f is continuous on $[-8, 8]$. f is not differentiable on $(-8, 8)$ because $f'(0)$ does not exist. Rolle's Theorem does not apply.

16. $f(x) = 3 - |x - 3|, [0, 6]$

$f(0) = f(6) = 0$

f is continuous on $[0, 6]$. f is not differentiable on $(0, 6)$ because $f'(3)$ does not exist. Rolle's Theorem does not apply.

17. $f(x) = \dfrac{x^2 - 2x - 3}{x + 2}, [-1, 3]$

$f(-1) = \dfrac{(-1)^2 - 2(-1) - 3}{-1 + 2} = 0$

$f(3) = \dfrac{3^2 - 2(3) - 3}{3 + 2} = 0$

f is continuous on $[-1, 3]$.

(**Note:** The discontinuity $x = -2$, is not in the interval.) f is differentiable on $(-1, 3)$. Rolle's Theorem applies.

$$f'(x) = \frac{(x + 2)(2x - 2) - (x^2 - 2x - 3)(1)}{(x + 2)^2} = 0$$

$$\frac{x^2 + 4x - 1}{(x + 2)^2} = 0$$

$$x = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$$

(**Note:** $x = -2 - \sqrt{5}$ is not in the interval.)

c-value: $-2 + \sqrt{5}$

18. $f(x) = \dfrac{x^2 - 4}{x - 1}, [-2, 2]$

$f(-2) = \dfrac{(-2)^2 - 4}{-2 - 1} = 0$

$f(2) = \dfrac{2^2 - 4}{2 - 1} = 0$

f is not continuous on $[-2, 2]$ because $f(1)$ does not exist. Rolle's Theorem does not apply.

19. $f(x) = \sin x, [0, 2\pi]$

$f(0) = \sin 0 = 0$

$f(2\pi) = \sin(2\pi) = 0$

f is continuous on $[0, 2\pi]$. f is differentiable on $(0, 2\pi)$. Rolle's Theorem applies.

$f'(x) = \cos x = 0 \Rightarrow x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

c-values: $\dfrac{\pi}{2}, \dfrac{3\pi}{2}$

20. $f(x) = \cos x, [\pi, 3\pi]$

$f(\pi) = f(3\pi) = -1$

f is continuous on $[\pi, 3\pi]$ and differentiable on $(\pi, 3\pi)$. Rolle's Theorem applies.

$f'(x) = -\sin x = 0 \Rightarrow x = \pi$

c-value: π

21. $f(x) = \cos \pi x, [0, 2]$

$f(0) = 1 = f(2)$

f is continuous on $[0, 2]$ and differentiable on $(0, 2)$.

Rolle's Theorem applies.

$f'(x) = -\pi \sin \pi x = 0 \Rightarrow x = 1$

c-value: 1

22. $f(x) = \sin 3x, \left[\frac{\pi}{2}, \frac{7\pi}{6}\right]$

$f\left(\frac{\pi}{2}\right) = -1 = f\left(\frac{7\pi}{6}\right)$

f is continuous on $\left[\frac{\pi}{2}, \frac{7\pi}{6}\right]$ and differentiable on $\left(\frac{\pi}{2}, \frac{7\pi}{6}\right)$. Rolle's Theorem applies.

$f'(x) = 3\cos 3x = 0 \Rightarrow 3x = \frac{5\pi}{2} \Rightarrow x = \frac{5\pi}{6}$

c-value: $\frac{5\pi}{6}$

23. $f(x) = \tan x, [0, \pi]$

$f(0) = \tan 0 = 0$

$f(\pi) = \tan \pi = 0$

f is not continuous on $[0, \pi]$ because $f(\pi/2)$ does not exist. Rolle's Theorem does not apply.

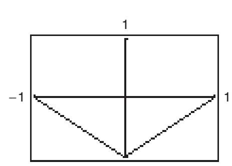
24. $f(x) = \sec x, [\pi, 2\pi]$

f is not continuous on $[\pi, 2\pi]$ because $f(3\pi/2) = \sec(3\pi/2)$ does not exist. Rolle's Theorem does not apply.

25. $f(x) = |x| - 1, [-1, 1]$

$f(-1) = f(1) = 0$

f is continuous on $[-1, 1]$. f is not differentiable on $(-1, 1)$ because $f'(0)$ does not exist. Rolle's Theorem does not apply.



26. $f(x) = x - x^{-1/3}, [0, 1]$

$f(0) = f(1) = 0$

f is continuous on $[0, 1]$. f is differentiable on $(0, 1)$. (**Note:** f is not differentiable at $x = 0$.)
Rolle's Theorem applies.

$$f'(x) = 1 - \frac{1}{3\sqrt[3]{x^2}} = 0$$

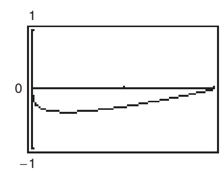
$$1 = \frac{1}{3\sqrt[3]{x^2}}$$

$$\sqrt[3]{x^2} = \frac{1}{3}$$

$$x^2 = \frac{1}{27}$$

$$x = \sqrt{\frac{1}{27}} = \frac{\sqrt{3}}{9}$$

c-value: $\frac{\sqrt{3}}{9} \approx 0.1925$



27. $f(x) = x - x^{-1/3}, [0, 1]$

$f(0) = f(1) = 0$

f is continuous on $[-1, 0]$. f is differentiable on $(-1, 0)$.
Rolle's Theorem applies.

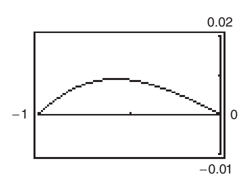
$$f'(x) = \frac{1}{2} - \frac{\pi}{6}\cos\frac{\pi x}{6} = 0$$

$$\cos\frac{\pi x}{6} = \frac{3}{\pi}$$

$$x = -\frac{6}{\pi}\arccos\frac{3}{\pi}\ \left[\text{Value needed in } (-1, 0).\right]$$

$$\approx -0.5756 \text{ radian}$$

c-value: –0.5756

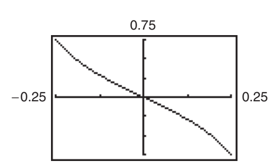


28. $f(x) = x - \tan \pi x, \left[-\frac{1}{4}, \frac{1}{4}\right]$

$f\left(-\frac{1}{4}\right) = -\frac{1}{4} + 1 = \frac{3}{4}$

$f\left(\frac{1}{4}\right) = \frac{1}{4} - 1 = -\frac{3}{4}$

Rolle's Theorem does not apply.



29. $f(t) = -16t^2 + 48t + 6$

(a) $f(1) = f(2) = 38$

(b) $v = f'(t)$ must be 0 at some time in $(1, 2)$.

$$f'(t) = -32t + 48 = 0$$
$$t = \tfrac{3}{2} \text{ sec}$$

30. $C(x) = 10\left(\dfrac{1}{x} + \dfrac{x}{x+3}\right)$

(a) $C(3) = C(6) = \dfrac{25}{3}$

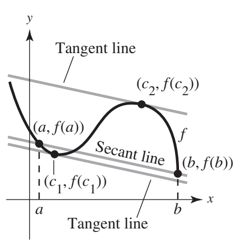
(b) $$C'(x) = 10\left(-\frac{1}{x^2} + \frac{3}{(x+3)^2}\right) = 0$$

$$\frac{3}{x^2 + 6x + 9} = \frac{1}{x^2}$$
$$2x^2 - 6x - 9 = 0$$
$$x = \frac{6 \pm \sqrt{108}}{4}$$
$$= \frac{6 \pm 6\sqrt{3}}{4} = \frac{3 \pm 3\sqrt{3}}{2}$$

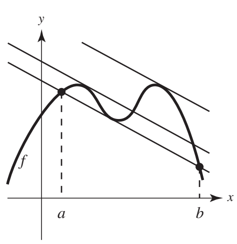
In the interval

$(3, 6)$: $c = \dfrac{3 + 3\sqrt{3}}{2} \approx 4.098 \approx 410$ components

31.



32.



33. f is not continuous on the interval $[0, 6]$. (f is not continuous at $x = 2$.)

34. f is not differentiable at $x = 2$. The graph of f is not smooth at $x = 2$.

35. $f(x) = \dfrac{1}{x-3}, [0, 6]$

f has a discontinuity at $x = 3$.

36. $f(x) = |x - 3|, [0, 6]$

f is not differentiable at $x = 3$.

37. $f(x) = -x^2 + 5$

(a) Slope $= \dfrac{1 - 4}{2 + 1} = -1$

Secant line: $y - 4 = -(x + 1)$

$$y = -x + 3$$
$$x + y - 3 = 0$$

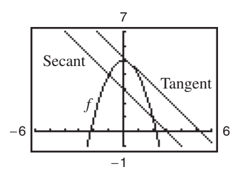
(b) $f'(x) = -2x = -1 \;\Rightarrow\; x = c = \dfrac{1}{2}$

(c) $f(c) = f\left(\dfrac{1}{2}\right) = -\dfrac{1}{4} + 5 = \dfrac{19}{4}$

Tangent line: $y - \dfrac{19}{4} = -\left(x - \dfrac{1}{2}\right)$

$$4y - 19 = -4x + 2$$
$$4x + 4y - 21 = 0$$

(d)



38. $f(x) = x^2 - x - 12$

(a) Slope $= \dfrac{-6 - 0}{-2 - 4} = 1$

Secant line: $y - 0 = x - 4$

$$x - y - 4 = 0$$

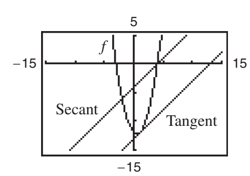
(b) $f'(x) = 2x - 1 = 1 \;\Rightarrow\; x = c = 1$

(c) $f(c) = f(1) = -12$

Tangent line: $y + 12 = x - 1$

$$x - y - 13 = 0$$

(d)



39. $f(x) = 6x^3$ is continuous on $[1, 2]$ and differentiable on $(1, 2)$.

$$\frac{f(2) - f(1)}{2 - 1} = \frac{48 - 6}{1} = 42$$

$$f'(x) = 18x^2 = 42$$

$$x^2 = \frac{7}{3} \Rightarrow x = \pm\sqrt{\frac{7}{3}}$$

On the interval $(1, 2)$, $c = \sqrt{\frac{7}{3}} = \frac{\sqrt{21}}{3}$.

$$f'\left(\frac{\sqrt{21}}{3}\right) = 42$$

40. $f(x) = x^6$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$.

$$\frac{f(1) - f(-1)}{1 - (-1)} = 0$$

$$f'(x) = 6x^5 = 0 \Rightarrow x = 0$$

On the interval $(-1, 1)$, $c = 0$ and $f'(c) = 0$.

41. $f(x) = x^3 + 2x + 4$ is continuous on $[-1, 0]$ and differentiable on $(-1, 0)$.

$$\frac{f(0) - f(-1)}{0 - (-1)} = \frac{4 - 1}{1} = 3$$

$$f'(x) = 3x^2 + 2 = 3$$

$$3x^2 = 1$$

$$x^2 = \frac{1}{3} \Rightarrow x = \pm\frac{\sqrt{3}}{3}$$

On the interval $(-1, 0)$, $c = -\frac{\sqrt{3}}{3}$ and

$$f'\left(-\frac{\sqrt{3}}{3}\right) = 3.$$

42. $f(x) = x^3 - 3x^2 + 9x + 5$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

$$\frac{f(1) - f(0)}{1 - 0} = \frac{12 - 5}{1} = 7$$

$$f'(x) = 3x^2 - 6x + 9 = 7$$

$$3x^2 - 6x + 2 = 0$$

$$x = \frac{6 + \sqrt{36 - 24}}{6} = 1 \pm \frac{\sqrt{3}}{3}$$

On the interval $(0, 1)$, $c = 1 - \frac{\sqrt{3}}{3}$ and

$$f'\left(1 - \frac{\sqrt{3}}{3}\right) = 7.$$

43. $f(x) = \frac{x + 2}{x - 1}$ is not continuous at $x = 1$.

The Mean Value Theorem does not apply on the interval $[-3, 3]$.

44. $f(x) = \frac{x}{x - 5}$ is continuous on $[1, 4]$ and differentiable on $(1, 4)$.

$$\frac{f(4) - f(1)}{4 - 1} = \frac{-4 - \left(-\frac{1}{4}\right)}{3} = -\frac{15}{12} = -\frac{5}{4}$$

$$f'(x) = \frac{(x - 5) - x}{(x - 5)^2} = \frac{-5}{(x - 5)^2} = -\frac{5}{4}$$

$$4 = (x - 5)^2$$

$$\pm 2 = x - 5$$

$$x = 5 \pm 2 \Rightarrow x = 3 \ (x = 7 \text{ is outside the interval.})$$

On the interval $(1, 4)$, $c = 3$ and $f'(3) = -\frac{5}{4}$.

45. $f(x) = |2x + 1|$ is not differentiable at $x = -1/2$. The Mean Value Theorem does not apply on the interval $[-1, 3]$.

46. $f(x) = \sqrt{2 - x}$ is continuous on $[-7, 2]$ and differentiable on $(-7, 2)$.

$$\frac{f(2) - f(-7)}{2 - (-7)} = \frac{0 - 3}{9} = -\frac{1}{3}$$

$$f'(x) = \frac{-1}{2\sqrt{2 - x}} = -\frac{1}{3}$$
$$2\sqrt{2 - x} = 3$$
$$\sqrt{2 - x} = \frac{3}{2}$$
$$2 - x = \frac{9}{4}$$
$$x = -\frac{1}{4}$$

On the interval $(-1, 3)$, $c = -\frac{1}{4}$ and $f'\left(-\frac{1}{4}\right) = -\frac{1}{3}$.

47. $f(x) = \sin x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

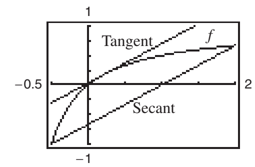
$$f'(x) = \cos x = 0$$
$$x = \frac{\pi}{2}$$

On the interval $(0, \pi)$, $c = \frac{\pi}{2}$ and $f'\left(\frac{\pi}{2}\right) = 0$.

48. $f(x) = \cos x + \tan x$ is not continuous at $x = \pi/2$. The Mean Value Theorem does not apply on the interval $[0, \pi]$.

49. $f(x) = \frac{x}{x + 1}, \left[-\frac{1}{2}, 2\right]$

(a)–(c)



(b) Secant line:

$$\text{slope} = \frac{f(2) - f(-1/2)}{2 - (-1/2)} = \frac{2/3 - (-1)}{5/2} = \frac{2}{3}$$

$$y - \frac{2}{3} = \frac{2}{3}(x - 2)$$

$$y = \frac{2}{3}(x - 1)$$

(c) $f'(x) = \frac{1}{(x + 1)^2} = \frac{2}{3}$

$$(x + 1)^2 = \frac{3}{2}$$

$$x = -1 \pm \sqrt{\frac{3}{2}} = -1 \pm \frac{\sqrt{6}}{2}$$

In the interval $[-1/2, 2]$: $c = -1 + \left(\sqrt{6}/2\right)$

$$f(c) = \frac{-1 + \left(\sqrt{6}/2\right)}{\left[-1 + \left(\sqrt{6}/2\right)\right] + 1}$$
$$= \frac{-2 + \sqrt{6}}{\sqrt{6}}$$
$$= \frac{-2}{\sqrt{6}} + 1$$

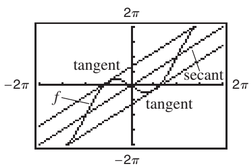
Tangent line: $y - 1 + \frac{2}{\sqrt{6}} = \frac{2}{3}\left(x - \frac{\sqrt{6}}{2} + 1\right)$

$$y - 1 + \frac{\sqrt{6}}{3} = \frac{2}{3}x - \frac{\sqrt{6}}{3} + \frac{2}{3}$$

$$y = \frac{1}{3}\left(2x + 5 - 2\sqrt{6}\right)$$

50. $f(x) = x - 2\sin x, [-\pi, \pi]$

(a)–(c)



(b) Secant line:

$$\text{slope} = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)} = \frac{\pi - (-\pi)}{2\pi} = 1$$

$$y - \pi = 1(x - \pi)$$
$$y = x$$

(c) $f'(x) = 1 - 2\cos x = 1$

$$\cos x = 0$$
$$x = c = \pm\frac{\pi}{2}$$

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 2$$

$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} + 2$$

Tangent lines: $y - \left(\frac{\pi}{2} - 2\right) = 1\left(x - \frac{\pi}{2}\right)$

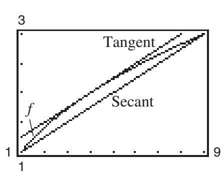
$$y = x - 2$$

$$y - \left(-\frac{\pi}{2} + 2\right) = 1\left(x + \frac{\pi}{2}\right)$$

$$y = x + 2$$

51. $f(x) = \sqrt{x}, [1, 9]$

(a)–(c)



(b) Secant line:

$$\text{slope} = \frac{f(9) - f(1)}{9 - 1} = \frac{3 - 1}{8} = \frac{1}{4}$$

$$y - 1 = \frac{1}{4}(x - 1)$$

$$y = \frac{1}{4}x + \frac{3}{4}$$

(c) $f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{4}$

$$x = c = 4$$

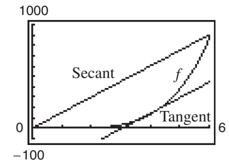
$$f(4) = 2$$

Tangent line: $y - 2 = \frac{1}{4}(x - 4)$

$$y = \frac{1}{4}x + 1$$

52. $f(x) = x^4 - 2x^3 + x^2, [0, 6]$

(a)–(c)



(b) Secant line:

$$\text{slope} = \frac{f(6) - f(0)}{6 - 0} = \frac{900 - 0}{6} = 150$$

$$y - 0 = 150(x - 0)$$
$$y = 150x$$

(c) $f'(x) = 4x^3 - 6x^2 + 2x = 150$

Using a graphing utility, there is one solution in $(0, 6)$, $x = c \approx 3.8721$ and $f(c) \approx 123.6721$

Tangent line: $y - 123.6721 = 150(x - 3.8721)$

$$y = 150x - 457.143$$

53. $s(t) = -4.9t^2 + 300$

(a) $v_{\text{avg}} = \frac{s(3) - s(0)}{3 - 0} = \frac{255.9 - 300}{3} = -14.7 \text{ m/sec}$

(b) $s(t)$ is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Therefore, the Mean Value Theorem applies.

$$v(t) = s'(t) = -9.8t = -14.7 \text{ m/sec}$$

$$t = \frac{-14.7}{-9.8} = 1.5 \text{ sec}$$

54. $S(t) = 200\left(5 - \frac{9}{2 + t}\right)$

(a) $\frac{S(12) - S(0)}{12 - 0} = \frac{200[5 - (9/14)] - 200[5 - (9/2)]}{12}$

$$= \frac{450}{7}$$

(b) $S'(t) = 200\left(\frac{9}{(2 + t)^2}\right) = \frac{450}{7}$

$$\frac{1}{(2 + t)^2} = \frac{1}{28}$$

$$2 + t = 2\sqrt{7}$$

$$t = 2\sqrt{7} - 2 \approx 3.2915 \text{ months}$$

$S'(t)$ is equal to the average value in April.

55. No. Let $f(x) = x^2$ on $[-1, 2]$.

$f'(x) = 2x$

$f'(0) = 0$ and zero is in the interval $(-1, 2)$ but $f(-1) \neq f(2)$.

56. $f(a) = f(b)$ and $f'(c) = 0$ where c is in the interval (a, b).

(a) $g(x) = f(x) + k$

$g(a) = g(b) = f(a) + k$

$g'(x) = f'(x) \Rightarrow g'(c) = 0$

Interval: $[a, b]$

Critical number of g: c

(b) $g(x) = f(x - k)$

$g(a + k) = g(b + k) = f(a)$

$g'(x) = f'(x - k)$

$g'(c + k) = f'(c) = 0$

Interval: $[a + k, b + k]$

Critical number of g: $c + k$

(c) $g(x) = f(kx)$

$g\left(\frac{a}{k}\right) = g\left(\frac{b}{k}\right) = f(a)$

$g'(x) = kf'(kx)$

$g'\left(\frac{c}{k}\right) = kf'(c) = 0$

Interval: $\left[\frac{a}{k}, \frac{b}{k}\right]$

Critical number of g: $\frac{c}{k}$

57. $f(x) = \begin{cases} 0, & x = 0 \\ 1 - x, & 0 < x \le 1 \end{cases}$

No, this does not contradict Rolle's Theorem. f is not continuous on $[0, 1]$.

58. No. If such a function existed, then the Mean Value Theorem would say that there exists $c \in (-2, 2)$ such that

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{6 + 2}{4} = 2.$$

But, $f'(x) < 1$ for all x.

59. Let $S(t)$ be the position function of the plane. If $t = 0$ corresponds to 2 P.M., $S(0) = 0$, $S(5.5) = 2500$ and the Mean Value Theorem says that there exists a time t_0, $0 < t_0 < 5.5$, such that

$$S'(t_0) = v(t_0) = \frac{2500 - 0}{5.5 - 0} \approx 454.54.$$

Applying the Intermediate Value Theorem to the velocity function on the intervals $[0, t_0]$ and $[t_0, 5.5]$, you see that there are at least two times during the flight when the speed was 400 miles per hour. $(0 < 400 < 454.54)$

60. Let $T(t)$ be the temperature of the object. Then $T(0) = 1500°$ and $T(5) = 390°$. The average temperature over the interval $[0, 5]$ is

$$\frac{390 - 1500}{5 - 0} = -222° \text{ F/h}.$$

By the Mean Value Theorem, there exist a time t_0, $0 < t_0 < 5$, such that $T'(t_0) = -222°\text{F/h}$.

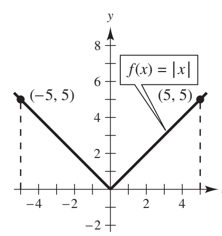
61. Let $S(t)$ be the difference in the positions of the 2 bicyclists, $S(t) = S_1(t) - S_2(t)$. Because $S(0) = S(2.25) = 0$, there must exist a time $t_0 \in (0, 2.25)$ such that $S'(t_0) = v(t_0) = 0$.

At this time, $v_1(t_0) = v_2(t_0)$.

62. Let $t = 0$ correspond to 9:13 A.M. By the Mean Value Theorem, there exists t_0 in $\left(0, \frac{1}{30}\right)$ such that

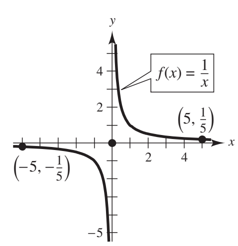
$$v'(t_0) = a(t_0) = \frac{85 - 35}{1/30} = 1500 \text{ mi/h}^2.$$

63. (a) f is continuous on $[-5, 5]$ and does not satisfy the conditions of the Mean Value Theorem. $\Rightarrow$ f is not differentiable on $(-5, 5)$. Example: $f(x) = |x|$



(b) f is not continuous on $[-5, 5]$.

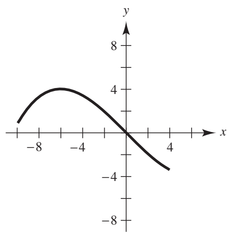
Example: $f(x) = \begin{cases} 1/x, & x \ne 0 \\ 0, & x = 0 \end{cases}$



64. (a) f is continuous on $[-10, 4]$ and changes sign, $(f(-8) > 0, f(3) < 0)$. By the Intermediate Value Theorem, there exists at least one value of x in $[-10, 4]$ satisfying $f(x) = 0$.

(b) There exist real numbers a and b such that $-10 < a < b < 4$ and $f(a) = f(b) = 2$. Therefore, by Rolle's Theorem there exists at least one number c in $(-10, 4)$ such that $f'(c) = 0$. This is called a critical number.

(c)



65. $f(x) = x^5 + x^3 + x + 1$

f is differentiable for all x.

$f(-1) = -2$ and $f(0) = 1$, so the Intermediate Value Theorem implies that f has at least one zero c in $[-1, 0]$, $f(c) = 0$.

Suppose f had 2 zeros, $f(c_1) = f(c_2) = 0$. Then Rolle's Theorem would guarantee the existence of a number a such that

$f'(a) = f(c_2) - f(c_1) = 0$.

But, $f'(x) = 5x^4 + 3x^2 + 1 > 0$ for all x. So, f has exactly one real solution.

66. $f(x) = 2x^5 + 7x - 1$

f is differentiable for all x.

$f(0) = -1$ and $f(1) = 8$, so the Intermediate Value Theorem implies that f has at least one zero c in $[0, 1]$, $f(c) = 0$.

Suppose f had 2 zeros, $f(c_1) = f(c_2) = 0$. Then Rolle's Theorem would guarantee the existence of a number a such that

$f'(a) = f(c_2) - f(c_1) = 0$.

But $f'(x) = 10x^4 + 7 > 0$ for all x. So, $f(x) = 0$ has exactly one real solution.

67. $f(x) = 3x + 1 - \sin x$

f is differentiable for all x.

$f(-\pi) = -3\pi + 1 < 0$ and $f(0) = 1 > 0$, so the Intermediate Value Theorem implies that f has at least one zero c in $[-\pi, 0]$, $f(c) = 0$.

Suppose f had 2 zeros, $f(c_1) = f(c_2) = 0$. Then Rolle's Theorem would guarantee the existence of a number a such that

$f'(a) = f(c_2) - f(c_1) = 0$.

But $f'(x) = 3 - \cos x > 0$ for all x. So, $f(x) = 0$ has exactly one real solution.

68. $f(x) = 2x - 2 - \cos x$

$f(0) = -3, f(\pi) = 2\pi - 2 + 1 = 2\pi - 1 > 0$. By the Intermediate Value Theorem, f has at least one zero.

Suppose f had 2 zeros, $f(c_1) = f(c_2) = 0$. Then Rolle's Theorem would guarantee the existence of a number a such that

$f'(a) = f(c_2) - f(c_1) = 0$.

But, $f'(x) = 2 + \sin x \geq 1$ for all x. So, f has exactly one real solution.

69. $f'(x) = 0$

$f(x) = c$

$f(2) = 5$

So, $f(x) = 5$.

70. $f'(x) = 4$

$f(x) = 4x + c$

$f(0) = 1 \Rightarrow c = 1$

So, $f(x) = 4x + 1$.

71. $f'(x) = 2x$

$f(x) = x^2 + c$

$f(1) = 0 \Rightarrow 0 = 1 + c \Rightarrow c = -1$

So, $f(x) = x^2 - 1$.

72. $f'(x) = 6x - 1$

$f(x) = 3x^2 - x + c$

$f(2) = 7 \Rightarrow 7 = 3(2^2) - 2 + c = 10 + c \Rightarrow c = -3$

So, $f(x) = 3x^2 - x - 3$.

73. False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

74. False. f must also be continuous *and* differentiable on each interval. For example, consider the function

$$f(x) = \frac{x^3 - 4x}{x^2 - 1}.$$

75. True. A polynomial is continuous and differentiable everywhere.

76. True. The tangent function is continuous on $\left[0, \frac{\pi}{4}\right]$ and differentiable on $\left(0, \frac{\pi}{4}\right)$.

77. Suppose that $p(x) = x^{2n+1} + ax + b$ has two real roots x_1 and x_2. Then by Rolle's Theorem, because $p(x_1) = p(x_2) = 0$, there exists c in (x_1, x_2) such that $p'(c) = 0$. But $p'(x) = (2n + 1)x^{2n} + a \neq 0$, because $n > 0, a > 0$. Therefore, $p(x)$ cannot have two real roots.

78. Suppose $f(x)$ is not constant on (a, b). Then there exists x_1 and x_2 in (a, b) such that $f(x_1) \neq f(x_2)$. Then by the Mean Value Theorem, there exists c in (a, b) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0.$$

This contradicts the fact that $f'(x) = 0$ for all x in (a, b).

79. If $p(x) = Ax^2 + Bx + C$, then

$$\begin{aligned} p'(x) = 2Ax + B &= \frac{f(b) - f(a)}{b - a} \\ &= \frac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b - a} \\ &= \frac{A(b^2 - a^2) + B(b - a)}{b - a} \\ &= \frac{(b - a)[A(b + a) + B]}{b - a} \\ &= A(b + a) + B. \end{aligned}$$

So, $2Ax = A(b + a)$ and $x = (b + a)/2$ which is the midpoint of $[a, b]$.

80. (a) $f(x) = x^2, g(x) = -x^3 + x^2 + 3x + 2$

$f(-1) = g(-1) = 1, f(2) = g(2) = 4$

Let $h(x) = f(x) - g(x)$. Then, $h(-1) = h(2) = 0$. So, by Rolle's Theorem these exists $c \in (-1, 2)$ such that

$h'(c) = f'(c) - g'(c) = 0.$

So, at $x = c$, the tangent line to f is parallel to the tangent line to g.

$h(x) = x^3 - 3x - 2, h'(x)$

$= 3x^2 - 3 = 0 \Rightarrow x = c = 1$

(b) Let $h(x) = f(x) - g(x)$. Then $h(a) = h(b) = 0$ by Rolle's Theorem, there exists c in (a, b) such that

$h'(c) = f'(c) - g'(c) = 0.$

So, at $x = c$, the tangent line to f is parallel to the tangent line to g.

81. Suppose $f(x)$ has two fixed points c_1 and c_2. Then, by the Mean Value Theorem, there exists c such that

$$f'(c) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = \frac{c_2 - c_1}{c_2 - c_1} = 1.$$

This contradicts the fact that $f'(x) < 1$ for all x.

82. $f(x) = \frac{1}{2}\cos x$ differentiable on $(-\infty, \infty)$.

$f'(x) = -\frac{1}{2}\sin x$

$-\frac{1}{2} \leq f'(x) \leq \frac{1}{2} \Rightarrow f'(x) < 1$ for all real numbers.

So, from Exercise 62, f has, at most, one fixed point. $(x \approx 0.4502)$

83. Let $f(x) = \cos x$. f is continuous and differentiable for all real numbers. By the Mean Value Theorem, for any interval $[a, b]$, there exists c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\frac{\cos b - \cos a}{b - a} = -\sin c$$

$$\cos b - \cos a = (-\sin c)(b - a)$$

$$|\cos b - \cos a| = |-\sin c||b - a|$$

$$|\cos b - \cos a| \leq |b - a| \text{ since } |-\sin c| \leq 1.$$

84. Let $f(x) = \sin x. f$ is continuous and differentiable for all real numbers. By the Mean Value Theorem, for any interval $[a, b]$, there exists c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$
$$\sin(b) - \sin(a) = (b - a)\cos(c)$$
$$|\sin(b) - \sin(a)| = |b - a||\cos(c)|$$
$$|\sin a - \sin b| \le |a - b|.$$

85. Let $0 < a < b$. $f(x) = \sqrt{x}$ satisfies the hypotheses of the Mean Value Theorem on $[a, b]$. Hence, there exists c in (a, b) such that

$$f'(c) = \frac{1}{2\sqrt{c}} = \frac{f(b) - f(a)}{b - a} = \frac{\sqrt{b} - \sqrt{a}}{b - a}.$$

So, $\sqrt{b} - \sqrt{a} = (b - a)\dfrac{1}{2\sqrt{c}} < \dfrac{b - a}{2\sqrt{a}}$.

Section 3.3 Increasing and Decreasing Functions and the First Derivative Test

1. A positive derivative of a function on an open interval implies that the function is increasing on the interval. A negative derivative implies that the function is decreasing. A zero derivative implies that the function is constant.

2. If c is in an open interval I and is a critical number of a function f, which is differentiable on I, then the function has a relative maximum at $(c, f(c))$ if f is increasing immediately to the left of c and decreasing immediately to the right of c. Similarly, the function has a relative minimum at $(c, f(c))$ if f is decreasing immediately to the left of c and increasing immediately to the right of c. If f is increasing both to the left and right of c or decreasing both to the left and right of c, then the function has neither a relative minimum or a relative maximum.

3. (a) Increasing: $(0, 6)$ and $(8, 9)$. Largest: $(0, 6)$

(b) Decreasing: $(6, 8)$ and $(9, 10)$. Largest: $(6, 8)$

4. (a) Increasing: $(4, 5), (6, 7)$. Largest: $(4, 5), (6, 7)$

(b) Decreasing: $(-3, 1), (1, 4), (5, 6)$. Largest: $(-3, 1)$

5. $y = -(x + 1)^2$

From the graph, f is increasing on $(-\infty, -1)$ and decreasing on $(-1, \infty)$.

Analytically, $y' = -2(x + 1)$.

Critical number: $x = -1$

Test intervals:	$-\infty < x < -1$	$-1 < x < \infty$
Sign of y':	$y' > 0$	$y' < 0$
Conclusion:	Increasing	Decreasing

6. $f(x) = x^2 - 6x + 8$

From the graph, f is decreasing on $(-\infty, 3)$ and increasing on $(3, \infty)$.

Analytically, $f'(x) = 2x - 6$.

Critical number: $x = 3$

Test intervals:	$-\infty < x < 3$	$3 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

7. $y = \dfrac{x^3}{4} - 3x$

From the graph, y is increasing on $(-\infty, -2)$ and $(2, \infty)$, and decreasing on $(-2, 2)$.

Analytically, $y' = \dfrac{3x^2}{4} - 3 = \dfrac{3}{4}(x^2 - 4) = \dfrac{3}{4}(x - 2)(x + 2)$

Critical numbers: $x = \pm 2$

Test intervals:	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Sign of y':	$y' > 0$	$y' < 0$	$y' > 0$
Conclusion:	Increasing	Decreasing	Increasing

8. $f(x) = x^4 - 2x^2$

From the graph, f is decreasing on $(-\infty, -1)$ and $(0, 1)$, and increasing on $(-1, 0)$ and $(1, \infty)$.

Analytically, $f'(x) = 4x^3 - 4x = 4x(x - 1)(x + 1)$.

Critical numbers: $x = 0, \pm 1$.

Test intervals:	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Sign of f':	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing	Decreasing	Increasing

9. $f(x) = \dfrac{1}{(x + 1)^2}$

From the graph, f is increasing on $(-\infty, -1)$ and decreasing on $(-1, \infty)$.

Analytically, $f'(x) = \dfrac{-2}{(x + 1)^3}$.

No critical numbers. Discontinuity: $x = -1$

Test intervals:	$-\infty < x < -1$	$-1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

10. $y = \dfrac{x^2}{2x - 1}$

From the graph, y is increasing on $(-\infty, 0)$ and $(1, \infty)$, and decreasing on $(0, 1/2)$ and $(1/2, 1)$.

Analytically, $y' = \dfrac{(2x - 1)2x - x^2(2)}{(2x - 1)^2} = \dfrac{2x^2 - 2x}{(2x - 1)^2} = \dfrac{2x(x - 1)}{(2x - 1)^2}$

Critical numbers: $x = 0, 1$

Discontinuity: $x = 1/2$

Test intervals:	$-\infty < x < 0$	$0 < x < 1/2$	$1/2 < x < 1$	$1 < x < \infty$
Sign of y':	$y' > 0$	$y' < 0$	$y' < 0$	$y' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

11. $g(x) = x^2 - 2x - 8$

$g'(x) = 2x - 2$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $g'(x)$:	$g' < 0$	$g' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(1, \infty)$

Decreasing on: $(-\infty, 1)$

12. $h(x) = 12x - x^3$

$h'(x) = 12 - 3x^2 = 3(4 - x^2) = 3(2 - x)(2 + x)$

Critical numbers: $x = \pm 2$

Test intervals:	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Sign of $h'(x)$:	$h' < 0$	$h' > 0$	$h' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(-2, 2)$

Decreasing on: $(-\infty, -2), (2, \infty)$

13. $y = x\sqrt{16 - x^2}$ Domain: $[-4, 4]$

$$y' = \frac{-2(x^2 - 8)}{\sqrt{16 - x^2}} = \frac{-2}{\sqrt{16 - x^2}}(x - 2\sqrt{2})(x + 2\sqrt{2})$$

Critical numbers: $x = \pm 2\sqrt{2}$

Test intervals:	$-4 < x < -2\sqrt{2}$	$-2\sqrt{2} < x < 2\sqrt{2}$	$2\sqrt{2} < x < 4$
Sign of y':	$y' < 0$	$y' > 0$	$y' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(-2\sqrt{2}, 2\sqrt{2})$

Decreasing on: $(-4, -2\sqrt{2}), (2\sqrt{2}, 4)$

14. $y = x + \dfrac{9}{x}$

$$y' = \frac{1 - 9}{x^2} = \frac{x^2 - 9}{x^2} = \frac{(x - 3)(x + 3)}{x^2}$$

Critical numbers: $x = \pm 3$

Discontinuity: $x = 0$

Test intervals:	$-\infty < x < -3$	$-3 < x < 0$	$0 < x < 3$	$3 < x < \infty$
Sign of y':	$y' > 0$	$y' < 0$	$y' < 0$	$y' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on: $(-\infty, -3), (3, \infty)$

Decreasing on: $(-3, 0), (0, 3)$

15. $f(x) = \sin x - 1, \quad 0 < x < 2\pi$

$f'(x) = \cos x$

Critical numbers: $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

Test intervals:	$0 < x < \dfrac{\pi}{2}$	$\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$	$\dfrac{3\pi}{2} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

16. $f(x) = \cos\dfrac{3x}{2}, \ 0 < x < 2\pi$

$f'(x) = -\dfrac{3}{2}\sin\dfrac{3x}{2} = 0 \Rightarrow \dfrac{3x}{2} = n\pi \Rightarrow x = \dfrac{2}{3}n\pi$

In the interval $0 < x < 2\pi$, the critical numbers are $\dfrac{2\pi}{3}$ and $\dfrac{4\pi}{3}$.

Test intervals:	$0 < x < \dfrac{2\pi}{3}$	$\dfrac{2\pi}{3} < x < \dfrac{4\pi}{3}$	$\dfrac{4\pi}{3} < x < 2\pi$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $\left(\dfrac{2\pi}{3}, \dfrac{4\pi}{3}\right)$

Decreasing on: $\left(0, \dfrac{2\pi}{3}\right), \left(\dfrac{4\pi}{3}, 2\pi\right)$

17. $y = x - 2\cos x, \quad 0 < x < 2\pi$

$y' = 1 + 2\sin x$

$y' = 0\text{: } \sin x = -\dfrac{1}{2}$

Critical numbers: $x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

Test intervals:	$0 < x < \dfrac{7\pi}{6}$	$\dfrac{7\pi}{6} < x < \dfrac{11\pi}{6}$	$\dfrac{11\pi}{6} < x < 2\pi$
Sign of y':	$y' > 0$	$y' < 0$	$y' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \dfrac{7\pi}{6}\right), \left(\dfrac{11\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\dfrac{7\pi}{6}, \dfrac{11\pi}{6}\right)$

18. $f(x) = \sin^2 x + \sin x,\ 0 < x < 2\pi$

$f'(x) = 2\sin x\cos x + \cos x = \cos x(2\sin x + 1)$

$2\sin x + 1 = 0 \Rightarrow \sin x = -\frac{1}{2} \Rightarrow x = \frac{7\pi}{6}, \frac{11\pi}{6}$

$\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

Critical numbers: $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{7\pi}{6}$	$\frac{7\pi}{6} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < \frac{11\pi}{6}$	$\frac{11\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \frac{\pi}{2}\right), \left(\frac{7\pi}{6}, \frac{3\pi}{2}\right), \left(\frac{11\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\frac{\pi}{2}, \frac{7\pi}{6}\right), \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)$

19. (a) $f(x) = x^2 - 8x$

$f'(x) = 2x - 8$

Critical number: $x = 4$

(b)

Test intervals:	$-\infty < x < 4$	$4 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Decreasing on: $(-\infty, 4)$

Increasing on: $(4, \infty)$

(c) Relative minimum: $(4, -16)$

20. (a) $f(x) = x^2 + 6x + 10$

$f'(x) = 2x + 6$

Critical number: $x = -3$

(b)

Test intervals:	$-\infty < x < -3$	$-3 < x < \infty$
Sign of f':	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Decreasing on: $(-\infty, -3)$

Increasing on: $(-3, \infty)$

(c) Relative minimum: $(-3, 1)$

21. (a) $f(x) = -2x^2 + 4x + 3$

$f'(x) = -4x + 4 = 0$

Critical number: $x = 1$

(b)

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, 1)$

Decreasing on: $(1, \infty)$

(c) Relative maximum: $(1, 5)$

22. (a) $f(x) = -3x^2 - 4x - 2$

$f'(x) = -6x - 4 = 0$

Critical number: $x = -\frac{2}{3}$

(b)

Test intervals:	$-\infty < x < -\frac{2}{3}$	$-\frac{2}{3} < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $\left(-\infty, -\frac{2}{3}\right)$

Decreasing on: $\left(-\frac{2}{3}, \infty\right)$

(c) Relative maximum: $\left(-\frac{2}{3}, -\frac{2}{3}\right)$

23. (a) $f(x) = -7x^3 + 21x + 3$

$f'(x) = -21x^2 + 21 = 0$

$x^2 = 1$

Critical numbers: $x = \pm 1$

(b)

Test intervals:	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(-1, 1)$

Decreasing on: $(-\infty, -1), (1, \infty)$

(c) Relative minimum: $(-1, -11)$

Relative maximum: $(1, 17)$

24. (a) $f(x) = x^3 - 6x^2 + 15$

$f'(x) = 3x^2 - 12x = 3x(x - 4)$

Critical numbers: $x = 0, 4$

(b)

Test intervals:	$-\infty < x < 0$	$0 < x < 4$	$4 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, 0), (4, \infty)$

Decreasing on: $(0, 4)$

(c) Relative maximum: $(0, 15)$

Relative minimum: $(4, -17)$

25. (a) $f(x) = (x - 1)^2(x + 3) = x^3 + x^2 - 5x + 3$

$f'(x) = 3x^2 + 2x - 5 = (x - 1)(3x + 5)$

Critical numbers: $x = 1, -\frac{5}{3}$

(b)

Test intervals:	$-\infty < x < -\frac{5}{3}$	$-5/3 < x < 1$	$1 < x < \infty$
Sign of f':	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $\left(-\infty, -\frac{5}{3}\right)$ and $(1, \infty)$

Decreasing on: $\left(-\frac{5}{3}, 1\right)$

(c) Relative maximum: $\left(-\frac{5}{3}, \frac{256}{27}\right)$

Relative minimum: $(1, 0)$

26. (a) $f(x) = (8 - x)(x + 1)^2$

$f'(x) = (8 - x)2(x + 1) + (x + 1)^2(-1)$

$= (x + 1)(16 - 2x - x - 1)$

$= (x + 1)(15 - 3x)$

$= -3(x + 1)(x - 5)$

Critical numbers: $x = -1, 5$

(b)

Test intervals:	$-\infty < x < -1$	$-1 < x < 5$	$5 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(-1, 5)$

Decreasing on: $(-\infty, -1), (5, \infty)$

(c) Relative maximum: $(5, 108)$

Relative minimum: $(-1, 0)$

27. (a) $f(x) = \dfrac{x^5 - 5x}{5}$

$f'(x) = x^4 - 1$

Critical numbers: $x = -1, 1$

(b)

Test intervals:	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -1), (1, \infty)$

Decreasing on: $(-1, 1)$

(c) Relative maximum: $\left(-1, \dfrac{4}{5}\right)$

Relative minimum: $\left(1, -\dfrac{4}{5}\right)$

28. (a) $f(x) = \dfrac{-x^6 + 6x}{10}$

$f'(x) = \dfrac{1}{10}(6 - 6x^5) = \dfrac{3}{5}(1 - x^5)$

Critical number: $x = 1$

(b)

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Decreasing on: $(1, \infty)$

Increasing on: $(-\infty, 1)$

(c) Relative maximum: $\left(1, \dfrac{1}{2}\right)$

29. (a) $f(x) = x^{1/3} + 1$

$f'(x) = \dfrac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$

Critical number: $x = 0$

(b)

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, \infty)$

(c) No relative extrema

30. (a) $f(x) = x^{2/3} - 4$

$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$

Critical number: $x = 0$

(b)

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(0, \infty)$

Decreasing on: $(-\infty, 0)$

(c) Relative minimum: $(0, -4)$

31. (a) $f(x) = (x + 2)^{2/3}$

$f'(x) = \frac{2}{3}(x + 2)^{-1/3} = \frac{2}{3(x + 2)^{1/3}}$

Critical number: $x = -2$

(b)

Test intervals:	$-\infty < x < -2$	$-2 < x < \infty$
Sign of f':	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Decreasing on: $(-\infty, -2)$

Increasing on: $(-2, \infty)$

(c) Relative minimum: $(-2, 0)$

32. (a) $f(x) = (x - 3)^{1/3}$

$f'(x) = \frac{1}{3}(x - 3)^{-2/3} = \frac{1}{3(x - 3)^{2/3}}$

Critical number: $x = 3$

(b)

Test intervals:	$-\infty < x < 3$	$3 < x < \infty$
Sign of f':	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, \infty)$

(c) No relative extrema

33. (a) $f(x) = 5 - |x - 5|$

$f'(x) = -\frac{x - 5}{|x - 5|} = \begin{cases} 1, & x < 5 \\ -1, & x > 5 \end{cases}$

Critical number: $x = 5$

(b)

Test intervals:	$-\infty < x < 5$	$5 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, 5)$

Decreasing on: $(5, \infty)$

(c) Relative maximum: $(5, 5)$

34. (a) $f(x) = |x + 3| - 1$

$f'(x) = \frac{x + 3}{|x + 3|} = \begin{cases} 1, & x > -3 \\ -1, & x < -3 \end{cases}$

Critical number: $x = -3$

(b)

Test intervals:	$-\infty < x < -3$	$-3 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(-3, \infty)$

Decreasing on: $(-\infty, -3)$

(c) Relative minimum:

35. (a) $f(x) = 2x + \dfrac{1}{x}$

$f'(x) = 2 - \dfrac{1}{x^2} = \dfrac{2x^2 - 1}{x^2}$

Critical numbers: $x = \pm\dfrac{\sqrt{2}}{2}$

Discontinuity: $x = 0$

(b)

Test intervals:	$-\infty < x < -\dfrac{\sqrt{2}}{2}$	$-\dfrac{\sqrt{2}}{2} < x < 0$	$0 < x < \dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{2}}{2} < x < \infty$
Sign of f':	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on: $\left(-\infty, -\dfrac{\sqrt{2}}{2}\right)$ and $\left(\dfrac{\sqrt{2}}{2}, \infty\right)$

Decreasing on: $\left(-\dfrac{\sqrt{2}}{2}, 0\right)$ and $\left(0, \dfrac{\sqrt{2}}{2}\right)$

(c) Relative maximum: $\left(-\dfrac{\sqrt{2}}{2}, -2\sqrt{2}\right)$

Relative minimum: $\left(\dfrac{\sqrt{2}}{2}, 2\sqrt{2}\right)$

36. (a) $f(x) = \dfrac{x}{x - 5}$

$f'(x) = \dfrac{(x - 5) - x}{(x - 5)^2} = \dfrac{-5}{(x - 5)^2}$

No critical numbers

Discontinuity: $x = 5$

(b)

Test intervals:	$-\infty < x < 5$	$5 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' < 0$
Conclusion:	Decreasing	Decreasing

Decreasing on: $(-\infty, 5), (5, \infty)$

(c) No relative extrema

37. (a) $f(x) = \dfrac{x^2}{x^2 - 9}$

$f'(x) = \dfrac{(x^2 - 9)(2x) - (x^2)(2x)}{(x^2 - 9)^2} = \dfrac{-18x}{(x^2 - 9)^2}$

Critical number: $x = 0$

Discontinuities: $x = -3, 3$

(b)

Test intervals:	$-\infty < x < -3$	$-3 < x < 0$	$0 < x < 3$	$3 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$	$f' < 0$	$f' < 0$
Conclusion:	Increasing	Increasing	Decreasing	Decreasing

Increasing on: $(-\infty, -3), (-3, 0)$

Decreasing on: $(0, 3), (3, \infty)$

(c) Relative maximum: $(0, 0)$

38. (a) $f(x) = \dfrac{x^2 - 2x + 1}{x + 1}$

$$f'(x) = \frac{(x + 1)(2x - 2) - (x^2 - 2x + 1)(1)}{(x + 1)^2} = \frac{x^2 + 2x - 3}{(x + 1)^2} = \frac{(x + 3)(x - 1)}{(x + 1)^2}$$

Critical numbers: $x = -3, 1$

Discontinuity: $x = -1$

(b)

Test intervals:	$-\infty < x < -3$	$-3 < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on: $(-\infty, -3), (1, \infty)$

Decreasing on: $(-3, -1), (-1, 1)$

(c) Relative maximum: $(-3, -8)$

Relative minimum: $(1, 0)$

39. (a) $f(x) = \begin{cases} 4 - x^2, & x \le 0 \\ -2x, & x > 0 \end{cases}$

$$f'(x) = \begin{cases} -2x, & x < 0 \\ -2, & x > 0 \end{cases}$$

Critical number: $x = 0$

(b)

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of f':	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, 0)$

Decreasing on: $(0, \infty)$

(c) Relative maximum: $(0, 4)$

40. (a) $f(x) = \begin{cases} 2x + 1, & x \le -1 \\ x^2 - 2, & x > -1 \end{cases}$

$f'(x) = \begin{cases} 2, & x < -1 \\ 2x, & x > -1 \end{cases}$

Critical numbers: $x = -1, 0$

(b)

Test intervals:	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < \infty$
Sign of f':	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -1)$ and $(0, \infty)$

Decreasing on: $(-1, 0)$

(c) Relative maximum: $(-1, -1)$

Relative minimum: $(0, -2)$

41. (a) $f(x) = x - 2 \sin x, (0, 2\pi)$

$f'(x) = 1 - 2 \cos x$

Critical numbers: $x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

Test intervals:	$0 < x < \dfrac{\pi}{3}$	$\dfrac{\pi}{3} < x < \dfrac{5\pi}{3}$	$\dfrac{5\pi}{3} < x < 2\pi$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

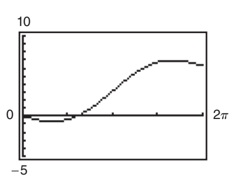
Increasing on: $\left(\dfrac{\pi}{3}, \dfrac{5\pi}{3}\right)$

Decreasing on: $\left(0, \dfrac{\pi}{3}\right), \left(\dfrac{5\pi}{3}, 2\pi\right)$

(b) Relative minimum: $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3} - \sqrt{3}\right)$

Relative maximum: $\left(\dfrac{5\pi}{3}, \dfrac{5\pi}{3} + \sqrt{3}\right)$

(c)



42. (a) $f(x) = \sin x \cos x + 5 = \frac{1}{2}\sin 2x + 5, 0 < x < 2\pi$

$f'(x) = \cos 2x$

Critical numbers: $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Test intervals:	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < \frac{7\pi}{4}$	$\frac{7\pi}{4} < x < 2\pi$
Sign of f':	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

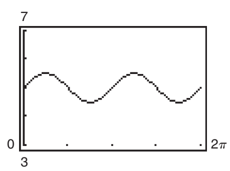
Increasing on: $\left(0, \frac{\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{7\pi}{4}, 2\pi\right)$

Decreasing on: $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right), \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

(b) Relative maxima: $\left(\frac{\pi}{4}, \frac{11}{2}\right), \left(\frac{5\pi}{4}, \frac{11}{2}\right)$

Relative minima: $\left(\frac{3\pi}{4}, \frac{9}{2}\right), \left(\frac{7\pi}{4}, \frac{9}{2}\right)$

(c)



43. (a) $f(x) = \sin x + \cos x, \quad 0 < x < 2\pi$

$f'(x) = \cos x - \sin x = 0 \Rightarrow \sin x = \cos x$

Critical numbers: $x = \frac{\pi}{4}, \frac{5\pi}{4}$

Test intervals:	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

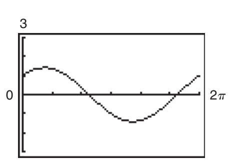
Increasing on: $\left(0, \frac{\pi}{4}\right), \left(\frac{5\pi}{4}, 2\pi\right)$

Decreasing on: $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

(b) Relative maximum: $\left(\frac{\pi}{4}, \sqrt{2}\right)$

Relative minimum: $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$

(c)



44. (a) $f(x) = \dfrac{x}{2} + \cos x, 0 < x < 2\pi$

$f'(x) = \dfrac{1}{2} - \sin x = 0$

Critical numbers: $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

Test intervals:	$0 < x < \dfrac{\pi}{6}$	$\dfrac{\pi}{6} < x < \dfrac{5\pi}{6}$	$\dfrac{5\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

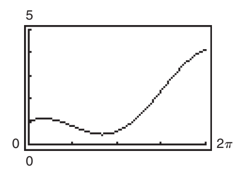
Increasing on: $\left(0, \dfrac{\pi}{6}\right), \left(\dfrac{5\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{6}, \dfrac{5\pi}{6}\right)$

(b) Relative maximum: $\left(\dfrac{\pi}{6}, \dfrac{\pi + 6\sqrt{3}}{12}\right)$

Relative minimum: $\left(\dfrac{5\pi}{6}, \dfrac{5\pi - 6\sqrt{3}}{12}\right)$

(c)



45. (a) $f(x) = \cos^2(2x), \quad 0 < x < 2\pi$

$f'(x) = -4\cos 2x \sin 2x = 0 \Rightarrow \cos 2x = 0$ or $\sin 2x = 0$

Critical numbers: $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$

Test intervals:	$0 < x < \dfrac{\pi}{4}$	$\dfrac{\pi}{4} < x < \dfrac{\pi}{2}$	$\dfrac{\pi}{2} < x < \dfrac{3\pi}{4}$	$\dfrac{3\pi}{4} < x < \pi$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing	Decreasing	Increasing

Test intervals:	$\pi < x < \dfrac{5\pi}{4}$	$\dfrac{5\pi}{4} < x < \dfrac{3\pi}{2}$	$\dfrac{3\pi}{2} < x < \dfrac{7\pi}{4}$	$\dfrac{7\pi}{4} < x < 2\pi$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing	Decreasing	Increasing

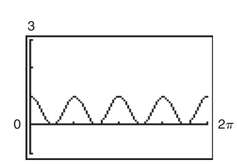
Increasing on: $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{4}, \pi\right), \left(\dfrac{5\pi}{4}, \dfrac{3\pi}{2}\right), \left(\dfrac{7\pi}{4}, 2\pi\right)$

Decreasing on: $\left(0, \dfrac{\pi}{4}\right), \left(\dfrac{\pi}{2}, \dfrac{3\pi}{4}\right), \left(\pi, \dfrac{5\pi}{4}\right), \left(\dfrac{3\pi}{2}, \dfrac{7\pi}{4}\right)$

(b) Relative maxima: $\left(\dfrac{\pi}{2}, 1\right), (\pi, 1), \left(\dfrac{3\pi}{2}, 1\right)$

Relative minima: $\left(\dfrac{\pi}{4}, 0\right), \left(\dfrac{3\pi}{4}, 0\right), \left(\dfrac{5\pi}{4}, 0\right), \left(\dfrac{7\pi}{4}, 0\right)$

(c)



46. (a) $f(x) = \sin x - \sqrt{3}\cos x,\ 0 < x < 2\pi$

$f'(x) = \cos x + \sqrt{3}\sin x = 0 \Rightarrow \sqrt{3}\sin x = -\cos x$

$\tan x = \dfrac{-1}{\sqrt{3}} = \dfrac{-\sqrt{3}}{3}$

Critical numbers: $x = \dfrac{5\pi}{6}, \dfrac{11\pi}{6}$

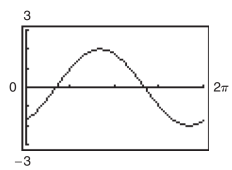
Test intervals:	$0 < x < \frac{5\pi}{6}$	$\frac{5\pi}{6} < x < \frac{11\pi}{6}$	$\frac{11\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \dfrac{5\pi}{6}\right), \left(\dfrac{11\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\dfrac{5\pi}{6}, \dfrac{11\pi}{6}\right)$

(b) Relative maximum: $\left(\dfrac{5\pi}{6}, 2\right)$

Relative minimum: $\left(\dfrac{11\pi}{6}, -2\right)$

(c) 

47. (a) $f(x) = \sin^2 x + \sin x, \quad 0 < x < 2\pi$

$f'(x) = 2\sin x\cos x + \cos x = \cos x(2\sin x + 1) = 0$

Critical numbers: $x = \dfrac{\pi}{2}, \dfrac{7\pi}{6}, \dfrac{3\pi}{2}, \dfrac{11\pi}{6}$

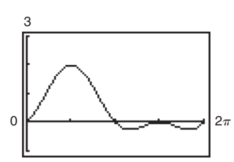
Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{7\pi}{6}$	$\frac{7\pi}{6} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < \frac{11\pi}{6}$	$\frac{11\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \dfrac{\pi}{2}\right), \left(\dfrac{7\pi}{6}, \dfrac{3\pi}{2}\right), \left(\dfrac{11\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{2}, \dfrac{7\pi}{6}\right), \left(\dfrac{3\pi}{2}, \dfrac{11\pi}{6}\right)$

(b) Relative minima: $\left(\dfrac{7\pi}{6}, -\dfrac{1}{4}\right), \left(\dfrac{11\pi}{6}, -\dfrac{1}{4}\right)$

Relative maxima: $\left(\dfrac{\pi}{2}, 2\right), \left(\dfrac{3\pi}{2}, 0\right)$

(c) 

48. (a) $f(x) = \dfrac{\sin x}{1 + \cos^2 x}, 0 < x < 2\pi$

$$f'(x) = \frac{\cos x(2 + \sin^2 x)}{(1 + \cos^2 x)^2} = 0$$

Critical numbers: $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

Test intervals:	$0 < x < \dfrac{\pi}{2}$	$\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$	$\dfrac{3\pi}{2} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

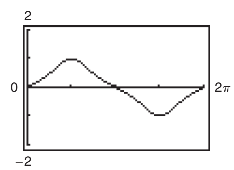
Increasing on: $\left(0, \dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

(b) Relative maximum: $\left(\dfrac{\pi}{2}, 1\right)$

Relative minimum: $\left(\dfrac{3\pi}{2}, -1\right)$

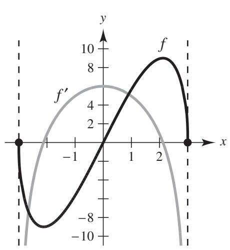
(c)



49. $f(x) = 2x\sqrt{9 - x^2}, [-3, 3]$

(a) $f'(x) = \dfrac{2(9 - 2x^2)}{\sqrt{9 - x^2}}$

(b)



(c) $\dfrac{2(9 - 2x^2)}{\sqrt{9 - x^2}} = 0$

Critical numbers: $x = \pm\dfrac{3}{\sqrt{2}} = \pm\dfrac{3\sqrt{2}}{2}$

(d) Intervals:

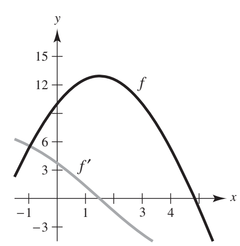
$\left(-3, -\dfrac{3\sqrt{2}}{2}\right)$	$\left(-\dfrac{3\sqrt{2}}{2}, \dfrac{3\sqrt{2}}{2}\right)$	$\left(\dfrac{3\sqrt{2}}{2}, 3\right)$
$f'(x) < 0$	$f'(x) > 0$	$f'(x) < 0$
Decreasing	Increasing	Decreasing

f is increasing when f' is positive and decreasing when f' is negative.

50. $f(x) = 10\left(5 - \sqrt{x^2 - 3x + 16}\right), [0, 5]$

(a) $f'(x) = -\dfrac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}}$

(b)



(c) $-\dfrac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}} = 0$

Critical number: $x = \dfrac{3}{2}$

(d) Intervals:

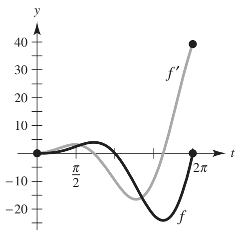
$\left(0, \dfrac{3}{2}\right)$	$\left(\dfrac{3}{2}, 5\right)$
$f'(x) > 0$	$f'(x) < 0$
Increasing	Decreasing

f is increasing when f' is positive and decreasing when f' is negative.

51. $f(t) = t^2 \sin t, [0, 2\pi]$

(a) $f'(t) = t^2 \cos t + 2t \sin t = t(t \cos t + 2 \sin t)$

(b)



(c)

$$\begin{aligned} t(t\cos t + 2\sin t) &= 0 \\ t = 0 \text{ or } t &= -2\tan t \\ t\cot t &= -2 \\ t &\approx 2.2889, 5.0870 \text{ (graphing utility)} \end{aligned}$$

Critical numbers: $t = 2.2889, 5.0870$

(d) Intervals:

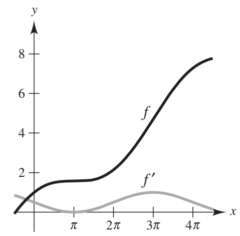
$(0, 2.2889)$	$(2.2889, 5.0870)$	$(5.0870, 2\pi)$
$f'(t) > 0$	$f'(t) < 0$	$f'(t) > 0$
Increasing	Decreasing	Increasing

f is increasing when f' is positive and decreasing when f' is negative.

52. $f(x) = \dfrac{x}{2} + \cos \dfrac{x}{2}, [0, 4\pi]$

(a) $f'(x) = \dfrac{1}{2} - \dfrac{1}{2} \sin \dfrac{x}{2}$

(b)



(c)
$$\begin{aligned} \frac{1}{2} - \frac{1}{2}\sin\frac{x}{2} &= 0 \\ \sin\frac{x}{2} &= 1 \\ \frac{x}{2} &= \frac{\pi}{2} \end{aligned}$$

Critical number: $x = \pi$

(d) Intervals:

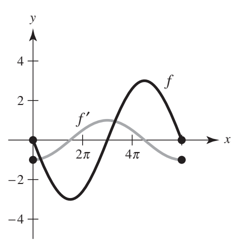
$(0, \pi)$	$(\pi, 4\pi)$
$f'(x) > 0$	$f'(x) > 0$
Increasing	Increasing

f is increasing when f' is positive.

53. (a) $f(x) = -3 \sin \dfrac{x}{3}, [0, 6\pi]$

$f'(x) = -\cos \dfrac{x}{3}$

(b)



(c) Critical numbers: $x = \dfrac{3\pi}{2}, \dfrac{9\pi}{2}$

(d) Intervals:

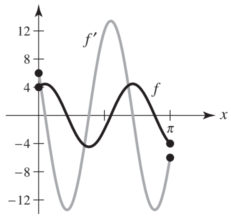
$\left(0, \frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2}, \frac{9\pi}{2}\right)$	$\left(\frac{9\pi}{2}, 6\pi\right)$
$f' < 0$	$f' > 0$	$f' < 0$
Decreasing	Increasing	Decreasing

f is increasing when f' is positive and decreasing when f' is negative.

54. (a) $f(x) = 2 \sin 3x + 4 \cos 3x, [0, \pi]$

$f'(x) = 6 \cos 3x - 12 \sin 3x$

(b)



(c) $f'(x) = 0 \Rightarrow \tan 3x = \dfrac{1}{2}$

Critical numbers: $x \approx 0.1545, 1.2017, 2.2489$

(d) Intervals:

$(0, 0.1545)$	$(0.1545, 1.2017)$	$(1.2017, 2.2489)$	$(2.2489, \pi)$
$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$
Increasing	Decreasing	Increasing	Decreasing

f is increasing when f' is positive and decreasing when f' is negative.

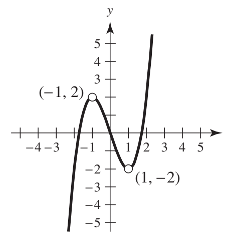
55. $f(x) = \dfrac{x^5 - 4x^3 + 3x}{x^2 - 1} = \dfrac{(x^2 - 1)(x^3 - 3x)}{x^2 - 1} = x^3 - 3x,\ x \neq \pm 1$

$f(x) = g(x) = x^3 - 3x$ for all $x \neq \pm 1$.

$f'(x) = 3x^2 - 3 = 3(x^2 - 1),\ x \neq \pm 1 \Rightarrow f'(x) \neq 0$

f symmetric about origin

zeros of f: $(0, 0), (\pm\sqrt{3}, 0)$



$g(x)$ is continuous on $(-\infty, \infty)$ and $f(x)$ has holes at $(-1, 2)$ and $(1, -2)$.

56. $f(t) = \cos^2 t - \sin^2 t = 1 - 2\sin^2 t = g(t)$

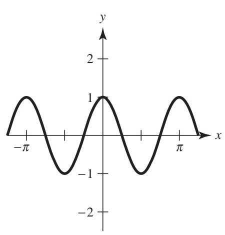
$f'(t) = -4\sin t\cos t = -2\sin 2t$

f symmetric with respect to y-axis

zeros of f: $\pm\dfrac{\pi}{4}$

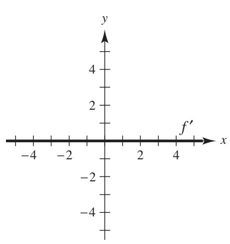
Relative maximum: $(0, 1)$

Relative minimum: $\left(-\dfrac{\pi}{2}, -1\right), \left(\dfrac{\pi}{2}, -1\right)$

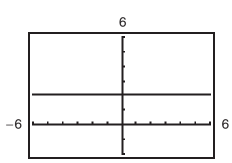


The graphs of $f(x)$ and $g(x)$ are the same.

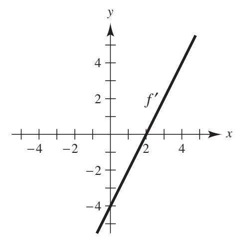
57. $f(x) = c$ is constant $\Rightarrow f'(x) = 0$.



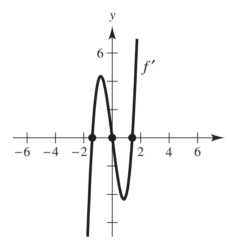
58. $f(x)$ is a line of slope $\approx 2 \Rightarrow f'(x) = 2$.



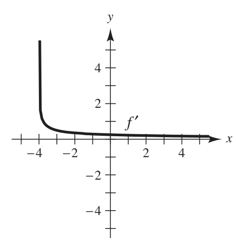
59. f is quadratic $\Rightarrow f'$ is a line.



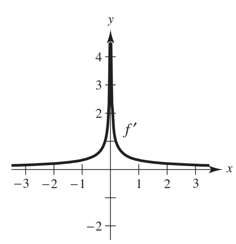
60. f is a 4th degree polynomial $\Rightarrow f'$ is a cubic polynomial.



61. f has positive, but decreasing slope.



62. f has positive slope.



In Exercises 63–66, $f'(x) > 0$ on $(-\infty, -4)$, $f'(x) < 0$ on $(-4, 6)$ and $f'(x) > 0$ on $(6, \infty)$.

63. $g(x) = f(x) + 5$

$g'(x) = f'(x)$

$g'(0) = f'(0) < 0$

64. $g(x) = 3f(x) - 3$

$g'(x) = 3f'(x)$

$g'(-5) = 3f'(-5) > 0$

65. $g(x) = -f(x)$

$g'(x) = -f'(x)$

$g'(-6) = -f'(-6) < 0$

66. $g(x) = f(x - 10)$

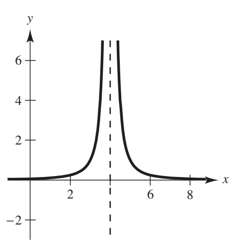
$g'(x) = f'(x - 10)$

$g'(0) = f'(-10) > 0$

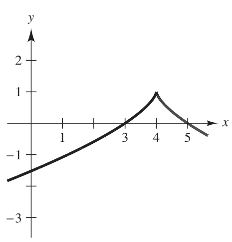
67. $f'(x)\begin{cases} > 0, & x < 4 \Rightarrow f \text{ is increasing on } (-\infty, 4). \\ \text{undefined}, & x = 4 \\ < 0, & x > 4 \Rightarrow f \text{ is decreasing on } (4\ \infty). \end{cases}$

Two possibilities for $f(x)$ are given below.

(a)



(b)



68. Yes. If $h(x) = f(x) + g(x)$ where f and g are increasing, then $h'(x) = f'(x) + g'(x) > 0$.

So, h is increasing.

69. No. For example, the product of $f(x) = x$ and $g(x) = x$ is $f(x) \cdot g(x) = x^2$, which is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

70. (i) (a) Critical number: $x = 2$ (Because $f'(2) = 0$)

(b) f increasing on

$(2, \infty)$ (Because $f' > 0$ on $(2, \infty)$)

f decreasing on

$(-\infty, 2)$ (Because $f' < 0$ on $(-\infty, 2)$)

(c) f has a relative minimum at $x = 2$.

(ii) (a) Critical numbers:
$x = 0, 1$ (Because $f'(1) = 0$)

(b) f increasing on $(-\infty, 0)$ and $(1, \infty)$

(Because $f' > 0$ on these intervals)

f decreasing on

$(0, 1)$ (Because $f' < 0$ on $(0, 1)$)

(c) f has a relative maximum at $x = 0$, and a relative minimum at $x = 1$.

(iii) (a) Critical numbers: $x = -1, 0, 1$

(Because $f'(-1) = f'(0) = f'(1) = 0$)

(b) f increasing on $(-\infty, -1)$ and $(0, 1)$

(Because $f' > 0$ on these intervals)

f decreasing on $(-1, 0)$ and $(1, \infty)$

(Because $f' < 0$ on these intervals)

(c) f has a relative maximum at $x = -1$ and $x = 1$. f has a relative minimum at $x = 0$.

(iv) (a) Critical numbers: $x = -3, 1, 5$

(Because $f'(-3) = f'(1) = f'(5) = 0$)

(b) f increasing on $(-3, 1)$ and $(1, 5)$

(Because $f' > 0$ on these intervals). In fact, f is increasing on $(-3, 5)$.

f decreasing on $(-\infty, -3)$ and $(5, \infty)$

(Because $f' < 0$ on these intervals)

(c) f has a relative minimum at $x = -3$, and a relative maximum at $x = 5$.

$x = 1$ is not a relative extremum.

71. Critical number: $x = 5$

$f'(4) = -2.5 \Rightarrow f$ is decreasing at $x = 4$.

$f'(6) = 3 \Rightarrow f$ is increasing at $x = 6$.

$(5, f(5))$ is a relative minimum.

72. Critical number: $x = 2$

$f'(1) = 2 \Rightarrow f$ is increasing at $x = 1$.

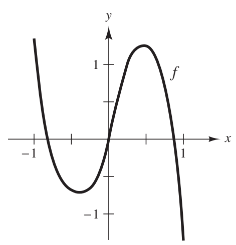
$f'(3) = 6 \Rightarrow f$ is increasing at $x = 3$.

$(2, f(2))$ is not a relative extremum.

In Exercises 73 and 74, answers will vary.

Sample answers:

73. (a)

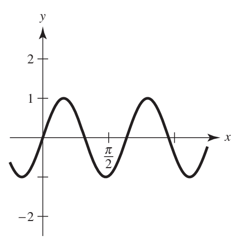


(b) The critical numbers are in intervals $(-0.50, -0.25)$ and $(0.25, 0.50)$ because the sign of f' changes in these intervals. f is decreasing on approximately $(-1, -0.40)$, $(0.48, 1)$, and increasing on $(-0.40, 0.48)$.

(c) Relative minimum when $x \approx -0.40$: $(-0.40, 0.75)$

Relative maximum when $x \approx 0.48$: $(0.48, 1.25)$

74. (a)



(b) The critical numbers are in the intervals $\left(0, \frac{\pi}{6}\right), \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, and $\left(\frac{3\pi}{4}, \frac{5\pi}{6}\right)$ because the sign of f' changes in these intervals. f is increasing on approximately $\left(0, \frac{\pi}{7}\right)$ and $\left(\frac{3\pi}{7}, \frac{6\pi}{7}\right)$ and decreasing on $\left(\frac{\pi}{7}, \frac{3\pi}{7}\right)$ and $\left(\frac{6\pi}{7}, \pi\right)$.

(c) Relative minima when $x \approx \frac{3\pi}{7}, \pi$

Relative maxima when $x \approx \frac{\pi}{7}, \frac{6\pi}{7}$

75. $s(t) = 4.9(\sin\theta)t^2$

(a) $s'(t) = 4.9(\sin\theta)(2t) = 9.8(\sin\theta)t$

speed $= |s'(t)| = |9.8(\sin\theta)t|$

(b)

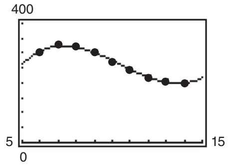
θ	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π
$\lvert s'(t)\rvert$	0	$4.9\sqrt{2}t$	$4.9\sqrt{3}t$	$9.8t$	$4.9\sqrt{3}t$	$4.9\sqrt{2}t$	0

The speed is maximum for $\theta = \frac{\pi}{2}$.

76. (a) Using a graphing utility,

$M = 0.9954t^3 - 31.267t^2 + 297.38t - 566.9.$

(b)



(c) $M'(t) = 2.9862t^2 - 62.534t + 297.38$

Using a graphing utility, $M'(t) = 0$ for $t \approx 7.3$ and $t \approx 13.6$.

The maximum is approximately $(7.3, 325.0)$, which compares well with the actual maximum: $(7, 326)$.

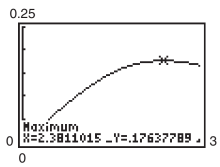
77. $C = \dfrac{3t}{27 + t^3}, t \geq 0$

(a)

t	0	0.5	1	1.5	2	2.5	3
$C(t)$	0	0.055	0.107	0.148	0.171	0.176	0.167

The concentration seems greatest near $t = 2.5$ hours.

(b)



The concentration is greatest when $t \approx 2.38$ hours.

(c) $C' = \dfrac{(27 + t^3)(3) - (3t)(3t^2)}{(27 + t^3)^2} = \dfrac{3(27 - 2t^3)}{(27 + t^3)^2}$

$C' = 0$ when $t = 3/\sqrt[3]{2} \approx 2.38$ hours.

By the First Derivative Test, this is a maximum.

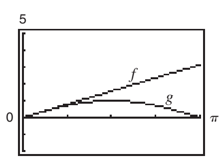
78. $f(x) = x, g(x) = \sin x, 0 < x < \pi$

(a)

x	0.5	1	1.5	2	2.5	3
$f(x)$	0.5	1	1.5	2	2.5	3
$g(x)$	0.479	0.841	0.997	0.909	0.598	0.141

$f(x)$ seems greater than $g(x)$ on $(0, \pi)$.

(b)



$x > \sin x$ on $(0, \pi)$ so, $f(x) > g(x)$.

(c) Let $h(x) = f(x) - g(x) = x - \sin x$

$h'(x) = 1 - \cos x > 0$ on $(0, \pi)$.

Therefore, $h(x)$ is increasing on $(0, \pi)$. Because $h(0) = 0$ and $h'(x) > 0$ on $(0, \pi)$,

$$h(x) > 0$$
$$x - \sin x > 0$$
$$x > \sin x$$
$$f(x) > g(x) \text{ on } (0, \pi)$$

79. $v = k(R - r)r^2 = k(Rr^2 - r^3)$

$v' = k(2Rr - 3r^2) = kr(2R - 3r) = 0$

$r = 0$ or $\frac{2}{3}R$

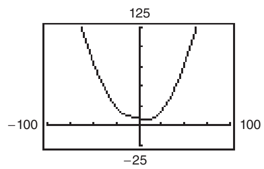
Maximum when $r = \frac{2}{3}R$.

80. $R = \sqrt{0.001T^4 - 4T + 100}$

(a) $R' = \dfrac{0.004T^3 - 4}{2\sqrt{0.001T^4 - 4T + 100}} = 0$

Critical number: $T = 10°$

Minimum resistance: $R \approx 8.3666$ ohms

(b) 

The minimum resistance is approximately $R \approx 8.37$ ohms at $T = 10°$.

81. (a) $s(t) = 6t - t^2, t \geq 0$

$v(t) = 6 - 2t$

(b) $v(t) = 0$ when $t = 3$.

Moving in positive direction for $0 \leq t < 3$ because $v(t) > 0$ on $0 \leq t < 3$.

(c) Moving in negative direction when $t > 3$.

(d) The particle changes direction at $t = 3$.

82. (a) $s(t) = t^2 - 10t + 29, t \geq 0$

$v(t) = 2t - 10$

(b) $v(t) = 0$ when $t = 5$

Particle moving in positive direction for $t > 5$ because $v'(t) > 0$ on $(5, \infty)$.

(c) Particle moving in negative direction on $[0, 5)$.

(d) The particle changes direction at $t = 5$.

83. (a) $s(t) = t^3 - 5t^2 + 4t, t \geq 0$

$v(t) = 3t^2 - 10t + 4$

(b) $v(t) = 0$ for $t = \dfrac{10 \pm \sqrt{100 - 48}}{6} = \dfrac{5 \pm \sqrt{13}}{3}$

Particle is moving in a positive direction on

$\left[0, \dfrac{5 - \sqrt{13}}{3}\right) \approx [0, 0.4648)$ and $\left(\dfrac{5 + \sqrt{13}}{3}, \infty\right) \approx (2.8685, \infty)$ because $v > 0$ on these intervals.

(c) Particle is moving in a negative direction on

$\left(\dfrac{5 - \sqrt{13}}{3}, \dfrac{5 + \sqrt{13}}{3}\right) \approx (0.4648, 2.8685)$

(d) The particle changes direction at $t = \dfrac{5 \pm \sqrt{13}}{3}$.

84. (a) $s(t) = t^3 - 20t^2 + 128t - 280$

$v(t) = 3t^2 - 40t + 128$

(b) $v(t) = (3t - 16)(t - 8)$

$v(t) = 0$ when $t = \frac{16}{3}, 8$

$v(t) > 0$ for $\left[0, \frac{16}{3}\right)$ and $(8, \infty)$

(c) $v(t) < 0$ for $\left(\frac{16}{3}, 8\right)$

(d) The particle changes direction at $t = \frac{16}{3}$ and 8.

85. Answers will vary.

86. Answers will vary.

87. (a) Use a cubic polynomial

$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

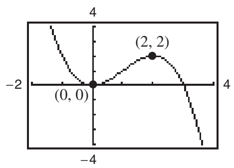
(b) $f'(x) = 3a_3x^2 + 2a_2x + a_1.$

$$\begin{aligned}
f(0) = 0:&\quad a_3(0)^3 + a_2(0)^2 + a_1(0) + a_0 = 0 \Rightarrow &a_0 &= 0\\
f'(0) = 0:&\quad 3a_3(0)^2 + 2a_2(0) + a_1 = 0 \Rightarrow &a_1 &= 0\\
f(2) = 2:&\quad a_3(2)^3 + a_2(2)^2 + a_1(2) + a_0 = 2 \Rightarrow &8a_3 + 4a_2 &= 2\\
f'(2) = 0:&\quad 3a_3(2)^2 + 2a_2(2) + a_1 = 0 \Rightarrow &12a_3 + 4a_2 &= 0
\end{aligned}$$

(c) The solution is $a_0 = a_1 = 0, a_2 = \frac{3}{2}, a_3 = -\frac{1}{2}$:

$f(x) = -\frac{1}{2}x^3 + \frac{3}{2}x^2.$

(d)



88. (a) Use a cubic polynomial

$f(x) = 3a_3x^3 + a_2x^2 + a_1x + a_0$

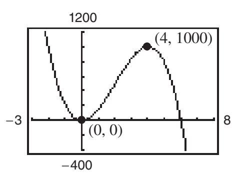
(b) $f'(x) = 3a_3x^2 + 2a_2x + a_1$

$$\begin{aligned}
f(0) = 0:&\quad a_3(0)^3 + a_2(0)^2 + a_1(0) + a_0 = 0 \Rightarrow &a_0 &= 0\\
f'(0) = 0:&\quad 3a_3(0)^2 + 2a_2(0) + a_1 = 0 \Rightarrow &a_1 &= 0\\
f(4) = 1000:&\quad a_3(4)^3 + a_2(4)^2 + a_1(4) + a_0 = 1000 \Rightarrow &64a_3 + 16a_2 &= 100\\
f'(4) = 0:&\quad 3a_3(4)^2 + 2a_2(4) + a_1 = 0 \Rightarrow &48a_3 + 8a_2 &= 0
\end{aligned}$$

(c) The solution is $a_0 = a_1 = 0, a_2 = \frac{375}{2}, a_3 = -\frac{125}{4}$

$f(x) = -\frac{125}{4}x^3 + \frac{375}{2}x^2.$

(d)



89. (a) Use a fourth degree polynomial

$f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$

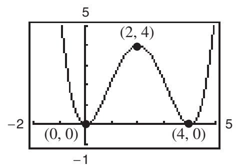
(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

$$\begin{aligned}
f(0) = 0:&\quad a_4(0)^4 + a_3(0)^3 + a_2(0)^2 + a_1(0) + a_0 = 0 \Rightarrow &a_0 &= 0\\
f'(0) = 0:&\quad 4a_4(0)^3 + 3a_3(0)^2 + 2a_2(0) + a_1 = 0 \Rightarrow &a_1 &= 0\\
f(4) = 0:&\quad a_4(4)^4 + a_3(4)^3 + a_2(4)^2 + a_1(4) + a_0 = 0 \Rightarrow &256a_4 + 64a_3 + 16a_2 &= 0\\
f'(4) = 0:&\quad 4a_4(4)^3 + 3a_3(4)^2 + 2a_2(4) + a_1 = 0 \Rightarrow &256a_4 + 48a_3 + 8a_2 &= 0\\
f(2) = 4:&\quad a_4(2)^4 + a_3(2)^3 + a_2(2)^2 + a_1(2) + a_0 = 4 \Rightarrow &16a_4 + 8a_3 + 4a_2 &= 4\\
f'(2) = 0:&\quad 4a_4(2)^3 + 3a_3(2)^2 + 2a_2(2) + a_1 = 0 \Rightarrow &32a_4 + 12a_3 + 4a_2 &= 0
\end{aligned}$$

(c) The solution is $a_0 = a_1 = 0, a_2 = 4, a_3 = -2, \quad a_4 = \frac{1}{4}$.

$f(x) = \frac{1}{4}x^4 - 2x^3 + 4x^2$

(d)



90. (a) Use a fourth-degree polynomial

$f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

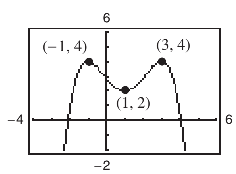
(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

$$
\begin{aligned}
f(1) = 2: &\quad a_4(1)^4 + a_3(1)^3 + a_2(1)^2 + a_1(1) + a_0 = 2 \Rightarrow &\quad a_4 + a_3 + a_2 + a_1 + a_0 = 2\\
f'(1) = 0: &\quad 4a_4(1)^3 + 3a_3(1)^2 + 2a_2(1) + a_1 = 0 \Rightarrow &\quad 4a_4 + 3a_3 + 2a_2 + a_1 = 0\\
f(-1) = 4: &\quad a_4(-1)^4 + a_3(-1)^3 + a_2(-1)^2 + a_1(-1) + a_0 = 4 \Rightarrow &\quad a_4 - a_3 + a_2 - a_1 + a_0 = 4\\
f'(-1) = 0: &\quad 4a_4(-1)^3 + 3a_3(-1)^2 + 2a_2(-1) + a_1 = 0 \Rightarrow &\quad -4a_4 + 3a_3 - 2a_2 + a_1 = 0\\
f(3) = 4: &\quad a_4(3)^4 + a_3(3)^3 + a_2(3)^2 + a_1(3) + a_0 = 4 \Rightarrow &\quad 81a_4 + 27a_3 + 9a_2 + a_1 + a_0 = 4\\
f'(3) = 0: &\quad 4a_4(3)^3 + 3a_3(3)^2 + 2a_2(3) + a_1 = 0 \Rightarrow &\quad 108a_4 + 27a_3 + 6a_2 + a_1 = 0
\end{aligned}
$$

(c) The solution is $a_0 = \frac{23}{8}, a_1 = -\frac{3}{2}, a_2 = \frac{1}{4}, a_3 = \frac{1}{2}, a_4 = -\frac{1}{8}$

$f(x) = -\frac{1}{8}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2 - \frac{3}{2}x + \frac{23}{8}$.

(d)



91. False. For example, $f(x) = \sin x$ has an infinite number of critical points at $x = n\pi$, where n is an integer.

92. True. $f'(x) = 1$ for all x.

93. False.

Let $f(x) = x^3$, then $f'(x) = 3x^2$ and f only has one critical number. Or, let $f(x) = x^3 + 3x + 1$, then $f'(x) = 3(x^2 + 1)$ has no critical numbers.

94. True.

If $f(x)$ is an nth-degree polynomial, then the degree of $f'(x)$ is $n - 1$.

95. False. For example, $f(x) = x^3$ does not have a relative extrema at the critical number $x = 0$.

96. False. The function might not be continuous on the interval.

97. Assume that $f'(x) < 0$ for all x in the interval (a, b) and let $x_1 < x_2$ be any two points in the interval. By the Mean Value Theorem, you know there exists a number c such that $x_1 < c < x_2$, and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Because $f'(c) < 0$ and $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) < 0$, which implies that $f(x_2) < f(x_1)$. So, f is decreasing on the interval.

98. Suppose $f'(x)$ changes from positive to negative at c. Then there exists a and b in I such that $f'(x) > 0$ for all x in (a, c) and $f'(x) < 0$ for all x in (c, b). By Theorem 3.5, f is increasing on (a, c) and decreasing on (c, b). Therefore, $f(c)$ is a maximum of f on (a, b) and so, a relative maximum of f.

99. Let x_1 and x_2 be two real numbers, $x_1 < x_2$. Then $x_1^3 < x_2^3 \Rightarrow f(x_1) < f(x_2)$. So f is increasing on $(-\infty, \infty)$.

100. Let x_1 and x_2 be two positive real numbers, $0 < x_1 < x_2$. Then

$$\frac{1}{x_1} > \frac{1}{x_2}$$

$$f(x_1) > f(x_2)$$

So, f is decreasing on $(0, \infty)$.

101. First observe that

$$\begin{aligned}
\tan x + \cot x + \sec x + \csc x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} + \frac{1}{\cos x} + \frac{1}{\sin x} \\
&= \frac{\sin^2 x + \cos^2 x + \sin x + \cos x}{\sin x \cos x} \\
&= \frac{1 + \sin x + \cos x}{\sin x \cos x}\left(\frac{\sin x + \cos x - 1}{\sin x + \cos x - 1}\right) \\
&= \frac{(\sin x + \cos x)^2 - 1}{\sin x \cos x(\sin x + \cos x - 1)} \\
&= \frac{2 \sin x \cos x}{\sin x \cos x(\sin x + \cos x - 1)} \\
&= \frac{2}{\sin x + \cos x - 1}
\end{aligned}$$

Let $t = \sin x + \cos x - 1$. The expression inside the absolute value sign is

$$\begin{aligned}
f(t) &= \sin x + \cos x + \frac{2}{\sin x + \cos x - 1} \\
&= (\sin x + \cos x - 1) + 1 + \frac{2}{\sin x + \cos x - 1} \\
&= t + 1 + \frac{2}{t}
\end{aligned}$$

Because $\sin\left(x + \frac{\pi}{4}\right) = \sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}(\sin x + \cos x)$,

$\sin x + \cos x \in \left[-\sqrt{2}, \sqrt{2}\right]$ and

$t = \sin x + \cos x - 1 \in \left[-1 - \sqrt{2}, -1 + \sqrt{2}\right]$.

$$f'(t) = 1 - \frac{2}{t^2} = \frac{t^2 - 2}{t^2} = \frac{\left(t + \sqrt{2}\right)\left(t - \sqrt{2}\right)}{t^2}$$

$$\begin{aligned}
f\left(-1 + \sqrt{2}\right) &= -1 + \sqrt{2} + 1 + \frac{2}{-1 + \sqrt{2}} = \sqrt{2} + \frac{2}{\sqrt{2} - 1} \\
&= \frac{4 - \sqrt{2}}{\sqrt{2} - 1}\left(\frac{\sqrt{2} + 1}{\sqrt{2} + 1}\right) = \frac{4\sqrt{2} - 2 + 4 - \sqrt{2}}{1} = 2 + 3\sqrt{2}
\end{aligned}$$

For $t > 0$, f is decreasing and $f(t) > f\left(-1 + \sqrt{2}\right) = 2 + 3\sqrt{2}$

For $t < 0$, f is increasing on $\left(-\sqrt{2} - 1, -\sqrt{2}\right)$, then decreasing on $\left(-\sqrt{2}, 0\right)$. So $f(t) < f\left(-\sqrt{2}\right) = 1 - 2\sqrt{2}$.

Finally, $\left|f(t)\right| \geq 2\sqrt{2} - 1$.

(You can verify this easily with a graphing utility.)

Section 3.4 Concavity and the Second Derivative Test

1. Find the second derivative of a function and form test intervals by using the values for which the second derivative is zero or does not exist and the values at which the function is not continuous. Determine the sign of the second derivative on these test intervals. If the second derivative is positive, then the graph is concave upward. If the second derivative is negative, then the graph is concave downward.

2. If the graph of a function f is concave upward on an open interval containing c, and $f'(c) = 0$, then $f(c)$ must be a relative minimum of f. Similarly, if the graph of a function f is concave downward on an open interval containing c, and $f'(c) = 0$, then $f(c)$ must be a relative maximum of f.

3. The graph of f is increasing and concave downward: $f' > 0, f'' < 0$

4. The graph of f is decreasing and concave upward: $f' < 0, f'' > 0$

5. $f(x) = x^2 - 4x + 8$

$f'(x) = 2x - 4$

$f''(x) = 2$

$f''(x) > 0$ for all x.

Concave upward: $(-\infty, \infty)$

6. $g(x) = 3x^2 - x^3$

$g'(x) = 6x - 3x^2$

$g''(x) = 6 - 6x$

$g''(x) = 0$ when $x = 1$.

Concave upward: $(-\infty, 1)$

Concave downward: $(1, \infty)$

Intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of g'':	$g'' > 0$	$g'' < 0$
Conclusion:	Concave upward	Concave downward

7. $f(x) = x^4 - 3x^3$

$f'(x) = 4x^3 - 9x^2$

$f''(x) = 12x^2 - 18x = 6x(2x - 3)$

$f''(x) = 0$ when $x = 0, \frac{3}{2}$.

Intervals:	$-\infty < x < 0$	$0 < x < \frac{3}{2}$	$\frac{3}{2} < x < \infty$
Sign of f'':	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Concave upward: $(-\infty, 0), \left(\frac{3}{2}, \infty\right)$

Concave downward: $\left(0, \frac{3}{2}\right)$

8. $h(x) = x^5 - 5x + 2$

$h'(x) = 5x^4 - 5$

$h''(x) = 20x^3$

$h''(x) = 0$ when $x = 0$.

Concave upward: $(0, \infty)$

Concave downward: $(-\infty, 0)$

Intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of h'':	$h'' < 0$	$h'' > 0$
Conclusion:	Concave downward	Concave upward

9. $f(x) = \dfrac{24}{x^2 + 12}$

$f'(x) = \dfrac{-48x}{(x^2 + 12)^2}$

$f''(x) = \dfrac{-144(4 - x^2)}{(x^2 + 12)^3}$

$f''(x) = 0$ when $x = \pm 2$.

Intervals:	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Sign of f'':	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Concave upward: $(-\infty, -2), (2, \infty)$

Concave downward: $(-2, 2)$

10. $f(x) = \dfrac{2x^2}{3x^2 + 1}$

$f'(x) = \dfrac{4x}{(3x^2 + 1)^2}$

$f''(x) = \dfrac{-4(3x - 1)(3x + 1)}{(3x^2 + 1)^3}$

$f''(x) = 0$ when $x = \pm\dfrac{1}{3}$.

Intervals:	$-\infty < x < -\frac{1}{3}$	$-\frac{1}{3} < x < \frac{1}{3}$	$\frac{1}{3} < x < \infty$
Sign of f'':	$f'' < 0$	$f'' > 0$	$f'' < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

Concave upward: $\left(-\dfrac{1}{3}, \dfrac{1}{3}\right)$

Concave downward: $\left(-\infty, -\dfrac{1}{3}\right)\left(\dfrac{1}{3}, \infty\right)$

11. $f(x) = \dfrac{x - 2}{6x + 1}$

$f'(x) = \dfrac{13}{(6x + 1)^2}$

$f''(x) = \dfrac{-156}{(6x + 1)^3}$

f is not continuous at $x = -\dfrac{1}{6}$.

Intervals:	$-\infty < x < -\frac{1}{6}$	$-\frac{1}{6} < x < \infty$
Sign of f'':	$f'' > 0$	$f'' < 0$
Conclusion:	Concave upward	Concave downward

Concave upward: $\left(-\infty, -\dfrac{1}{6}\right)$

Concave downward: $\left(-\dfrac{1}{6}, \infty\right)$

12. $f(x) = \dfrac{x + 8}{x - 7}$

$f'(x) = \dfrac{-15}{(x - 7)^2}$

$f''(x) = \dfrac{30}{(x - 7)^3}$

f is not continuous on $x = 7$.

Intervals:	$-\infty < x < 7$	$7 < x < \infty$
Sign of f'':	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward

Concave downward: $(-\infty, 7)$

Concave upward: $(7, \infty)$

13. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$

$f' = \dfrac{-4x}{\left(x^2 - 1\right)^2}$

$f'' = \dfrac{4\left(3x^2 + 1\right)}{\left(x^2 - 1\right)^3}$

f is not continuous at $x = \pm 1$.

Concave upward: $(-\infty, -1), (1, \infty)$

Concave downward: $(-1, 1)$

Intervals:	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of f'':	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

14. $h(x) = \dfrac{x^2 - 1}{2x - 1}$

$h'(x) = \dfrac{2\left(x^2 - x + 1\right)}{(2x - 1)^2}$

$h''(x) = \dfrac{-6}{(2x - 1)^3}$

f'' is not continuous at $x = \dfrac{1}{2}$.

Concave upward: $\left(-\infty, \dfrac{1}{2}\right)$

Concave downward: $\left(\dfrac{1}{2}, \infty\right)$

Intervals:	$-\infty < x < \frac{1}{2}$	$\frac{1}{2} < x < \infty$
Sign of h'':	$h'' > 0$	$h'' < 0$
Conclusion:	Concave upward	Concave downward

15. $y = 2x - \tan x, \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$y' = 2 - \sec^2 x$

$y'' = -2\sec^2 x \tan x$

$y'' = 0$ when $x = 0$.

Concave upward: $\left(-\dfrac{\pi}{2}, 0\right)$

Concave downward: $\left(0, \dfrac{\pi}{2}\right)$

Intervals:	$-\frac{\pi}{2} < x < 0$	$0 < x < \frac{\pi}{2}$
Sign of y'':	$y'' > 0$	$y'' < 0$
Conclusion:	Concave upward	Concave downward

16. $y = x + 2\csc x, \quad (-\pi, \pi)$

$y' = 1 - 2\csc x \cot x$

$y'' = -2\csc x\left(-\csc^2 x\right) - 2\cot x(-\csc x \cot x)$

$= 2\left(\csc^3 x + \csc x \cot^2 x\right)$

$y'' = 0$ when $x = 0$.

Concave upward: $(0, \pi)$

Concave downward: $(-\pi, 0)$

Intervals:	$-\pi < x < 0$	$0 < x < \pi$
Sign of y'':	$y'' < 0$	$y'' > 0$
Conclusion:	Concave downward	Concave upward

17. $f(x) = x^3 - 9x^2 + 24x - 18$

$f'(x) = 3x^2 - 18x + 24$

$f''(x) = 6x - 18 = 0$ when $x = 3$.

Intervals:	$-\infty < x < 3$	$3 < x < \infty$
Sign of f'':	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward

Concave upward: $(3, \infty)$

Concave downward: $(-\infty, 3)$

Point of inflection: $(3, 0)$

18. $f(x) = -x^3 + 6x^2 - 5$

$f'(x) = -3x^2 + 12x$

$f''(x) = -6x + 12 = -6(x - 2) = 0$ when $x = 2$.

Intervals:	$-\infty < x < 2$	$2 < x < \infty$
Sign of f'':	$f'' > 0$	$f'' < 0$
Conclusion:	Concave upward	Concave downward

Concave upward: $(-\infty, 2)$

Concave downward: $(2, \infty)$

Point of inflection: $(2, 11)$

19. $f(x) = 2 - 7x^4$

$f'(x) = -28x^3$

$f''(x) = -84x^2 = 0$ when $x = 0$.

Intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of f'':	$f'' < 0$	$f'' < 0$
Conclusion:	Concave downward	Concave downward

Concave downward: $(-\infty, \infty)$

No points of inflection

20. $f(x) = 4 - x - 3x^4$

$f'(x) = -1 - 12x^3$

$f''(x) = -36x^2 = 0$ when $x = 0$.

Intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of f'':	$f'' < 0$	$f'' < 0$
Conclusion:	Concave downward	Concave downward

Concave downward: $(-\infty, \infty)$

No points of inflection

21. $f(x) = x(x - 4)^3$

$f'(x) = x\left[3(x - 4)^2\right] + (x - 4)^3 = (x - 4)^2(4x - 4)$

$f''(x) = 4(x - 1)\left[2(x - 4)\right] + 4(x - 4)^2 = 4(x - 4)\left[2(x - 1) + (x - 4)\right] = 4(x - 4)(3x - 6) = 12(x - 4)(x - 2)$

$f''(x) = 12(x - 4)(x - 2) = 0$ when $x = 2, 4$.

Intervals:	$-\infty < x < 2$	$2 < x < 4$	$4 < x < \infty$
Sign of $f''(x)$:	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Concave upward: $(-\infty, 2), (4, \infty)$

Concave downward: $(2, 4)$

Points of inflection: $(2, -16), (4, 0)$

22. $f(x) = (x-2)^3(x-1)$

$f'(x) = (x-2)^2(4x-5)$

$f''(x) = 6(x-2)(2x-3)$

$f''(x) = 0$ when $x = \frac{3}{2}, 2.$

Intervals:	$-\infty < x < \frac{3}{2}$	$\frac{3}{2} < x < 2$	$2 < x < \infty$
Sign of f'':	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Concave upward: $\left(-\infty, \frac{3}{2}\right), (2, \infty)$

Concave downward: $\left(\frac{3}{2}, 2\right)$

Points of inflection: $\left(\frac{3}{2}, -\frac{1}{16}\right), (2, 0)$

23. $f(x) = x\sqrt{x+3}$, Domain: $[-3, \infty)$

$f'(x) = x\left(\frac{1}{2}\right)(x+3)^{-1/2} + \sqrt{x+3} = \frac{3(x+2)}{2\sqrt{x+3}}$

$f''(x) = \frac{6\sqrt{x+3} - 3(x+2)(x+3)^{-1/2}}{4(x+3)}$

$= \frac{3(x+4)}{4(x+3)^{3/2}} = 0$ when $x = -4.$

$x = -4$ is not in the domain. f'' is not continuous at $x = -3.$

Interval:	$-3 < x < \infty$
Sign of f'':	$f'' > 0$
Conclusion:	Concave upward

Concave upward: $(-3, \infty)$

There are no points of inflection.

24. $f(x) = x\sqrt{9-x}$, Domain: $x \le 9$

$f'(x) = \frac{3(6-x)}{2\sqrt{9-x}}$

$f''(x) = \frac{3(x-12)}{4(9-x)^{3/2}} = 0$ when $x = 12.$

$x = 12$ is not in the domain. f'' is not continuous at $x = 9.$

Interval:	$-\infty < x < 9$
Sign of f'':	$f'' < 0$
Conclusion:	Concave downward

Concave downward: $(-\infty, 9)$

No point of inflection

25. $f(x) = \frac{6-x}{\sqrt{x}} = 6x^{-1/2} - x^{1/2}$

$f'(x) = \frac{-(x+6)}{2x^{3/2}}$

$f''(x) = \frac{x+18}{4x^{5/2}}$

Note that the domain of f is $x > 0.$

Furthermore, $f''(x) > 0$ on $(0, \infty).$

Concave upward: $(0, \infty)$

No points of inflection

26. $f(x) = \dfrac{x+3}{\sqrt{x}}$, Domain: $x > 0$

$f'(x) = \dfrac{x-3}{2x^{3/2}}$

$f''(x) = \dfrac{9-x}{4x^{5/2}} = 0$ when $x = 9$

Intervals:	$0 < x < 9$	$9 < x < \infty$
Sign of f'':	$f'' > 0$	$f'' < 0$
Conclusion:	Concave upward	Concave downward

Concave upward: $(0, 9)$

Concave downward: $(9, \infty)$

Points of inflection: $(9, 4)$

27. $f(x) = \sin \dfrac{x}{2}, 0 \le x \le 4\pi$

$f'(x) = \dfrac{1}{2}\cos\left(\dfrac{x}{2}\right)$

$f''(x) = -\dfrac{1}{4}\sin\left(\dfrac{x}{2}\right)$

$f''(x) = 0$ when $x = 0, 2\pi, 4\pi$.

Intervals:	$0 < x < 2\pi$	$2\pi < x < 4\pi$
Sign of f'':	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward

Concave upward: $(2\pi, 4\pi)$

Concave downward: $(0, 2\pi)$

Point of inflection: $(2\pi, 0)$

28. $f(x) = 2\csc \dfrac{3x}{2}, 0 < x < 2\pi$

$f'(x) = -3\csc \dfrac{3x}{2} \cot \dfrac{3x}{2}$

$f''(x) = \dfrac{9}{2}\left(\csc^3 \dfrac{3x}{2} + \csc \dfrac{3x}{2} \cot^2 \dfrac{3x}{2}\right) \ne 0$ for any x in the domain of f.

f'' is not continuous at $x = \dfrac{2\pi}{3}$ and $x = \dfrac{4\pi}{3}$.

Intervals:	$0 < x < \dfrac{2\pi}{3}$	$\dfrac{2\pi}{3} < x < \dfrac{4\pi}{3}$	$\dfrac{4\pi}{3} < x < 2\pi$
Sign of $f''(x)$:	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Concave upward: $\left(0, \dfrac{2\pi}{3}\right), \left(\dfrac{4\pi}{3}, 2\pi\right)$

Concave downward: $\left(\dfrac{2\pi}{3}, \dfrac{4\pi}{3}\right)$

No point of inflection

29. $f(x) = \sec\left(x - \dfrac{\pi}{2}\right), 0 < x < 4\pi$

$f'(x) = \sec\left(x - \dfrac{\pi}{2}\right)\tan\left(x - \dfrac{\pi}{2}\right)$

$f''(x) = \sec^3\left(x - \dfrac{\pi}{2}\right) + \sec\left(x - \dfrac{\pi}{2}\right)\tan^2\left(x - \dfrac{\pi}{2}\right) \ne 0$ for any x in the domain of f.

f'' is not continuous at $x = \pi, x = 2\pi$, and $x = 3\pi$.

Intervals:	$0 < x < \pi$	$\pi < x < 2\pi$	$2\pi < x < 3\pi$	$3\pi < x < 4\pi$
Sign of f'':	$f'' > 0$	$f'' < 0$	$f'' > 0$	$f'' < 0$
Conclusion:	Concave upward	Concave downward	Concave upward	Concave upward

Concave upward: $(0, \pi), (2\pi, 3\pi)$

Concave downward: $(\pi, 2\pi), (3\pi, 4\pi)$

No point of inflection

30. $f(x) = \sin x + \cos x, 0 \le x \le 2\pi$

$f'(x) = \cos x - \sin x$

$f''(x) = \sin x - \cos x$

$f''(x) = 0$ when $x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$.

Intervals:	$0 < x < \dfrac{3\pi}{4}$	$\dfrac{3\pi}{4} < x < \dfrac{7\pi}{4}$	$\dfrac{7\pi}{4} < x < 2\pi$
Sign of f'':	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

Concave upward: $\left(\dfrac{3\pi}{4}, \dfrac{7\pi}{4}\right)$

Concave downward: $\left(0, \dfrac{3\pi}{4}\right), \left(\dfrac{7\pi}{4}, 2\pi\right)$

Points of inflection: $\left(\dfrac{3\pi}{4}, 0\right), \left(\dfrac{7\pi}{4}, 0\right)$

31. $f(x) = 2\sin x + \sin 2x, 0 \le x \le 2\pi$

$f'(x) = 2\cos x + 2\cos 2x$

$f''(x) = -2\sin x - 4\sin 2x = -2\sin x(1 + 4\cos x)$

$f''(x) = 0$ when $x = 0, 1.823, \pi, 4.460$.

Intervals:	$0 < x < 1.823$	$1.823 < x < \pi$	$\pi < x < 4.460$	$4.460 < x < 2\pi$
Sign of f'':	$f'' < 0$	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward	Concave downward	Concave upward

Concave upward: $(1.823, \pi), (4.460, 2\pi)$

Concave downward: $(0, 1.823), (\pi, 4.460)$

Points of inflection: $(1.823, 1.452), (\pi, 0), (4.46, -1.452)$

32. $f(x) = x + 2\cos x, [0, 2\pi]$

$f'(x) = 1 - 2\sin x$

$f''(x) = -2\cos x$

$f''(x) = 0$ when $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Intervals:	$0 < x < \dfrac{\pi}{2}$	$\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$	$\dfrac{3\pi}{2} < x < 2\pi$
Sign of f'':	$f'' < 0$	$f'' > 0$	$f'' < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

Concave upward: $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

Concave downward: $\left(0, \dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, 2\pi\right)$

Points of inflection: $\left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, \dfrac{3\pi}{2}\right)$

33. $f(x) = 6x - x^2$

$f'(x) = 6 - 2x$

$f''(x) = -2$

Critical number: $x = 3$

$f''(3) = -2 < 0$

Therefore, $(3, 9)$ is a relative maximum.

34. $f(x) = x^2 + 3x - 8$

$f'(x) = 2x + 3$

$f''(x) = 2$

Critical number: $x = -\frac{3}{2}$

$f''\left(-\frac{3}{2}\right) = 2 > 0$

Therefore, $\left(-\frac{3}{2}, -\frac{41}{4}\right)$ is a relative minimum.

35. $f(x) = x^3 - 3x^2 + 3$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

$f''(x) = 6x - 6 = 6(x - 1)$

Critical numbers: $x = 0, x = 2$

$f''(0) = -6 < 0$

Therefore, $(0, 3)$ is a relative maximum.

$f''(2) = 6 > 0$

Therefore, $(2, -1)$ is a relative minimum.

36. $f(x) = -x^3 + 7x^2 - 15x$

$f'(x) = -3x^2 + 14x - 15 = -(x - 3)(3x - 5)$

$f''(x) = -6x + 14 = -2(3x - 7)$

Critical numbers: $x = 3, \frac{5}{3}$

$f''(3) = -4 < 0$

Therefore, $(3, 9)$ is a relative maximum.

$f''\left(\frac{5}{3}\right) = 4 > 0$

Therefore, $\left(\frac{5}{3}, -\frac{275}{27}\right)$ is a relative minimum.

37. $f(x) = x^4 - 4x^3 + 2$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$

$f''(x) = 12x^2 - 24x = 12x(x - 2)$

Critical numbers: $x = 0, x = 3$

However, $f''(0) = 0$, so you must use the First Derivative Test. $f'(x) < 0$ on the intervals $(-\infty, 0)$ and $(0, 3)$; so, $(0, 2)$ is not an extremum. $f''(3) > 0$ so $(3, -25)$ is a relative minimum.

38. $f(x) = -x^4 + 2x^3 + 8x$

$f'(x) = -4x^3 + 6x^2 + 8 = -2(x - 2)(2x^2 + x + 2)$

$f''(x) = -12x^2 + 12x$

Critical number: $x = 2$

$f''(2) = -48 + 24 = -24 < 0$

Therefore, $(2, 16)$ is a relative maximum.

39. $f(x) = x^{2/3} - 3$

$f'(x) = \dfrac{2}{3x^{1/3}}$

$f''(x) = -\dfrac{2}{9x^{4/3}}$

Critical number: $x = 0$

However, $f''(0)$ is undefined, so you must use the First Derivative Test. Because $f'(x) < 0$ on $(-\infty, 0)$ and $f'(x) > 0$ on $(0, \infty)$, $(0, -3)$ is a relative minimum.

40. $f(x) = \sqrt{x^2 + 1}$

$f'(x) = \dfrac{x}{\sqrt{x^2 + 1}}$

$f''(x) = \dfrac{1}{\left(x^2 + 1\right)^{3/2}}$

Critical number: $x = 0$

$f''(0) = 1 > 0$

Therefore, $(0, 1)$ is a relative minimum.

41. $f(x) = x + \frac{4}{x}$

$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2}$

$f''(x) = \frac{8}{x^3}$

Critical numbers: $x = \pm 2$

$f''(-2) = -1 < 0$

Therefore, $(-2, -4)$ is a relative maximum.

$f''(2) = 1 > 0$

Therefore, $(2, 4)$ is a relative minimum.

42. $f(x) = \frac{9x - 1}{x + 5}$

$f'(x) = \frac{46}{(x + 5)^2}$

$f''(x) = \frac{-92}{(x + 5)^3}$

Note that f is not defined at $x = -5$.

There are no critical numbers and no relative extrema.

43. $f(x) = \cos x - x, 0 \le x \le 4\pi$

$f'(x) = -\sin x - 1 \le 0$

Therefore, f is non-increasing and there are no relative extrema.

44. $f(x) = 2\sin x + \cos 2x, 0 \le x \le 2\pi$

$f'(x) = 2\cos x - 2\sin 2x = 2\cos x - 4\sin x\cos x$

$= 2\cos x(1 - 2\sin x) = 0$ when $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

$f''(x) = -2\sin x - 4\cos 2x$

$f''\left(\frac{\pi}{6}\right) = -3 < 0$

Therefore, $\left(\frac{\pi}{6}, \frac{3}{2}\right)$ is a relative maximum.

$f''\left(\frac{\pi}{2}\right) = 2 > 0$

Therefore, $\left(\frac{\pi}{2}, 1\right)$ is a relative minimum.

$f''\left(\frac{5\pi}{6}\right) = -3 < 0$

Therefore, $\left(\frac{5\pi}{6}, \frac{3}{2}\right)$ is a relative maximum.

$f''\left(\frac{3\pi}{2}\right) = 6 > 0$

Therefore, $\left(\frac{3\pi}{2}, -3\right)$ is a relative minimum.

45. $f(x) = 0.2x^2(x - 3)^3, [-1, 4]$

(a) $f'(x) = 0.2x(5x - 6)(x - 3)^2$

$f''(x) = (x - 3)(4x^2 - 9.6x + 3.6)$

$= 0.4(x - 3)(10x^2 - 24x + 9)$

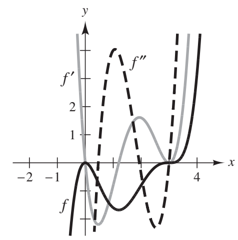
(b) $f''(0) < 0 \Rightarrow (0, 0)$ is a relative maximum.

$f''\left(\frac{6}{5}\right) > 0 \Rightarrow (1.2, -1.6796)$ is a relative minimum.

Points of inflection:

$(3, 0), (0.4652, -0.7048), (1.9348, -0.9049)$

(c)



f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

46. $f(x) = x^2\sqrt{6 - x^2}, \left[-\sqrt{6}, \sqrt{6}\right]$

(a) $f'(x) = \frac{3x(4 - x^2)}{\sqrt{6 - x^2}}$

$f'(x) = 0$ when $x = 0, x = \pm 2$.

$f''(x) = \frac{6(x^4 - 9x^2 + 12)}{(6 - x^2)^{3/2}}$

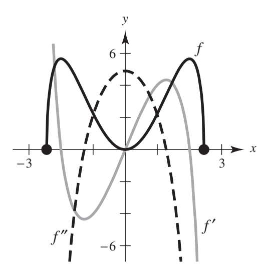
$f''(x) = 0$ when $x = \pm\sqrt{\frac{9 - \sqrt{33}}{2}}$.

(b) $f''(0) > 0 \Rightarrow (0, 0)$ is a relative minimum.

$f''(\pm 2) < 0 \Rightarrow \left(\pm 2, 4\sqrt{2}\right)$ are relative maxima.

Points of inflection: $(\pm 1.2758, 3.4035)$

(c)



The graph of f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

47. $f(x) = \sin x - \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x, \quad [0, \pi]$

(a) $f'(x) = \cos x - \cos 3x + \cos 5x$

$f'(x) = 0$ when $x = \frac{\pi}{6}, x = \frac{\pi}{2}, x = \frac{5\pi}{6}$.

$f''(x) = -\sin x + 3\sin 3x - 5\sin 5x$

$f''(x) = 0$ when $x = \frac{\pi}{6}, x = \frac{5\pi}{6}$,

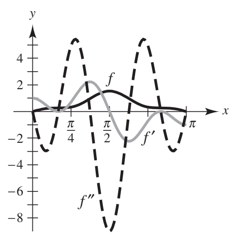
$x \approx 1.1731, x \approx 1.9685$

(b) $f''\left(\frac{\pi}{2}\right) < 0 \Rightarrow \left(\frac{\pi}{2}, 1.53333\right)$ is a relative maximum.

Points of inflection: $\left(\frac{\pi}{6}, 0.2667\right), (1.1731, 0.9638),$

$(1.9685, 0.9637), \left(\frac{5\pi}{6}, 0.2667\right)$

Note: $(0, 0)$ and $(\pi, 0)$ are not points of inflection because they are endpoints.

(c) 

The graph of f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

48. $f(x) = \sqrt{2x}\sin x, [0, 2\pi]$

(a) $f'(x) = \sqrt{2x}\cos x + \frac{\sin x}{\sqrt{2x}}$

Critical numbers: $x \approx 1.84, 4.82$

$$f''(x) = -\sqrt{2x}\sin x + \frac{\cos x}{\sqrt{2x}} + \frac{\cos x}{\sqrt{2x}} - \frac{\sin x}{2x\sqrt{2x}}$$

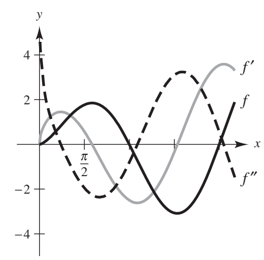
$$= \frac{2\cos x}{\sqrt{2x}} - \frac{(4x^2 + 1)\sin x}{2x\sqrt{2x}}$$

$$= \frac{4x\cos x - (4x^2 + 1)\sin x}{2x\sqrt{2x}}$$

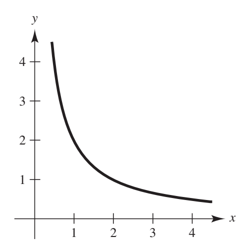
(b) Relative maximum: $(1.84, 1.85)$

Relative minimum: $(4.82, -3.09)$

Points of inflection: $(0.75, 0.83), (3.42, -0.72)$

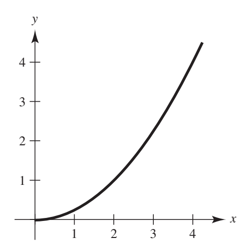
(c) 

f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

49. (a) 

$f' < 0$ means f decreasing

f' increasing means concave upward

(b) 

$f' > 0$ means f increasing

f' increasing means concave upward

50. (a) The rate of change of sales is increasing.

$S'' > 0$

(b) The rate of change of sales is decreasing.

$S' > 0, S'' < 0$

(c) The rate of change of sales is constant.

$S' = C, S'' = 0$

(d) Sales are steady.

$S = C, S' = 0, S'' = 0$

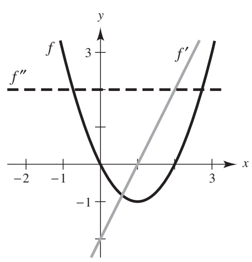
(e) Sales are declining, but at a lower rate.

$S' < 0, S'' > 0$

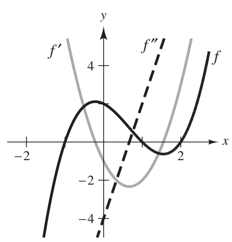
(f) Sales have bottomed out and have started to rise.

$S' > 0, S'' > 0$ Answers will vary.

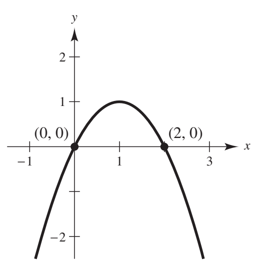
51.



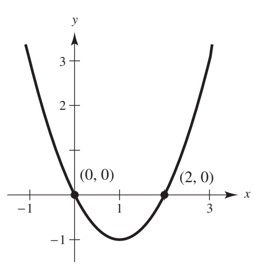
52.



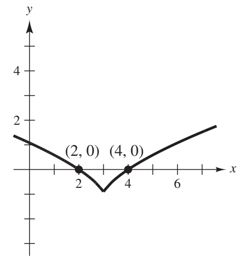
53.



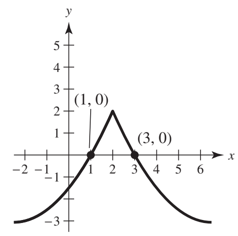
54.



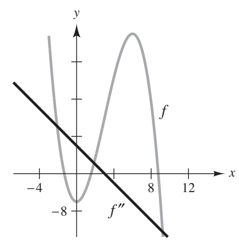
55.



56.



57.



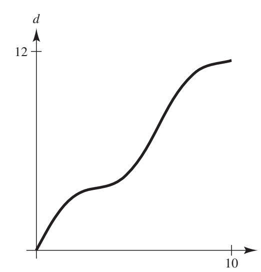
f'' is linear.

f' is quadratic.

f is cubic.

f concave upward on $(-\infty, 3)$, downward on $(3, \infty)$.

58. (a)



(b) Because the depth d is always increasing, there are no relative extrema. $f'(x) > 0$

(c) The rate of change of d is decreasing until you reach the widest point of the jug, then the rate increases until you reach the narrowest part of the jug's neck, then the rate decreases until you reach the top of the jug.

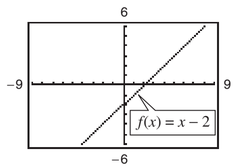
59. (a) $n = 1$:

$f(x) = x - 2$

$f'(x) = 1$

$f''(x) = 0$

No point of inflection



$n = 2$:

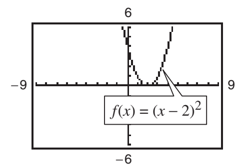
$f(x) = (x - 2)^2$

$f'(x) = 2(x - 2)$

$f''(x) = 2$

No point of inflection

Relative minimum: $(2, 0)$



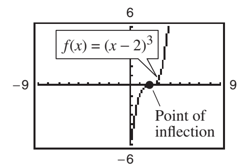
$n = 3$:

$f(x) = (x - 2)^3$

$f'(x) = 3(x - 2)^2$

$f''(x) = 6(x - 2)$

Point of inflection: $(2, 0)$



$n = 4$:

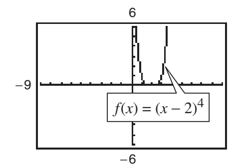
$f(x) = (x - 2)^4$

$f'(x) = 4(x - 2)^3$

$f''(x) = 12(x - 2)^2$

No point of inflection

Relative minimum: $(2, 0)$



Conclusion: If $n \ge 3$ and n is odd, then $(2, 0)$ is point of inflection. If $n \ge 2$ and n is even, then $(2, 0)$ is a relative minimum.

(b) Let $f(x) = (x - 2)^n, f'(x) = n(x - 2)^{n-1}, f''(x) = n(n - 1)(x - 2)^{n-2}$.

For $n \ge 3$ and odd, $n - 2$ is also odd and the concavity changes at $x = 2$.

For $n \ge 4$ and even, $n - 2$ is also even and the concavity does not change at $x = 2$.

So, $x = 2$ is point of inflection if and only if $n \ge 3$ is odd.

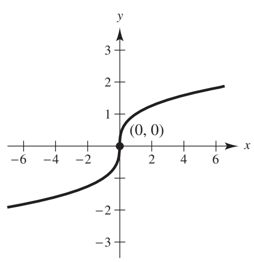
60. (a) $f(x) = \sqrt[3]{x}$

$f'(x) = \frac{1}{3}x^{-2/3}$

$f''(x) = -\frac{2}{9}x^{-5/3}$

Point of inflection: $(0, 0)$

(b) $f''(x)$ does not exist at $x = 0$.



61. $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: $(3, 3)$

Relative minimum: $(5, 1)$

Point of inflection: $(4, 2)$

$f'(x) = 3ax^2 + 2bx + c, f''(x) = 6ax + 2b$

$$\left.\begin{aligned} f(3) &= 27a + 9b + 3c + d = 3 \\ f(5) &= 125a + 25b + 5c + d = 1 \end{aligned}\right\} 98a + 16b + 2c = -2 \Rightarrow 49a + 8b + c = -1$$

$f'(3) = 27a + 6b + c = 0, f''(4) = 24a + 2b = 0$

$$\begin{array}{lll} 49a + 8b + c = -1 & \quad & 24a + 2b = 0 \\ \underline{27a + 6b + c = 0} & & \underline{22a + 2b = -1} \\ 22a + 2b = -1 & & 2a = 1 \end{array}$$

$a = \frac{1}{2}, b = -6, c = \frac{45}{2}, d = -24$

$f(x) = \frac{1}{2}x^3 - 6x^2 + \frac{45}{2}x - 24$

62. $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: $(2, 4)$

Relative minimum: $(4, 2)$

Point of inflection: $(3, 3)$

$f'(x) = 3ax^2 + 2bx + c, f''(x) = 6ax + 2b$

$$\left.\begin{aligned} f(2) &= 8a + 4b + 2c + d = 4 \\ f(4) &= 64a + 16b + 4c + d = 2 \end{aligned}\right\} 56a + 12b + 2c = -2 \Rightarrow 28a + 6b + c = -1$$

$f'(2) = 12a + 4b + c = 0, f'(4) = 48a + 8b + c = 0, f''(3) = 18a + 2b = 0$

$$\begin{aligned} 28a + 6b + c &= -1 \qquad & 18a + 2b &= 0 \\ 12a + 4b + c &= 0 \qquad & 16a + 2b &= -1 \\ 16a + 2b &= -1 \qquad & 2a &= 1 \end{aligned}$$

$a = \frac{1}{2}, b = -\frac{9}{2}, c = 12, d = -6$

$f(x) = \frac{1}{2}x^3 - \frac{9}{2}x^2 + 12x - 6$

63. $f(x) = ax^3 + bx^2 + cx + d$

Maximum: $(-4, 1)$

Minimum: $(0, 0)$

(a) $f'(x) = 3ax^2 + 2bx + c, \quad f''(x) = 6ax + 2b$

$$\begin{aligned} f(0) = 0 &\Rightarrow d = 0 \\ f(-4) = 1 &\Rightarrow -64a + 16b - 4c = 1 \\ f'(-4) = 0 &\Rightarrow 48a - 8b + c = 0 \\ f'(0) = 0 &\Rightarrow c = 0 \end{aligned}$$

Solving this system yields $a = \frac{1}{32}$ and $b = 6a = \frac{3}{16}$.

$f(x) = \frac{1}{32}x^3 + \frac{3}{16}x^2$

(b) The plane would be descending at the greatest rate at the point of inflection.

$f''(x) = 6ax + 2b = \frac{3}{16}x + \frac{3}{8} = 0 \Rightarrow x = -2.$

Two miles from touchdown.

64. (a) line OA: $y = -0.06x$ slope: -0.06

line CB: $y = 0.04x + 50$ slope: 0.04

$f(x) = ax^3 + bx^2 + cx + d$

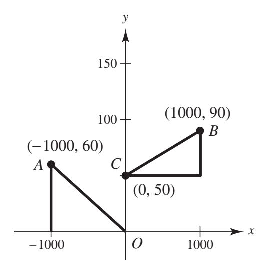
$f'(x) = 3ax^2 + 2bx + c$

$(-1000, 60)$: $60 = (-1000)^3 a + (1000)^2 b - 1000c + d$

$-0.06 = (1000)^2\, 3a - 2000b + c$

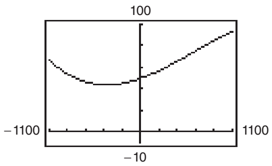
$(1000, 90)$: $90 = (1000)^3 a + (1000)^2 b + 1000c + d$

$0.04 = (1000)^2\, 3a + 2000b + c$

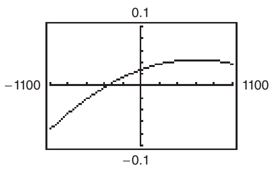


The solution to this system of four equations is $a = -1.25 \times 10^{-8}$, $b = 0.000025$, $c = 0.0275$, and $d = 50$.

(b) $y = -1.25 \times 10^{-8}x^3 + 0.000025x^2 + 0.0275x + 50$



(c)



(d) The steepest part of the road is 6% at the point A.

65. $C = 0.5x^2 + 15x + 5000$

$$\overline{C} = \frac{C}{x} = 0.5x + 15 + \frac{5000}{x}$$

$\overline{C}$ = average cost per unit

$$\frac{d\overline{C}}{dx} = 0.5 - \frac{5000}{x^2} = 0 \text{ when } x = 100$$

By the First Derivative Test, $\overline{C}$ is minimized when $x = 100$ units.

66. $S = \dfrac{5.755}{10^8}T^3 - \dfrac{8.521}{10^6}T^2 + \dfrac{6.540}{10^5}T + 0.99987, 0 < T < 25$

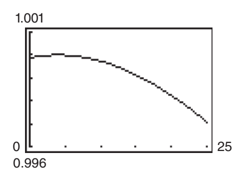
(a) $S' = \dfrac{17.265}{10^8}T^2 - \dfrac{17.042}{10^6}T + \dfrac{6.540}{10^5}$

$S'' = \dfrac{34.53}{10^8}T - \dfrac{17.042}{10^6} = 0$ when $T \approx 49.4$, which is not in the domain

$S'' < 0$ for $0 < T < 25 \Rightarrow$ Concave downward.

(b) The maximum is approximately $(4, 1)$.

(c)



(d) When $t = 20, S \approx 0.998$.

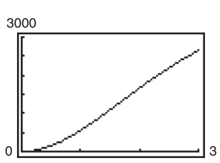
67. $S = \dfrac{5000t^2}{8 + t^2}, 0 \le t \le 3$

(a)

t	0.5	1	1.5	2	2.5	3
S	151.5	555.6	1097.6	1666.7	2193.0	2647.1

Increasing at greatest rate when $1.5 < t < 2$

(b)



Increasing at greatest rate when $t \approx 1.5$.

(c) $S = \dfrac{5000t^2}{8 + t^2}$

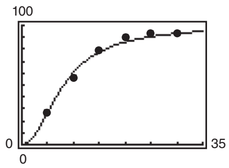
$$S'(t) = \frac{80{,}000t}{\left(8 + t^2\right)^2}$$

$$S''(t) = \frac{80{,}000\left(8 - 3t^2\right)}{\left(8 + t^2\right)^3}$$

$S''(t) = 0$ for $t = \pm\sqrt{\dfrac{8}{3}}$. So, $t = \dfrac{2\sqrt{6}}{3} \approx 1.633$ yrs.

68. $S = \dfrac{100t^2}{65 + t^2}, \; t > 0$

(a)



(b) $S'(t) = \dfrac{13{,}000t}{(t^2 + 65)^2}$

$S''(t) = \dfrac{13{,}000(65 - 3t^2)}{(t^2 + 65)^3}$

$S''(t) = 0$ when $t \approx 4.65$.

The graph of S is concave upward on $(0, 4.65)$ and downward on $(4.65, 30)$.

(c) $S'(t) > 0$ for $t > 0$.

As t increases, the typing speed increases.

69. $f(x) = 2(\sin x + \cos x), \qquad f\left(\dfrac{\pi}{4}\right) = 2\sqrt{2}$

$f'(x) = 2(\cos x - \sin x), \qquad f'\left(\dfrac{\pi}{4}\right) = 0$

$f''(x) = 2(-\sin x - \cos x), \quad f''\left(\dfrac{\pi}{4}\right) = -2\sqrt{2}$

$P_1(x) = 2\sqrt{2} + 0\left(x - \dfrac{\pi}{4}\right) = 2\sqrt{2}$

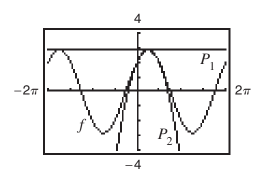
$P_1'(x) = 0$

$P_2(x) = 2\sqrt{2} + 0\left(x - \dfrac{\pi}{4}\right) + \dfrac{1}{2}\left(-2\sqrt{2}\right)\left(x - \dfrac{\pi}{4}\right)^2$

$= 2\sqrt{2} - \sqrt{2}\left(x - \dfrac{\pi}{4}\right)^2$

$P_2'(x) = -2\sqrt{2}\left(x - \dfrac{\pi}{4}\right)$

$P_2''(x) = -2\sqrt{2}$



The values of f, P_1, P_2, and their first derivatives are equal at $x = \pi/4$. The values of the second derivatives of f and P_2 are equal at $x = \pi/4$. The approximations worsen as you move away from $x = \pi/4$.

70. $f(x) = 2(\sin x + \cos x), \qquad f(0) = 2$

$f'(x) = 2(\cos x - \sin x), \qquad f'(0) = 2$

$f''(x) = 2(-\sin x - \cos x), \qquad f''(0) = -2$

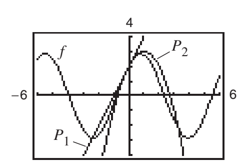
$P_1(x) = 2 + 2(x - 0) = 2(1 + x)$

$P_1'(x) = 2$

$P_2(x) = 2 + 2(x - 0) + \tfrac{1}{2}(-2)(x - 0)^2 = 2 + 2x - x^2$

$P_2'(x) = 2 - 2x$

$P_2''(x) = -2$



The values of f, P_1, P_2, and their first derivatives are equal at $x = 0$. The values of the second derivatives of f and P_2 are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

71. $f(x) = \sqrt{1-x}, \qquad f(0) = 1$

$f'(x) = -\dfrac{1}{2\sqrt{1-x}}, \qquad f'(0) = -\dfrac{1}{2}$

$f''(x) = -\dfrac{1}{4(1-x)^{3/2}}, \qquad f''(0) = -\dfrac{1}{4}$

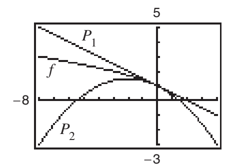
$P_1(x) = 1 + \left(-\dfrac{1}{2}\right)(x-0) = 1 - \dfrac{x}{2}$

$P_1'(x) = -\dfrac{1}{2}$

$P_2(x) = 1 + \left(-\dfrac{1}{2}\right)(x-0) + \dfrac{1}{2}\left(-\dfrac{1}{4}\right)(x-0)^2 = 1 - \dfrac{x}{2} - \dfrac{x^2}{8}$

$P_2'(x) = -\dfrac{1}{2} - \dfrac{x}{4}$

$P_2''(x) = -\dfrac{1}{4}$



The values of f, P_1, P_2, and their first derivatives are equal at $x = 0$. The values of the second derivatives of f and P_2 are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

72. $f(x) = \dfrac{\sqrt{x}}{x-1}, \qquad f(2) = \sqrt{2}$

$f'(x) = \dfrac{-(x+1)}{2\sqrt{x}(x-1)^2}, \qquad f'(2) = -\dfrac{3}{2\sqrt{2}} = -\dfrac{3\sqrt{2}}{4}$

$f''(x) = \dfrac{3x^2 + 6x - 1}{4x^{3/2}(x-1)^3}, \qquad f''(2) = \dfrac{23}{8\sqrt{2}} = \dfrac{23\sqrt{2}}{16}$

$P_1(x) = \sqrt{2} + \left(-\dfrac{3\sqrt{2}}{4}\right)(x-2) = -\dfrac{3\sqrt{2}}{4}x + \dfrac{5\sqrt{2}}{2}$

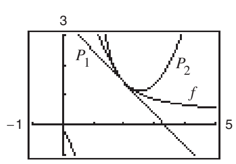
$P_1'(x) = -\dfrac{3\sqrt{2}}{4}$

$P_2(x) = \sqrt{2} + \left(-\dfrac{3\sqrt{2}}{4}\right)(x-2) + \dfrac{1}{2}\left(\dfrac{23\sqrt{2}}{16}\right)(x-2)^2 = \sqrt{2} - \dfrac{3\sqrt{2}}{4}(x-2) + \dfrac{23\sqrt{2}}{32}(x-2)^2$

$P_2'(x) = -\dfrac{3\sqrt{2}}{4} + \dfrac{23\sqrt{2}}{16}(x-2)$

$P_2''(x) = \dfrac{23\sqrt{2}}{16}$

The values of f, P_1, P_2 and their first derivatives are equal at $x = 2$. The values of the second derivatives of f and P_2 are equal at $x = 2$. The approximations worsen as you move away from $x = 2$.



73. $f(x) = x\sin\left(\frac{1}{x}\right)$

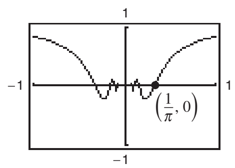
$$f'(x) = x\left[-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)\right] + \sin\left(\frac{1}{x}\right) = -\frac{1}{x}\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$$

$$f''(x) = -\frac{1}{x}\left[\frac{1}{x^2}\sin\left(\frac{1}{x}\right)\right] + \frac{1}{x^2}\cos\left(\frac{1}{x}\right) - \frac{1}{x^2}\cos\left(\frac{1}{x}\right) = -\frac{1}{x^3}\sin\left(\frac{1}{x}\right) = 0$$

$$x = \frac{1}{\pi}$$

Point of inflection: $\left(\frac{1}{\pi}, 0\right)$

When $x > 1/\pi$, $f'' < 0$, so the graph is concave downward.



74. $f(x) = x(x-6)^2 = x^3 - 12x^2 + 36x$

$f'(x) = 3x^2 - 24x + 36 = 3(x-2)(x-6) = 0$

$f''(x) = 6x - 24 = 6(x-4) = 0$

Relative extrema: $(2, 32)$ and $(6, 0)$

Point of inflection $(4, 16)$ is midway between the relative extrema of *f*.

75. True. Let $y = ax^3 + bx^2 + cx + d$, $a \neq 0$. Then $y'' = 6ax + 2b = 0$ when $x = -(b/3a)$, and the concavity changes at this point.

76. False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

77. False. Concavity is determined by f''. For example, let $f(x) = x$ and $c = 2$. $f'(c) = f'(2) > 0$, but *f* is not concave upward at $c = 2$.

78. False. For example, let $f(x) = (x-2)^4$.

79. *f* and *g* are concave upward on (a, b) implies that f' and g' are increasing on (a, b), and $f'' > 0$ and $g'' > 0$.

So, $(f+g)'' > 0 \Rightarrow f + g$ is concave upward on (a, b) by Theorem 3.7.

80. *f*, *g* are positive, increasing, and concave upward on $(a, b) \Rightarrow f(x) > 0$, $f'(x) \geq 0$ and $f''(x) > 0$, and $g(x) > 0$, $g'(x) \geq 0$ and $g''(x) > 0$ on (a, b). For $x \in (a, b)$,

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

$$(fg)''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x) > 0$$

So, *fg* is concave upward on (a, b).

Section 3.5 Limits at Infinity

1. (a) As *x* increases without bound, $f(x)$ approaches -5.

(b) As *x* decreases without bound, $f(x)$ approaches 3.

2. The line $y = L$ is a horizontal asymptote of the graph of *f* when $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

3. The maximum number is 2, one from the left and one from the right.

4. (1) If the degree of the numerator of a rational function is less than the degree of the denominator, then the limit of the function is 0.

(2) If the degree of the numerator of a rational function is equal to the degree of the denominator, then the limit of the function is the ratio of the leading coefficients.

(3) If the degree of the numerator of a rational function is greater than the degree of the denominator, then the limit of the function does not exist.

5. $f(x) = \dfrac{2x^2}{x^2 + 2}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (f).

6. $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$

No vertical asymptotes

Horizontal asymptotes: $y = \pm 2$

Matches (c).

7. $f(x) = \dfrac{x}{x^2 + 2}$

No vertical asymptotes

Horizontal asymptote: $y = 0$

$f(1) < 1$

Matches (d).

8. $f(x) = 2 + \dfrac{x^2}{x^4 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (a).

9. $f(x) = \dfrac{4 \sin x}{x^2 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 0$

$f(1) > 1$

Matches (b).

10. $f(x) = \dfrac{2x^2 - 3x + 5}{x^2 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (e).

11. (a) $h(x) = \dfrac{f(x)}{x^2} = \dfrac{5x^3 - 3x^2 + 10x}{x^2} = 5x - 3 + \dfrac{10}{x}$

$\lim_{x\to\infty} h(x) = \infty$ (Limit does not exist)

(b) $h(x) = \dfrac{f(x)}{x^3} = \dfrac{5x^3 - 3x^2 + 10x}{x^3} = 5 - \dfrac{3}{x} + \dfrac{10}{x^2}$

$\lim_{x\to\infty} h(x) = 5$

(c) $h(x) = \dfrac{f(x)}{x^4} = \dfrac{5x^3 - 3x^2 + 10x}{x^4} = \dfrac{5}{x} - \dfrac{3}{x^2} + \dfrac{10}{x^3}$

$\lim_{x\to\infty} h(x) = 0$

12. (a) $h(x) = \dfrac{f(x)}{x} = \dfrac{-4x^2 + 2x - 5}{x} = -4x + 2 - \dfrac{5}{x}$

$\lim_{x\to\infty} h(x) = -\infty$ (Limit does not exist)

(b) $h(x) = \dfrac{f(x)}{x^2} = \dfrac{-4x^2 + 2x - 5}{x^2} = -4 + \dfrac{2}{x} - \dfrac{5}{x^2}$

$\lim_{x\to\infty} h(x) = -4$

(c) $h(x) = \dfrac{f(x)}{x^3} = \dfrac{-4x^2 + 2x - 5}{x^3} = -\dfrac{4}{x} + \dfrac{2}{x^2} - \dfrac{5}{x^3}$

$\lim_{x\to\infty} h(x) = 0$

13. (a) $\lim_{x\to\infty} \dfrac{x^2 + 2}{x^3 - 1} = 0$

(b) $\lim_{x\to\infty} \dfrac{x^2 + 2}{x^2 - 1} = 1$

(c) $\lim_{x\to\infty} \dfrac{x^2 + 2}{x - 1} = \infty$ (Limit does not exist)

14. (a) $\lim_{x\to\infty} \dfrac{3 - 2x}{3x^3 - 1} = 0$

(b) $\lim_{x\to\infty} \dfrac{3 - 2x}{3x - 1} = -\dfrac{2}{3}$

(c) $\lim_{x\to\infty} \dfrac{3 - 2x^2}{3x - 1} = -\infty$ (Limit does not exist)

15. (a) $\lim_{x\to\infty} \dfrac{5 - 2x^{3/2}}{3x^2 - 4} = 0$

(b) $\lim_{x\to\infty} \dfrac{5 - 2x^{3/2}}{3x^{3/2} - 4} = -\dfrac{2}{3}$

(c) $\lim_{x\to\infty} \dfrac{5 - 2x^{3/2}}{3x - 4} = -\infty$ (Limit does not exist)

16. (a) $\lim_{x\to\infty} \dfrac{5x^{3/2}}{4x^2 + 1} = 0$

(b) $\lim_{x\to\infty} \dfrac{5x^{3/2}}{4x^{3/2} + 1} = \dfrac{5}{4}$

(c) $\lim_{x\to\infty} \dfrac{5x^{3/2}}{4\sqrt{x} + 1} = \infty$ (Limit does not exist)

17. $\lim_{x\to\infty}\left(4 + \dfrac{3}{x}\right) = 4 + 0 = 4$

18. $\lim_{x\to-\infty}\left(\dfrac{5}{x} - \dfrac{x}{3}\right) = \infty$ (Limit does not exist)

19. $\lim_{x\to\infty} \dfrac{7x + 6}{9x - 4} = \lim_{x\to\infty} \dfrac{7 + 6/x}{9 - 4/x} = \dfrac{7 + 0}{9 - 0} = \dfrac{7}{9}$

20. $\lim_{x\to-\infty} \frac{4x^2+5}{x^2+3} = \lim_{x\to-\infty} \frac{4+(5/x^2)}{1+(3/x^2)} = 4$

21. $\lim_{x\to-\infty} \frac{2x^2+x}{6x^3+2x^2+x} = \lim_{x\to-\infty} \frac{2/x+1/x^2}{6+2/x+1/x^2}$

$= \frac{0+0}{6+0+0} = 0$

22. $\lim_{x\to\infty} \frac{5x^3+1}{10x^3-3x^2+7} = \lim_{x\to\infty} \frac{5+1/x^3}{10-3/x+7/x^3}$

$= \frac{5+0}{10-0} = \frac{1}{2}$

23. $\lim_{x\to-\infty} \frac{5x^2}{x+3} = \lim_{x\to-\infty} \frac{5x}{1+(3/x)} = -\infty$

Limit does not exist.

24. $\lim_{x\to-\infty} \frac{x^3-4}{x^2+1} = \lim_{x\to-\infty} \frac{x-(4/x^2)}{1+(1/x^2)} = -\infty$

Limit does not exist.

25. $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2-x}}$

$= \lim_{x\to-\infty} \frac{1}{\left(\frac{\sqrt{x^2-x}}{-\sqrt{x^2}}\right)}$

$= \lim_{x\to-\infty} \frac{-1}{\sqrt{1-(1/x)}}$

$= -1$, $\left(\text{for } x < 0 \text{ we have } x = -\sqrt{x^2}\right)$

26. $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}}$

$= \lim_{x\to-\infty} \frac{1}{\left(\frac{\sqrt{x^2+1}}{-\sqrt{x^2}}\right)}$

$= \lim_{x\to-\infty} \frac{-1}{\sqrt{1+(1/x^2)}}$

$= -1$, $\left(\text{for } x < 0 \text{ we have } x = -\sqrt{x^2}\right)$

27. $\lim_{x\to-\infty} \frac{2x+1}{\sqrt{x^2-x}}$

$= \lim_{x\to-\infty} \frac{2+\frac{1}{x}}{\left(\frac{\sqrt{x^2-x}}{-\sqrt{x^2}}\right)}$

$= \lim_{x\to-\infty} \frac{-2-\left(\frac{1}{x}\right)}{\sqrt{1-\frac{1}{x}}}$

$= -2$, $\left(\text{for } x < 0,\ x = -\sqrt{x^2}\right)$

28. $\lim_{x\to\infty} \frac{5x^2+2}{\sqrt{x^2+3}}$

$= \lim_{x\to\infty} \frac{5x^2+2}{x\sqrt{1+(3/x^2)}}$

$= \lim_{x\to\infty} \frac{5x^2+(2/x)}{\sqrt{1+3/x^2}}$

$= \infty$

Limit does not exist.

29. $\lim_{x\to\infty} \frac{\sqrt{x^2-1}}{2x-1} = \lim_{x\to\infty} \frac{\sqrt{x^2-1}/\sqrt{x^2}}{2-1/x}$

$= \lim_{x\to\infty} \frac{\sqrt{1-1/x^2}}{2-1/x} = \frac{1}{2}$

30. $\lim_{x\to-\infty} \frac{\sqrt{x^4-1}}{x^3-1} = \lim_{x\to-\infty} \frac{\sqrt{x^4-1}}{x^3-1}\left(\frac{1/\left(-\sqrt{x^6}\right)}{1/x^3}\right)$

$= \lim_{x\to-\infty} \frac{\sqrt{1/x^2-1/x^6}}{-1+1/x^3} = 0,$

$\left(\text{for } x < 0, \text{ we have } -\sqrt{x^6} = x^3\right)$

31. $\lim_{x\to\infty} \frac{x+1}{\left(x^2+1\right)^{1/3}} = \lim_{x\to\infty} \frac{x+1}{\left(x^2+1\right)^{1/3}}\left(\frac{1/x^{2/3}}{1/\left(x^2\right)^{1/3}}\right)$

$= \lim_{x\to\infty} \frac{x^{1/3}+1/x^{2/3}}{\left(1+1/x^2\right)^{1/3}} = \infty$

Limit does not exist.

32. $\lim_{x\to-\infty} \frac{2x}{\left(x^6-1\right)^{1/3}} = \lim_{x\to-\infty} \frac{2x}{\left(x^6-1\right)^{1/3}} \frac{\left(1/x^2\right)}{\left(1/\left(x^6\right)^{1/3}\right)}$

$= \lim_{x\to-\infty} \frac{2/x}{\left(1-1/x^6\right)^{1/3}} = 0$

33. $\lim_{x\to\infty} \frac{1}{2x + \sin x} = 0$

34. $\lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \cos 0 = 1$

35. Because $(-1/x) \le (\sin 2x)/x \le (1/x)$ for all $x \ne 0$, you have by the Squeeze Theorem,

$$\lim_{x\to\infty} -\frac{1}{x} \le \lim_{x\to\infty} \frac{\sin 2x}{x} \le \lim_{x\to\infty} \frac{1}{x}$$

$$0 \le \lim_{x\to\infty} \frac{\sin 2x}{x} \le 0.$$

Therefore, $\lim_{x\to\infty} \frac{\sin 2x}{x} = 0.$

36. $\lim_{x\to\infty} \frac{x - \cos x}{x} = \lim_{x\to\infty}\left(1 - \frac{\cos x}{x}\right) = 1 - 0 = 1$

Note:

$\lim_{x\to\infty} \frac{\cos x}{x} = 0$ by the Squeeze Theorem because

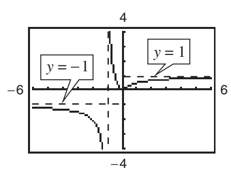
$-\frac{1}{x} \le \frac{\cos x}{x} \le \frac{1}{x}.$

37. $f(x) = \frac{|x|}{x + 1}$

$$\lim_{x\to\infty} \frac{|x|}{x + 1} = 1$$

$$\lim_{x\to-\infty} \frac{|x|}{x + 1} = -1$$

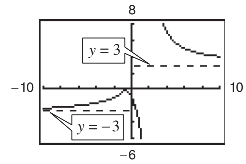
Therefore, $y = 1$ and $y = -1$ are both horizontal asymptotes.



38. $f(x) = \frac{|3x + 2|}{x - 2}$

$y = 3$ is a horizontal asymptote (to the right).

$y = -3$ is a horizontal asymptote (to the left).

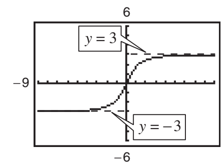


39. $f(x) = \frac{3x}{\sqrt{x^2 + 2}}$

$$\lim_{x\to\infty} f(x) = 3$$

$$\lim_{x\to-\infty} f(x) = -3$$

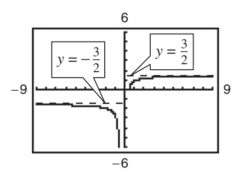
Therefore, $y = 3$ and $y = -3$ are both horizontal asymptotes.



40. $f(x) = \frac{\sqrt{9x^2 - 2}}{2x + 1}$

$y = \frac{3}{2}$ is a horizontal asymptote (to the right).

$y = -\frac{3}{2}$ is a horizontal asymptote (to the left).



41. $\lim_{x\to\infty} x \sin \frac{1}{x} = \lim_{t\to 0^+} \frac{\sin t}{t} = 1$

(Let $x = 1/t$.)

42. $\lim_{x\to\infty} x \tan \frac{1}{x} = \lim_{x\to 0^+} \frac{\tan t}{t} = \lim_{x\to 0^+}\left[\frac{\sin t}{t} \cdot \frac{1}{\cos t}\right] = (1)(1) = 1$

(Let $x = 1/t$.)

43. $\lim_{x\to-\infty}\left(x + \sqrt{x^2 + 3}\right) = \lim_{x\to-\infty}\left[\left(x + \sqrt{x^2 + 3}\right) \cdot \frac{x - \sqrt{x^2 + 3}}{x - \sqrt{x^2 + 3}}\right] = \lim_{x\to-\infty} \frac{-3}{x - \sqrt{x^2 + 3}} = 0$

44. $\lim_{x\to\infty}\left(x - \sqrt{x^2 + x}\right) = \lim_{x\to\infty}\left[\left(x - \sqrt{x^2 + x}\right) \cdot \frac{x + \sqrt{x^2 + x}}{x + \sqrt{x^2 + x}}\right] = \lim_{x\to\infty} \frac{-x}{x + \sqrt{x^2 + x}} = \lim_{x\to\infty} \frac{-1}{1 + \sqrt{1 + (1/x)}} = -\frac{1}{2}$

45. $\lim_{x\to-\infty}\left(3x+\sqrt{9x^2-x}\right) = \lim_{x\to-\infty}\left[\left(3x+\sqrt{9x^2-x}\right)\cdot\frac{3x-\sqrt{9x^2-x}}{3x-\sqrt{9x^2-x}}\right]$

$$= \lim_{x\to-\infty}\frac{x}{3x-\sqrt{9x^2-x}}$$

$$= \lim_{x\to-\infty}\frac{1}{3-\frac{\sqrt{9x^2-x}}{-\sqrt{x^2}}}\quad\left(\text{for } x<0 \text{ you have } x=-\sqrt{x^2}\right)$$

$$= \lim_{x\to-\infty}\frac{1}{3+\sqrt{9-(1/x)}} = \frac{1}{6}$$

46. $\lim_{x\to\infty}\left(4x-\sqrt{16x^2-x}\right)\frac{4x+\sqrt{16x^2-x}}{4x+\sqrt{16x^2-x}} = \lim_{x\to\infty}\frac{16x^2-\left(16x^2-x\right)}{4x+\sqrt{\left(16x^2-x\right)}}$

$$= \lim_{x\to\infty}\frac{x}{4x+\sqrt{16x^2-x}}$$

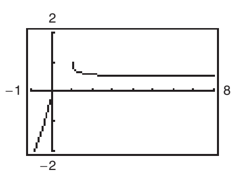
$$= \lim_{x\to\infty}\frac{1}{4+\sqrt{16-1/x}}$$

$$= \frac{1}{4+4} = \frac{1}{8}$$

47.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1	0.513	0.501	0.500	0.500	0.500	0.500

$$\lim_{x\to\infty}\left(x-\sqrt{x(x-1)}\right) = \lim_{x\to\infty}\frac{x-\sqrt{x^2-x}}{1}\cdot\frac{x+\sqrt{x^2-x}}{x+\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{x}{x+\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{1}{1+\sqrt{1-(1/x)}} = \frac{1}{2}$$

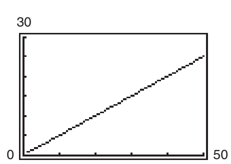


48.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1.0	5.1	50.1	500.1	5000.1	50,000.1	500,000.1

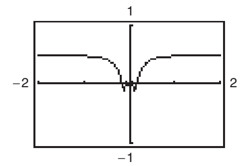
$$\lim_{x\to\infty}\frac{x^2-x\sqrt{x^2-x}}{1}\cdot\frac{x^2+x\sqrt{x^2-x}}{x^2+x\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{x^3}{x^2+x\sqrt{x^2-x}} = \infty$$

Limit does not exist.



49.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	0.479	0.500	0.500	0.500	0.500	0.500	0.500

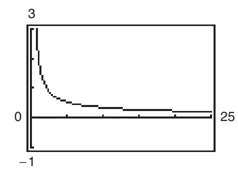


Let $x = 1/t$.

$$\lim_{x\to\infty} x \sin\left(\frac{1}{2x}\right) = \lim_{t\to 0^+} \frac{\sin(t/2)}{t} = \lim_{t\to 0^+} \frac{1}{2}\frac{\sin(t/2)}{t/2} = \frac{1}{2}$$

50.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	2.000	0.348	0.101	0.032	0.010	0.003	0.001



$$\lim_{x\to\infty} \frac{x+1}{x\sqrt{x}} = 0$$

51. $\displaystyle\lim_{v_1/v_2\to\infty} 100\left[1 - \frac{1}{(v_1/v_2)^c}\right] = 100[1 - 0] = 100\%$

52. $\displaystyle\lim_{t\to\infty} N(t) = \infty$

$\displaystyle\lim_{t\to\infty} E(t) = c$

53. An infinite limit describes how a limit fails to exist. A limit at infinity deals with the end behavior of a function.

54. Yes. Example 4 on page 203 illustrates such an occurrence. The graph of $f(x) = \dfrac{3x - 2}{\sqrt{2x^2 + 1}}$ crosses a horizontal asymptote.

55. (a) The function is even: $\displaystyle\lim_{x\to-\infty} f(x) = 5$

(b) The function is odd: $\displaystyle\lim_{x\to-\infty} f(x) = -5$

56. (a) $\displaystyle\lim_{t\to 0^+} T = 1700°$

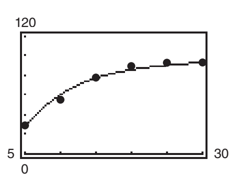
This is the temperature of the kiln.

(b) $\displaystyle\lim_{t\to\infty} T = 72°$

This is the temperature of the room.

57. $S = \dfrac{100t^2}{65 + t^2}, t > 0$

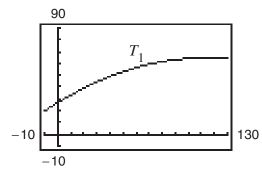
(a)



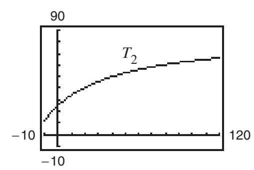
(b) Yes. $\displaystyle\lim_{t\to\infty} S = \frac{100}{1} = 100$

58. (a) $T_1(t) = -0.003t^2 + 0.68t + 26.6$

(b)



(c)



$T_2 = \dfrac{1451 + 86t}{58 + t}$

(d) $T_1(0) \approx 26.6°$

$T_2(0) \approx 25.0°$

(e) $\displaystyle\lim_{t\to\infty} T_2 = \frac{86}{1} = 86$

(f) No. The limiting temperature is $86°$. T_1 has no horizontal asymptote.

59. $f(x) = \dfrac{2x^2}{x^2 + 2}$

(a) $\lim_{x\to\infty} f(x) = 2 = L$

(b) $f(x_1) + \varepsilon = \dfrac{2x_1^2}{x_1^2 + 2} + \varepsilon = 2$

$$2x_1^2 + \varepsilon x_1^2 + 2\varepsilon = 2x_1^2 + 4$$
$$x_1^2\varepsilon = 4 - 2\varepsilon$$
$$x_1 = \sqrt{\frac{4 - 2\varepsilon}{\varepsilon}}$$
$$x_2 = -x_1 \text{ by symmetry}$$

(c) Let $M = \sqrt{\dfrac{4 - 2\varepsilon}{\varepsilon}} > 0$. For $x > M$:

$$x > \sqrt{\frac{4 - 2\varepsilon}{\varepsilon}}$$
$$x^2\varepsilon > 4 - 2\varepsilon$$
$$2x^2 + x^2\varepsilon + 2\varepsilon > 2x^2 + 4$$
$$\frac{2x^2}{x^2 + 2} + \varepsilon > 2$$
$$\left|\frac{2x^2}{x^2 + 2} - 2\right| > |-\varepsilon| = \varepsilon$$
$$|f(x) - L| > \varepsilon$$

(d) Similarly, $N = -\sqrt{\dfrac{4 - 2\varepsilon}{\varepsilon}}$.

60. $f(x) = \dfrac{6x}{\sqrt{x^2 + 2}}$

(a) $\lim_{x\to\infty} f(x) = 6 = L$

$\lim_{x\to-\infty} f(x) = -6 = K$

(b) $f(x_1) + \varepsilon = \dfrac{6x_1}{\sqrt{x_1^2 + 2}} + \varepsilon = 6$

$$6x_1 = (6 - \varepsilon)\sqrt{x_1^2 + 2}$$
$$36x_1^2 = (x_1^2 + 2)(6 - \varepsilon)^2$$
$$36x_1^2 - (6 - \varepsilon)^2 x_1^2 = 2(6 - \varepsilon)^2$$
$$x_1^2\left[36 - 36 + 12\varepsilon - \varepsilon^2\right] = 2(6 - \varepsilon)^2$$
$$x_1^2 = \frac{2(6 - \varepsilon)^2}{12\varepsilon - \varepsilon^2}$$
$$x_1 = (6 - \varepsilon)\sqrt{\frac{2}{12\varepsilon - \varepsilon^2}}$$
$$x_2 = -x_1 \text{ by symmetry}$$

(c) $M = x_1 = (6 - \varepsilon)\sqrt{\dfrac{2}{12\varepsilon - \varepsilon^2}}$

(d) $N = x_2 = (\varepsilon - 6)\sqrt{\dfrac{2}{12\varepsilon - \varepsilon^2}}$

61. $\lim_{x\to\infty} \dfrac{3x}{\sqrt{x^2 + 3}} = 3$

$f(x_1) + \varepsilon = \dfrac{3x_1}{\sqrt{x_1^2 + 3}} + \varepsilon = 3$

$$3x_1 = (3 - \varepsilon)\sqrt{x_1^2 + 3}$$
$$9x_1^2 = (3 - \varepsilon)^2(x_1^2 + 3)$$
$$9x_1^2 - (3 - \varepsilon)^2 x_1^2 = 3(3 - \varepsilon)^2$$
$$x_1^2(9 - 9 + 6\varepsilon - \varepsilon^2) = 3(3 - \varepsilon)^2$$
$$x_1^2 = \frac{3(3 - \varepsilon)^2}{6\varepsilon - \varepsilon^2}$$
$$x_1 = (3 - \varepsilon)\sqrt{\frac{3}{6\varepsilon - \varepsilon^2}}$$

Let $M = x_1 = (3 - \varepsilon)\sqrt{\dfrac{3}{6\varepsilon - \varepsilon^2}}$

(a) When $\varepsilon = 0.5$:

$$M = (3 - 0.5)\sqrt{\frac{3}{6(0.5) - (0.5)^2}} = \frac{5\sqrt{33}}{11}$$

(b) When $\varepsilon = 0.1$:

$$M = (3 - 0.1)\sqrt{\frac{3}{6(0.1) - (0.1)^2}} = \frac{29\sqrt{177}}{59}$$

62. $\lim_{x\to-\infty} \dfrac{3x}{\sqrt{x^2 + 3}} = -3$

$f(x_1) - \varepsilon = \dfrac{3x_1}{\sqrt{x_1^2 + 3}} - \varepsilon = -3$

$$3x_1 = (\varepsilon - 3)\sqrt{x_1^2 + 3}$$
$$9x_1^2 = (\varepsilon - 3)^2(x_1^2 + 3)$$
$$9x_1^2 - (\varepsilon - 3)^2 x_1^2 = 3(\varepsilon - 3)^2$$
$$x_1^2(9 - \varepsilon^2 + 6\varepsilon - 9) = 3(\varepsilon - 3)^2$$
$$x_1^2 = \frac{3(\varepsilon - 3)^2}{6\varepsilon - \varepsilon^2}$$
$$x_1 = (\varepsilon - 3)\sqrt{\frac{3}{6\varepsilon - \varepsilon^2}}$$

Let $x_1 = N = (\varepsilon - 3)\sqrt{\dfrac{3}{6\varepsilon - \varepsilon^2}}$

(a) When $\varepsilon = 0.5$:

$$N = (0.5 - 3)\sqrt{\frac{3}{6(0.5) - (0.5)^2}} = \frac{-5\sqrt{33}}{11}$$

(b) When $\varepsilon = 0.1$:

$$N = (0.1 - 3)\sqrt{\frac{3}{6(0.1) - (0.1)^2}} = \frac{-29\sqrt{177}}{59}$$

63. $\lim_{x\to\infty} \frac{1}{x^2} = 0$. Let $\varepsilon > 0$ be given. You need $M > 0$ such that

$$\left|f(x) - L\right| = \left|\frac{1}{x^2} - 0\right| = \frac{1}{x^2} < \varepsilon \text{ whenever } x > M.$$

$$x^2 > \frac{1}{\varepsilon} \Rightarrow x > \frac{1}{\sqrt{\varepsilon}}$$

Let $M = \frac{1}{\sqrt{\varepsilon}}$.

For $x > M$, you have

$$x > \frac{1}{\sqrt{\varepsilon}} \Rightarrow x^2 > \frac{1}{\varepsilon} \Rightarrow \frac{1}{x^2} < \varepsilon \Rightarrow \left|f(x) - L\right| < \varepsilon.$$

64. $\lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$. Let $\varepsilon > 0$ be given. You need $M > 0$ such that

$$\left|f(x) - L\right| = \left|\frac{2}{\sqrt{x}} - 0\right| = \frac{2}{\sqrt{x}} < \varepsilon \text{ whenever } x > M.$$

$$\frac{2}{\sqrt{x}} < \varepsilon \Rightarrow \frac{\sqrt{x}}{2} > \frac{1}{\varepsilon} \Rightarrow x > \frac{4}{\varepsilon^2}$$

Let $M = 4/\varepsilon^2$.

For $x > M = 4/\varepsilon^2$, you have

$$\sqrt{x} > 2/\varepsilon \Rightarrow \frac{2}{\sqrt{x}} < \varepsilon \Rightarrow \left|f(x) - L\right| < \varepsilon.$$

65. $\lim_{x\to-\infty} \frac{1}{x^3} = 0$. Let $\varepsilon > 0$. You need $N < 0$ such that

$$\left|f(x) - L\right| = \left|\frac{1}{x^3} - 0\right| = \frac{-1}{x^3} < \varepsilon \text{ whenever } x < N.$$

$$\frac{-1}{x^3} < \varepsilon \Rightarrow -x^3 > \frac{1}{\varepsilon} \Rightarrow x < \frac{-1}{\varepsilon^{1/3}}$$

Let $N = \frac{-1}{\sqrt[3]{\varepsilon}}$.

For $x < N = \frac{-1}{\sqrt[3]{\varepsilon}}$,

$$\frac{1}{x} > -\sqrt[3]{\varepsilon}$$

$$-\frac{1}{x} < \sqrt[3]{\varepsilon}$$

$$-\frac{1}{x^3} < \varepsilon$$

$$\Rightarrow \left|f(x) - L\right| < \varepsilon.$$

66. $\lim_{x\to-\infty} \frac{1}{x-2} = 0$. Let $\varepsilon > 0$ be given.

You need $N < 0$ such that

$$\left|f(x) - L\right| = \left|\frac{1}{x-2} - 0\right| = \frac{-1}{x-2} < \varepsilon \text{ whenever } x < N.$$

$$\frac{-1}{x-2} < \varepsilon \Rightarrow x - 2 < \frac{-1}{\varepsilon} \Rightarrow x < 2 - \frac{1}{\varepsilon}$$

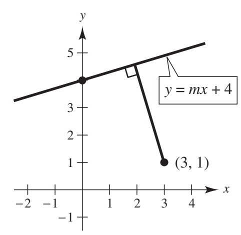
Let $N = 2 - \frac{1}{\varepsilon}$. For $x < N = 2 - \frac{1}{\varepsilon}$,

$$x - 2 < \frac{-1}{\varepsilon}$$

$$\frac{-1}{x-2} < \varepsilon$$

$$\Rightarrow \left|f(x) - L\right| < \varepsilon.$$

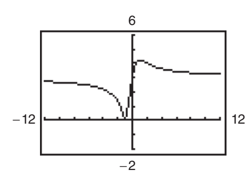
67. line: $mx - y + 4 = 0$



(a) $d = \frac{\left|Ax_1 + By_1 + C\right|}{\sqrt{A^2 + B^2}} = \frac{\left|m(3) - 1(1) + 4\right|}{\sqrt{m^2 + 1}}$

$$= \frac{\left|3m + 3\right|}{\sqrt{m^2 + 1}}$$

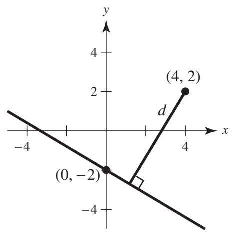
(b)



(c) $\lim_{m\to\infty} d(m) = 3 = \lim_{m\to-\infty} d(m)$

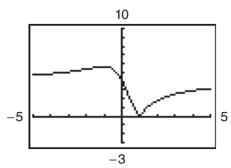
The line approaches the vertical line $x = 0$. So, the distance from $(3, 1)$ approaches 3.

68. line: $y + 2 = m(x - 0) \Rightarrow mx - y - 2 = 0$



(a) $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \dfrac{|m(4) - 1(2) - 2|}{\sqrt{m^2 + 1}} = \dfrac{|4m - 4|}{\sqrt{m^2 + 1}}$

(b)



(c) $\lim_{m \to \infty} d(m) = 4; \lim_{m \to -\infty} d(m) = 4$

The line approaches the vertical line $x = 0$. So, the distance from $(4, 2)$ approaches 4.

69. $\lim_{x \to \infty} \dfrac{p(x)}{q(x)} = \lim_{x \to \infty} \dfrac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$

Divide $p(x)$ and $q(x)$ by x^m.

Case 1: If $n < m$: $\lim_{x \to \infty} \dfrac{p(x)}{q(x)} = \lim_{x \to \infty} \dfrac{\dfrac{a_n}{x^{m-n}} + \cdots + \dfrac{a_1}{x^{m-1}} + \dfrac{a_0}{x^m}}{b_m + \cdots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}} = \dfrac{0 + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \dfrac{0}{b_m} = 0.$

Case 2: If $m = n$: $\lim_{x \to \infty} \dfrac{p(x)}{q(x)} = \lim_{x \to \infty} \dfrac{a_n + \cdots + \dfrac{a_1}{x^{m-1}} + \dfrac{a_0}{x^m}}{b_m + \cdots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}} = \dfrac{a_n + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \dfrac{a_n}{b_m}.$

Case 3: If $n > m$: $\lim_{x \to \infty} \dfrac{p(x)}{q(x)} = \lim_{x \to \infty} \dfrac{a_n x^{n-m} + \cdots + \dfrac{a_1}{x^{m-1}} + \dfrac{a_0}{x^m}}{b_m + \cdots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}} = \dfrac{\pm\infty + \cdots + 0}{b_m + \cdots + 0} = \pm\infty.$

70. $\lim_{x \to \infty} x^3 = \infty$. Let $M > 0$ be given. You need $N > 0$ such that $f(x) = x^3 > M$ whenever $x > N$.

$x^3 > M \Rightarrow x > M^{1/3}$. Let $N = M^{1/3}$. For $x > N = M^{1/3}, x > M^{1/3} \Rightarrow x^3 > M \Rightarrow f(x) > M$.

Section 3.6 A Summary of Curve Sketching

1. Domain, range, asymptotes, symmetry, end behavior, differentiability, relative extrema, points of inflection, concavity, increasing and decreasing, infinite limits at infinity

2. Across the top of the table, write y, y', y'' and Characteristic of Graph. Down the left side of the table, place the intervals and key points that are determined by analyzing y' and y''. Fill in the appropriate characteristic of the graph.

3. Rational functions can have slant asymptotes. Use long division to rewrite the rational function as the sum of a first-degree polynomial and another rational function.

4. A fifth-degree polynomial can have at most four critical numbers, and hence four relative extrema. It can have at most three points of inflection.

5. f has constant negative slope. Matches (d)

6. The slope of f approaches ∞ as $x \to 0^-$, and approaches $-\infty$ as $x \to 0^+$. Matches (c)

7. The slope is periodic, and zero at $x = 0$. Matches (a)

8. The slope is positive up to approximately $x = 1.5$. Matches (b)

9. $y = \dfrac{1}{x-2} - 3$

$y' = -\dfrac{1}{(x-2)^2} \Rightarrow$ undefined when $x = 2$

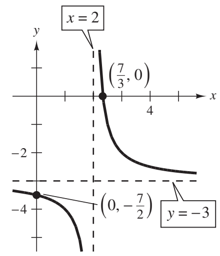
$y'' = \dfrac{2}{(x-2)^3} \Rightarrow$ undefined when $x = 2$

Intercepts: $\left(\frac{7}{3}, 0\right), \left(0, -\frac{7}{2}\right)$

Vertical asymptote: $x = 2$

Horizontal asymptote: $y = -3$

	y	y'	y''	Conclusion
$-\infty < x < 2$		$-$	$-$	Decreasing, concave down
$2 < x < \infty$		$-$	$+$	Decreasing, concave up



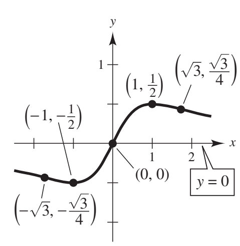
10. $y = \dfrac{x}{x^2+1}$

$y' = \dfrac{1-x^2}{(x^2+1)^2} = \dfrac{(1-x)(x+1)}{(x^2+1)^2} = 0$ when $x = \pm 1$.

$y'' = -\dfrac{2x(3-x^2)}{(x^2+1)^3} = 0$ when $x = 0, \pm\sqrt{3}$.

Horizontal asymptote: $y = 0$

	y	y'	y''	Conclusion
$-\infty < x < -\sqrt{3}$		$-$	$-$	Decreasing, concave down
$x = -\sqrt{3}$	$-\frac{\sqrt{3}}{4}$	$-$	0	Point of inflection
$-\sqrt{3} < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	$-\frac{1}{2}$	0	$+$	Relative minimum
$-1 < x < 0$		$+$	$+$	Increasing, concave up
$x = 0$	0	$+$	0	Point of inflection
$0 < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	$\frac{1}{2}$	0	$-$	Relative maximum
$1 < x < \sqrt{3}$		$-$	$-$	Decreasing, concave down
$x = \sqrt{3}$	$\frac{\sqrt{3}}{4}$	$-$	0	Point of inflection
$\sqrt{3} < x < \infty$		$-$	$+$	Decreasing, concave up



11. $y = \dfrac{x}{1 - x}$

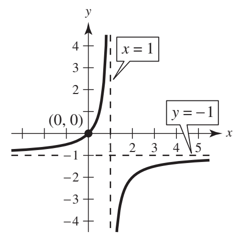
$y' = \dfrac{1}{(x-1)^2}$, undefined when $x = 1$

$y'' = -\dfrac{2}{(x-1)^3}$, undefined when $x = 1$

Horizontal asymptote: $y = -1$

Vertical asymptote: $x = 1$

	y	y'	y''	Conclusion
$-\infty < x < 1$		+	+	Increasing, concave up
$1 < x < \infty$		+	–	Increasing, concave down



12. $y = \dfrac{x - 4}{x - 3}$

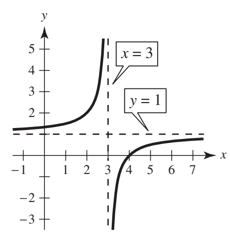
$y' = \dfrac{1}{(x-3)^2}$, undefined when $x = 3$

$y'' = -\dfrac{2}{(x-3)^3}$, undefined when $x = 3$

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 3$

	y	y'	y''	Conclusion
$-\infty < x < 3$		+	+	Increasing, concave up
$3 < x < \infty$		+	–	Increasing, concave down



13. $y = \dfrac{x + 1}{x^2 - 4}$

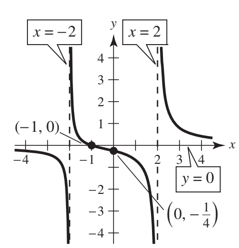
$y' = -\dfrac{x^2 + 2x + 4}{(x^2 - 4)^2}$, undefined when $x = \pm 2$

$y'' = \dfrac{2(x^3 + 3x^2 + 12x + 4)}{(x^2 - 4)^3} = 0$ when $x = \sqrt[3]{9} - \dfrac{3}{\sqrt[3]{9}} - 1 \approx -0.36$, undefined when $x = \pm 2$

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = \pm 2$

	y	y'	y''	Conclusion
$-\infty < x < -2$		–	–	Decreasing, concave down
$-2 < x < \sqrt[3]{9} - \dfrac{3}{\sqrt[3]{9}} - 1$		+	+	Increasing, concave up
$x = \sqrt[3]{9} - \dfrac{3}{\sqrt[3]{9}} - 1$	−0.16	+	0	Point of inflection
$\sqrt[3]{9} - \dfrac{3}{\sqrt[3]{9}} - 1 < x < 2$		–	–	Decreasing, concave down
$2 < x < \infty$		–	+	Decreasing, concave up



14. $y = \dfrac{2x}{9 - x^2}$

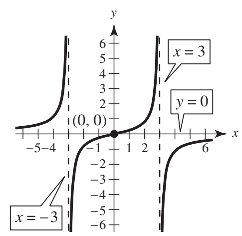
$y' = \dfrac{2(x^2 + 9)}{(x^2 - 9)^2}$, undefined when $x = \pm 3$

$y'' = -\dfrac{4x(x^2 + 27)}{(x^2 - 9)^3} = 0$ when $x = 0$, undefined when $x = \pm 3$

Horizontal asymptote: $y = 0$

Vertical asymptotes: $x = \pm 3$

	y	y'	y''	Conclusion
$-\infty < x < -3$		+	+	Increasing, concave up
$-3 < x < 0$		+	−	Increasing, concave down
$x = 0$	0	+	0	Point of inflection
$0 < x < 3$		+	+	Increasing, concave up
$3 < x < \infty$		+	−	Increasing, concave down



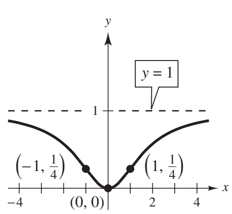
15. $y = \dfrac{x^2}{x^2 + 3}$

$y' = \dfrac{6x}{(x^2 + 3)^2} = 0$ when $x = 0$.

$y'' = \dfrac{18(1 - x^2)}{(x^2 + 3)^3} = 0$ when $x = \pm 1$.

Horizontal asymptote: $y = 1$

	y	y'	y''	Conclusion
$-\infty < x < -1$		−	−	Decreasing, concave down
$x = -1$	$\frac{1}{4}$	−	0	Point of inflection
$-1 < x < 0$		−	+	Decreasing, concave up
$x = 0$	0	0	+	Relative minimum
$0 < x < 1$		+	+	Increasing, concave up
$x = 1$	$\frac{1}{4}$	+	0	Point of inflection
$1 < x < \infty$		+	−	Increasing, concave down



16. $y = \dfrac{x^2 + 1}{x^2 - 4}$

$y' = \dfrac{-10x}{\left(x^2 - 4\right)^2} = 0$ when $x = 0$ and undefined when $x = \pm 2$.

$y'' = \dfrac{10\left(3x^2 + 4\right)}{\left(x^2 - 4\right)^3} < 0$ when $x = 0$.

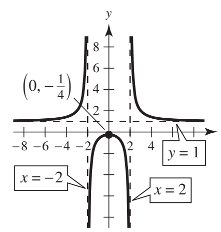
Intercept: $(0, -1/4)$

Symmetric about y-axis

Vertical asymptotes: $x = \pm 2$

Horizontal asymptote: $y = 1$

	y	y'	y''	Conclusion
$-\infty < x < -2$		$+$	$+$	Increasing, concave up
$-2 < x < 0$		$+$	$-$	Increasing, concave down
$x = 0$	$-\frac{1}{4}$			Relative maximum
$0 < x < 2$		$-$	$-$	Decreasing, concave down
$2 < x < \infty$		$-$	$+$	Decreasing, concave up



17. $y = 3 + \dfrac{2}{x}$

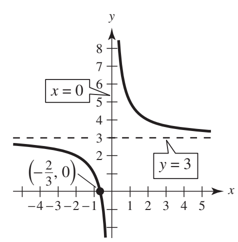
$y' = -\dfrac{2}{x^2}$, undefined when $x = 0$

$y'' = \dfrac{4}{x^3}$, undefined when $x = 0$

Horizontal asymptote: $y = 3$

Vertical asymptote: $x = 0$

	y	y'	y''	Conclusion
$-\infty < x < 0$		$-$	$-$	Decreasing, concave down
$0 < x < \infty$		$-$	$+$	Decreasing, concave up



18. $f(x) = \dfrac{x - 3}{x} = 1 - \dfrac{3}{x}$

$f'(x) = \dfrac{3}{x^2}$ undefined when $x = 0$

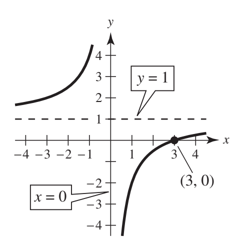
$f''(x) = -\dfrac{6}{x^3} \neq 0$

Vertical asymptote: $x = 0$

Intercept: $(3, 0)$

Horizontal asymptote: $y = 1$

	y	y'	y''	Conclusion
$-\infty < x < 0$		$+$	$+$	Increasing, concave up
$0 < x < \infty$		$+$	$-$	Increasing, concave down



19. $f(x) = x + \dfrac{32}{x^2}$

$f'(x) = 1 - \dfrac{64}{x^3} = \dfrac{(x-4)(x^2+4x+16)}{x^3} = 0$ when $x = 4$ and undefined when $x = 0$.

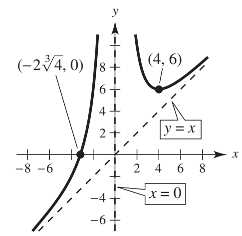
$f''(x) = \dfrac{192}{x^4}$

Intercept: $\left(-2\sqrt[3]{4}, 0\right)$

Vertical asymptote: $x = 0$

Slant asymptote: $y = x$

	y	y'	y''	Conclusion
$-\infty < x < 0$		+	+	Increasing, concave up
$0 < x < 4$		−	+	Decreasing, concave up
$x = 4$	6	0	+	Relative minimum
$4 < x < \infty$		+	+	Increasing, concave up



20. $y = \dfrac{4}{x^2} + 1 = \dfrac{4 + x^2}{x^2}$

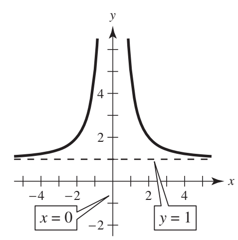
$y' = -\dfrac{8}{x^3}$, undefined when $x = 0$

$y'' = \dfrac{24}{x^4}$, undefined when $x = 0$

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 0$

	y	y'	y''	Conclusion
$-\infty < x < 0$		+	+	Increasing, concave up
$0 < x < \infty$		−	+	Decreasing, concave up



21. $y = \dfrac{3x}{x^2 - 1}$

$y' = \dfrac{-3(x^2+1)}{(x^2-1)^2}$ undefined when $x = \pm 1$

$y'' = \dfrac{6x(x^2+3)}{(x^2-1)^3}$

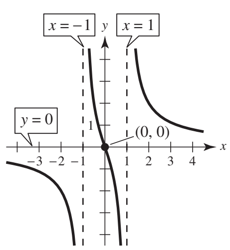
Intercept: $(0, 0)$

Symmetry with respect to origin

Vertical asymptotes: $x = \pm 1$

Horizontal asymptote: $y = 0$

	y	y'	y''	Conclusion
$-\infty < x < -1$		−	−	Decreasing, concave down
$-1 < x < 0$		−	+	Decreasing, concave up
$x = 0$	0	−3	0	Point of inflection
$0 < x < 1$		−	−	Decreasing, concave down
$1 < x < \infty$		−	+	Decreasing, concave up



22. $f(x) = \dfrac{x^3}{x^2 - 9} = x + \dfrac{9x}{x^2 - 9}$

$f'(x) = \dfrac{x^2(x^2 - 27)}{(x^2 - 9)^2} = 0$ when $x = 0, \pm 3\sqrt{3}$ and is undefined when $x = \pm 3$.

$f''(x) = \dfrac{18x(x^2 + 27)}{(x^2 - 9)^3} = 0$ when $x = 0$

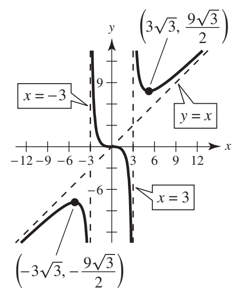
Intercept: $(0, 0)$

Symmetry: origin

Vertical asymptotes: $x = \pm 3$

Slant asymptote: $y = x$

	y	y'	y''	Conclusion
$-\infty < x < -3\sqrt{3}$		+	–	Increasing, concave down
$x = -3\sqrt{3}$	$-\dfrac{9\sqrt{3}}{2}$	0	–	Relative maximum
$-3\sqrt{3} < x < -3$		–	–	Decreasing, concave down
$-3 < x < 0$		–	+	Decreasing, concave up
$x = 0$	0	0	0	Point of inflection
$0 < x < 3$		–	–	Decreasing, concave down
$3 < x < 3\sqrt{3}$		–	+	Decreasing, concave up
$x = 3\sqrt{3}$	$\dfrac{9\sqrt{3}}{2}$	0	+	Relative minimum
$3\sqrt{3} < x < \infty$		+	+	Increasing, concave up



23. $y = \dfrac{x^2 - 6x + 12}{x - 4} = x - 2 + \dfrac{4}{x - 4}$

$y' = 1 - \dfrac{4}{(x - 4)^2}$

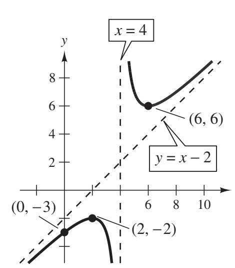
$= \dfrac{(x - 2)(x - 6)}{(x - 4)^2} = 0$ when $x = 2, 6$ and is undefined when $x = 4$.

$y'' = \dfrac{8}{(x - 4)^3}$

Vertical asymptote: $x = 4$

Slant asymptote: $y = x - 2$

	y	y'	y''	Conclusion
$-\infty < x < 2$		+	–	Increasing, concave down
$x = 2$	-2	0	–	Relative maximum
$2 < x < 4$		–	–	Decreasing, concave down
$4 < x < 6$		–	+	Decreasing, concave up
$x = 6$	6	0	+	Relative minimum
$6 < x < \infty$		+	+	Increasing, concave up



24. $y = \dfrac{-x^2 - 4x - 7}{x + 3} = -x - 1 - \dfrac{4}{x + 3}$

$y' = -\dfrac{x^2 + 6x + 5}{(x + 3)^2} = -\dfrac{(x + 1)(x + 5)}{(x + 3)^2} = 0$ when $x = -1, -5$ and is undefined when $x = -3$.

$y'' = \dfrac{-8}{(x + 3)^3}$

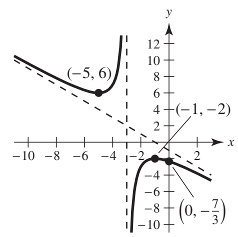
Intercept: $\left(0, -\frac{7}{3}\right)$

No symmetry

Vertical asymptote: $x = -3$

Slant asymptote: $y = -x - 1$

	y	y'	y''	Conclusion
$-\infty < x < -5$		$-$	$+$	Decreasing, concave up
$x = -5$	6	0	$+$	Relative minimum
$-5 < x < -3$		$+$	$+$	Increasing, concave up
$-3 < x < -1$		$+$	$-$	Increasing, concave down
$x = -1$	-2	0	$-$	Relative maximum
$-1 < x < \infty$		$-$	$-$	Decreasing, concave down



25. $y = \dfrac{x^3}{\sqrt{x^2 - 4}}$, Domain: $(-\infty, 2), (2, \infty)$

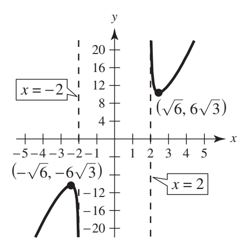
$y' = \dfrac{2x^2(x^2 - 6)}{(x^2 - 4)^{3/2}} = 0$ when $x = 0, \pm\sqrt{6}$, undefined when $x = \pm 2$

$y'' = \dfrac{2x^5 - 20x^3 + 96x}{(x^2 - 4)^{5/2}} = 0$ when $x = 0$, undefined when $x = \pm 2$

Vertical asymptotes: $x = \pm 2$

Symmetric with respect to the origin

	y	y'	y''	Conclusion
$-\infty < x < -\sqrt{6}$		$+$	$-$	Increasing, concave down
$x = -\sqrt{6}$	$-6\sqrt{3}$	0	$-$	Relative maximum
$-\sqrt{6} < x < -2$		$-$	$-$	Decreasing, concave down
$2 < x < \sqrt{6}$		$-$	$+$	Decreasing, concave up
$x = \sqrt{6}$	$6\sqrt{3}$	0	$+$	Relative minimum
$\sqrt{6} < x < \infty$		$+$	$+$	Increasing, concave up



26. $y = \dfrac{x}{\sqrt{x^2 - 4}}$, Domain: $(-\infty, 2), (2, \infty)$

$y' = -\dfrac{4}{(x^2 - 4)^{3/2}}$, undefined when $x = \pm 2$

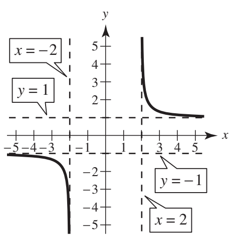
$y'' = \dfrac{12x}{(x^2 - 4)^{5/2}} = 0$ when $x = 0$, undefined when $x = \pm 2$

Horizontal asymptotes: $y = \pm 1$ because $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2 - 4}} = 1$, $\lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2 - 4}} = -1$.

Vertical asymptotes: $x = \pm 2$

Symmetric with respect to the origin

	y	y'	y''	Conclusion
$-\infty < x < -2$		$-$	$-$	Decreasing, concave down
$2 < x < \infty$		$-$	$+$	Decreasing, concave up



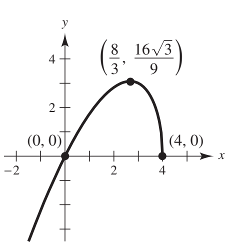
27. $y = x\sqrt{4 - x}$, Domain: $(-\infty, 4]$

$y' = \dfrac{8 - 3x}{2\sqrt{4 - x}} = 0$ when $x = \dfrac{8}{3}$ and undefined when $x = 4$.

$y'' = \dfrac{3x - 16}{4(4 - x)^{3/2}} = 0$ when $x = \dfrac{16}{3}$ and undefined when $x = 4$.

Note: $x = \dfrac{16}{3}$ is not in the domain.

	y	y'	y''	Conclusion
$-\infty < x < \dfrac{8}{3}$		$+$	$-$	Increasing, concave down
$x = \dfrac{8}{3}$	$\dfrac{16}{3\sqrt{3}}$	0	$-$	Relative maximum
$\dfrac{8}{3} < x < 4$		$-$	$-$	Decreasing, concave down
$x = 4$	0	Undefined	Undefined	Endpoint



28. $h(x) = x\sqrt{9 - x^2}$, Domain: $-3 \le x \le 3$

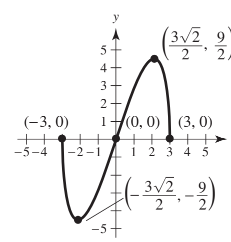
$h'(x) = \dfrac{9 - 2x^2}{\sqrt{9 - x^2}} = 0$ when $x = \pm\dfrac{3}{\sqrt{2}} = \pm\dfrac{3\sqrt{2}}{2}$ and undefined when $x = \pm 3$.

$h''(x) = \dfrac{x(2x^2 - 27)}{(9 - x^2)^{3/2}} = 0$ when $x = 0$ and undefined when $x = \pm 3$.

Intercepts: $(0, 0)$, $(\pm 3, 0)$

Symmetric with respect to the origin

	y	y'	y''	Conclusion
$x = -3$	0	Undefined	Undefined	Endpoint
$-3 < x < -\dfrac{3}{\sqrt{2}}$		−	+	Decreasing, concave up
$x = -\dfrac{3}{\sqrt{2}}$	$-\dfrac{9}{2}$	0	+	Relative minimum
$-\dfrac{3}{\sqrt{2}} < x < 0$		+	+	Increasing, concave up
$x = 0$	0	3	0	Point of inflection
$0 < x < \dfrac{3}{\sqrt{2}}$		+	−	Increasing, concave down
$x = \dfrac{3}{\sqrt{2}}$	$\dfrac{9}{2}$	0	−	Relative maximum
$\dfrac{3}{\sqrt{2}} < x < 3$		−	−	Decreasing, concave down
$x = 3$	0	Undefined	Undefined	Endpoint



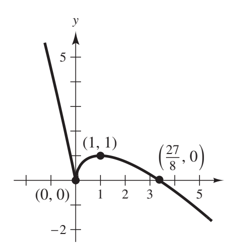
29. $y = 3x^{2/3} - 2x$

$y' = 2x^{-1/3} - 2 = \dfrac{2(1 - x^{1/3})}{x^{1/3}}$

$= 0$ when $x = 1$ and undefined when $x = 0$.

$y'' = \dfrac{-2}{3x^{4/3}} < 0$ when $x \ne 0$.

	y	y'	y''	Conclusion
$-\infty < x < 0$		−	−	Decreasing, concave down
$x = 0$	0	Undefined	Undefined	Relative minimum
$0 < x < 1$		+	−	Increasing, concave down
$x = 1$	1	0	−	Relative maximum
$1 < x < \infty$		−	−	Decreasing, concave down



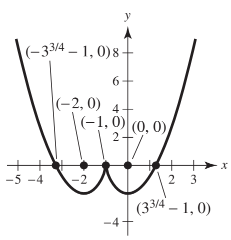
30. $y = (x+1)^2 - 3(x+1)^{2/3}$

$y' = 2(x+1) - 2(x+1)^{-1/3} = \dfrac{2(x+1)^{4/3} - 2}{(x+1)^{1/3}} = 0$ when $x = 0, -2$ and undefined when $x = -1$.

$y'' = 2 + \dfrac{2}{3}(x+1)^{-4/3} = \dfrac{6(x+1)^{4/3} + 2}{3(x+1)^{4/3}}$

Intercepts: $(-1, 0), \left(\pm 3^{3/4} - 1, 0\right)$

	y	y'	y''	Conclusion
$-\infty < x < -2$		$-$	$+$	Decreasing, concave up
$x = -2$	-2	0	$+$	Relative minimum
$-2 < x < -1$		$+$	$+$	Increasing, concave up
$x = -1$	0	Undefined	$+$	Relative maximum
$-1 < x < 0$		$-$	$+$	Decreasing, concave up
$x = 0$	0	0	$+$	Relative minimum
$0 < x < \infty$		$+$	$+$	Increasing, concave up



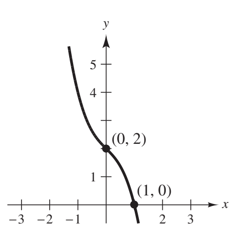
31. $y = 2 - x - x^3$

$y' = -1 - 3x^2$

No critical numbers

$y'' = -6x = 0$ when $x = 0$.

	y	y'	y''	Conclusion
$-\infty < x < 0$		$-$	$+$	Decreasing, concave up
$x = 0$	2	$-$	0	Point of inflection
$0 < x < \infty$		$-$	$-$	Decreasing, concave down

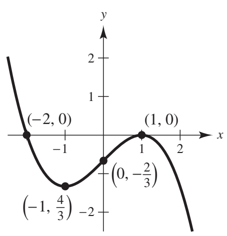


32. $y = -\frac{1}{3}(x^3 - 3x + 2)$

$y' = -x^2 + 1 = 0$ when $x = \pm 1$.

$y'' = -2x = 0$ when $x = 0$.

	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	$-\frac{4}{3}$	0	$+$	Relative minimum
$-1 < x < 0$		$+$	$+$	Increasing, concave up
$x = 0$	$-\frac{2}{3}$	$+$	0	Point of inflection
$0 < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	0	0	$-$	Relative maximum
$1 < x < \infty$		$-$	$-$	Decreasing, concave down

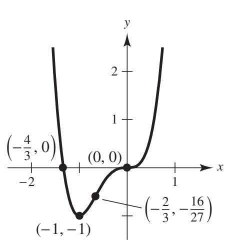


33. $y = 3x^4 + 4x^3$

$y' = 12x^3 + 12x^2 = 12x^2(x + 1) = 0$ when $x = 0, x = -1$.

$y'' = 36x^2 + 24x = 12x(3x + 2) = 0$ when $x = 0, x = -\frac{2}{3}$.

	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	-1	0	$+$	Relative minimum
$-1 < x < -\frac{2}{3}$		$+$	$+$	Increasing, concave up
$x = -\frac{2}{3}$	$-\frac{16}{27}$	$+$	0	Point of inflection
$-\frac{2}{3} < x < 0$		$+$	$-$	Increasing, concave down
$x = 0$	0	0	0	Point of inflection
$0 < x < \infty$		$+$	$+$	Increasing, concave up



34. $y = -2x^4 + 3x^2$

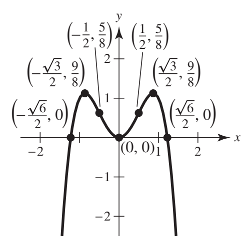
$y' = -8x^3 + 6x = 0$ when $x = 0, \pm\dfrac{\sqrt{3}}{2}$.

$y'' = -24x^2 + 6 = 0$ when $x = \pm\dfrac{1}{2}$.

Symmetry: y-axis

Intercepts: $\left(\pm\dfrac{\sqrt{6}}{2}, 0\right)$

	y	y'	y''	Conclusion
$-\infty < x < -\dfrac{\sqrt{3}}{2}$		+	–	Increasing, concave down
$x = -\dfrac{\sqrt{3}}{2}$	$\dfrac{9}{8}$	0	–	Relative maximum
$-\dfrac{\sqrt{3}}{2} < x < -\dfrac{1}{2}$		–	–	Decreasing, concave down
$x = -\dfrac{1}{2}$	$\dfrac{5}{8}$	-2	0	Point of inflection
$-\dfrac{1}{2} < x < 0$		–	+	Decreasing, concave up
$x = 0$	0	0	+	Relative minimum
$0 < x < \dfrac{1}{2}$		+	+	Increasing, concave up
$x = \dfrac{1}{2}$	$\dfrac{5}{8}$	2	0	Point of inflection
$\dfrac{1}{2} < x < \dfrac{\sqrt{3}}{2}$		+	–	Increasing, concave down
$x = \dfrac{\sqrt{3}}{2}$	$\dfrac{9}{8}$	0	–	Relative maximum
$\dfrac{\sqrt{3}}{2} < x < \infty$		–	–	Decreasing, concave down



35. $xy^2 = 9$

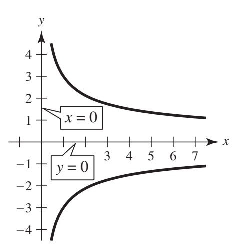
$y^2 = \dfrac{9}{x}$

$y = \pm\dfrac{3}{\sqrt{x}}, x > 0$

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = 0$

Symmetric with respect to x-axis



36. $x^2y = 9 \Rightarrow y = \dfrac{9}{x^2}$

$y' = -\dfrac{18}{x^3}$, undefined when $x = 0$

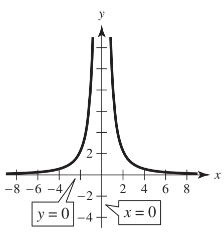
$y'' = \dfrac{54}{x^4}$, undefined when $x = 0$

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = 0$

Symmetric with respect to y-axis

	y	y'	y''	Conclusion
$-\infty < x < 0$		+	+	Increasing, concave up
$0 < x < \infty$		–	+	Decreasing, concave down

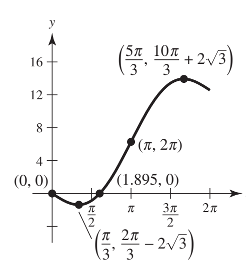


37. $f(x) = 2x - 4\sin x,\ 0 \le x \le 2\pi$

$f'(x) = 2 - 4\cos x = 0$ when $\cos x = \dfrac{1}{2} \Rightarrow x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

$f''(x) = 4\sin x = 0$ when $x = 0, \pi, 2\pi$

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$x = 0$	0	–	0	Endpoint, point of inflection
$0 < x < \dfrac{\pi}{3}$		–	+	Decreasing, concave up
$x = \dfrac{\pi}{3}$	$\dfrac{2\pi}{3} - 2\sqrt{3}$	0	+	Relative minimum
$\dfrac{\pi}{3} < x < \pi$		+	+	Increasing, concave up
$x = \pi$	2π	+	0	Point of inflection
$\pi < x < \dfrac{5\pi}{3}$		+	–	Increasing, concave down
$x = \dfrac{5\pi}{3}$	$\dfrac{10\pi}{3} + 2\sqrt{3}$	0	–	Relative maximum
$\dfrac{5\pi}{3} < x < 2\pi$		–	–	Decreasing, concave down
$x = 2\pi$	4π	–	0	Endpoint, point of inflection

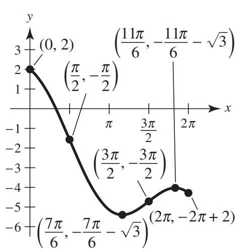


38. $f(x) = -x + 2\cos x,\ 0 \le x \le 2\pi$

$f'(x) = -2\sin x - 1 = 0$ when $\sin x = -\frac{1}{2} \Rightarrow x = \frac{7\pi}{6}, \frac{11\pi}{6}$

$f''(x) = -2\cos x = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$x = 0$	2	–	–	Endpoint
$0 < x < \frac{\pi}{2}$		–	–	Decreasing, concave down
$x = \frac{\pi}{2}$	$-\frac{\pi}{2}$	–	0	Point of inflection
$\frac{\pi}{2} < x < \frac{7\pi}{6}$		–	+	Decreasing, concave up
$x = \frac{7\pi}{6}$	$-\frac{7\pi}{6} - \sqrt{3}$	0	+	Relative minimum
$\frac{7\pi}{6} < x < \frac{3\pi}{2}$		+	+	Increasing, concave up
$x = \frac{3\pi}{2}$	$-\frac{3\pi}{2}$	+	0	Point of inflection
$\frac{3\pi}{2} < x < \frac{11\pi}{6}$		+	–	Increasing, concave down
$x = \frac{11\pi}{6}$	$-\frac{11\pi}{6} + \sqrt{3}$	0	–	Relative maximum
$\frac{11\pi}{6} < x < 2\pi$		–	–	Decreasing, concave down
$x = 2\pi$	$2\pi - 2$	–	–	Endpoint

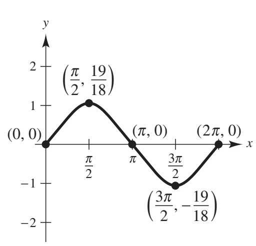


39. $y = \sin x - \frac{1}{18}\sin 3x,\ 0 \le x \le 2\pi$

$y' = \cos x\left(\frac{5}{6} + \frac{2}{3}\sin^2 x\right) = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$

$y'' = \sin x\left(4\cos^2 x - 1\right) = 0$ when $x = 0, \pi, 2\pi, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

	y	y'	y''	Conclusion
$x = 0$	0	+	0	Endpoint, point of inflection
$0 < x < \frac{\pi}{6}$		+	+	Increasing, concave up
$x = \frac{\pi}{6}$	$\frac{4}{9}$	+	0	Point of inflection
$\frac{\pi}{6} < x < \frac{\pi}{2}$		+	–	Increasing, concave down
$x = \frac{\pi}{2}$	$\frac{19}{18}$	0	–	Relative maximum
$\frac{\pi}{2} < x < \frac{5\pi}{6}$		–	–	Decreasing, concave down
$x = \frac{5\pi}{6}$	$\frac{4}{9}$	–	0	Point of inflection
$\frac{5\pi}{6} < x < \pi$		–	+	Decreasing, concave up
$x = \pi$	0	–	0	Point of inflection
$\pi < x < \frac{7\pi}{6}$		–	–	Decreasing, concave down
$x = \frac{7\pi}{6}$	$-\frac{4}{9}$	–	0	Point of inflection
$\frac{7\pi}{6} < x < \frac{3\pi}{2}$		–	+	Decreasing, concave up
$x = \frac{3\pi}{2}$	$-\frac{19}{18}$	0	+	Relative minimum
$\frac{3\pi}{2} < x < \frac{11\pi}{6}$		+	+	Increasing, concave up
$x = \frac{11\pi}{6}$	$-\frac{4}{9}$	+	0	Point of inflection
$\frac{11\pi}{6} < x < 2\pi$		+	–	Increasing, concave down
$x = 2\pi$	0	+	0	Endpoint, point of inflection

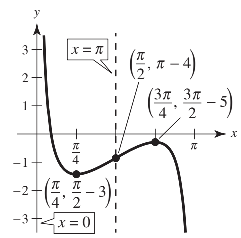


40. $y = 2(x - 2) + \cot x,\ 0 < x < \pi$

$y' = 2 - \csc^2 x = 0$ when $x = \frac{\pi}{4}, \frac{3\pi}{4}$

$y'' = 2 \cot x \csc^2 x = 0$ when $x = \frac{\pi}{2}$

	y	y'	y''	Conclusion
$x = 0$	Undef.			Vertical asymptote
$0 < x < \frac{\pi}{4}$		$-$	$+$	Decreasing, concave up
$x = \frac{\pi}{4}$	$\frac{\pi}{2} - 3$	0	$+$	Relative minimum
$\frac{\pi}{4} < x < \frac{\pi}{2}$		$+$	$+$	Increasing, concave up
$x = \frac{\pi}{2}$	$\pi - 4$	$+$	0	Point of inflection
$\frac{\pi}{2} < x < \frac{3\pi}{4}$		$+$	$-$	Increasing, concave down
$x = \frac{3\pi}{4}$	$\frac{3\pi}{2} - 5$	0	$-$	Relative maximum
$\frac{3\pi}{4} < x < \pi$		$-$	$-$	Decreasing, concave down
$x = \pi$	Undef.			Vertical asymptote

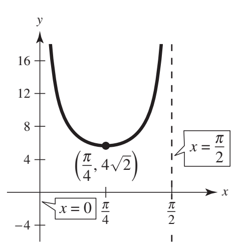


41. $y = 2(\csc x + \sec x),\ 0 < x < \frac{\pi}{2}$

$y' = 2(\sec x \tan x - \csc x \cot x) = 0$ when $x = \frac{\pi}{4}$

$y'' = 2(\sec x \tan^2 x + \sec^3 x + \csc^3 x + \cot^2 x \csc x) \neq 0$ in the given interval

	y	y'	y''	Conclusion
$x = 0$	Undef.			Vertical asymptote
$0 < x < \frac{\pi}{4}$		$-$	$+$	Decreasing, concave up
$x = \frac{\pi}{4}$	$4\sqrt{2}$	0	$+$	Relative minimum
$\frac{\pi}{4} < x < \frac{\pi}{2}$		$+$	$+$	Increasing, concave up
$x = \frac{\pi}{2}$	Undef.			Vertical asymptote

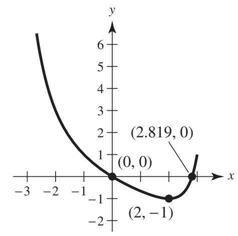


42. $y = \sec^2 \dfrac{\pi x}{8} - 2 \tan \dfrac{\pi x}{8} - 1, \ -3 < x < 3$

$y' = \dfrac{\pi}{4} \sec^2 \dfrac{\pi x}{8}\left(\tan \dfrac{\pi x}{8} - 1\right) = 0$ when $x = 2$

$y'' = \dfrac{\pi^2}{32} \sec^2\left(\dfrac{\pi x}{8}\right)\left(2 \tan^2 \dfrac{\pi x}{8} - 2 \tan \dfrac{\pi x}{8} + \sec^2 \dfrac{\pi x}{8}\right) \neq 0$ in the given interval

	y	y'	y''	Conclusion
$-3 < x < 2$		$-$	$+$	Decreasing, concave up
$x = 2$	-1	0	$+$	Relative minimum
$2 < x < 3$		$+$	$+$	Increasing, concave up

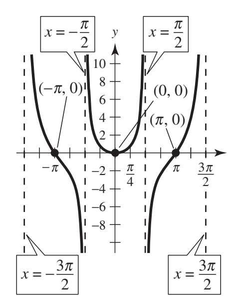


43. $g(x) = x \tan x, \ -\dfrac{3\pi}{2} < x < \dfrac{3\pi}{2}$

$g'(x) = \dfrac{x + \sin x \cos x}{\cos^2 x} = 0$ when $x = 0$

$g''(x) = \dfrac{2(\cos x + x \sin x)}{\cos^3 x} = 0$ when $x \approx \pm 2.798$

	$g(x)$	$g'(x)$	$g''(x)$	Conclusion
$x = -\dfrac{3\pi}{2}$	Undef.			Vertical asymptote
$-\dfrac{3\pi}{2} < x < -2.798$		$-$	$+$	Decreasing, concave up
$x \approx -2.798$	-1.001	$-$	0	Point of inflection
$-2.798 < x < -\dfrac{\pi}{2}$		$-$	$-$	Decreasing, concave down
$x = -\dfrac{\pi}{2}$	Undef.			Vertical asymptote
$-\dfrac{\pi}{2} < x < 0$		$-$	$+$	Decreasing, concave up
$x = 0$	0	0	$+$	Relative minimum
$0 < x < \dfrac{\pi}{2}$		$+$	$+$	Increasing, concave up
$\dfrac{\pi}{2} < x < 2.798$		$+$	$-$	Increasing, concave down
$x = 2.798$	-1.001	$+$	0	Point of inflection
$2.798 < x < \dfrac{3\pi}{2}$		$+$	$+$	Increasing, concave up
$x = \dfrac{3\pi}{2}$	Undef.			Point of inflection

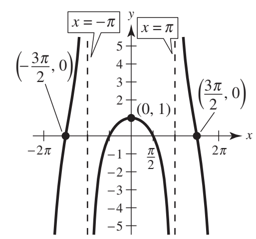


44. $g(x) = x \cot x, \ -2\pi < x < 2\pi$

$g'(x) = \cot x - x\csc^2 x = 0$ when $x = 0$

$g''(x) = \csc^2 x(2x \cot x - 2)$ when $x \approx \pm 4.493$

	$g(x)$	$g'(x)$	$g''(x)$	Conclusion
$-2\pi < x < -4.493$		+	–	Increasing, concave down
$x \approx -4.493$	–1.002	+	0	Point of inflection
$-4.493 < x < -\pi$		+	+	Increasing, concave up
$x = -\pi$	Undef.			Vertical asymptote
$-\pi < x < 0$		+	–	Increasing, concave down
$x = 0$	1	0	–	Relative maximum
$0 < x < \pi$		–	–	Decreasing, concave down
$x = \pi$	Undef.			Vertical asymptote
$\pi < x < 4.493$		–	+	Decreasing, concave up
$x \approx 4.493$	1.002	–	0	Point of inflection
$4.493 < x < 2\pi$		–	–	Decreasing, concave down



45. $f(x) = \dfrac{20x}{x^2+1} - \dfrac{1}{x} = \dfrac{19x^2 - 1}{x(x^2+1)}$

$$f'(x) = \frac{-(19x^4 - 22x^2 - 1)}{x^2(x^2+1)^2} = 0 \text{ for } x \approx \pm 1.10$$

$$f''(x) = \frac{2(19x^6 - 63x^9 - 3x^2 - 1)}{x^3(x^2+1)^3} = 0 \text{ for } x \approx \pm 1.84$$

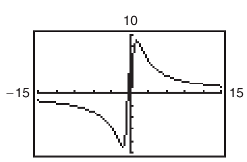
Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 0$

Minimum: $(-1.10, -9.05)$

Maximum: $(1.10, 9.05)$

Points of inflection: $(-1.84, -7.86), (1.84, 7.86)$



46. $f(x) = x + \dfrac{4}{x^2+1} = \dfrac{x^3 + x + 4}{x^2+1} = 0$ for $x \approx -1.379$

$$f'(x) = \frac{x^4 + 2x^2 - 8x + 1}{(x^2+1)^2} = 0 \text{ for } x \approx 1.608, x \approx 0.129$$

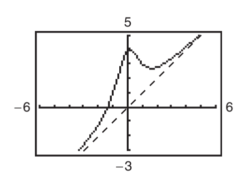
$$f''(x) = \frac{8(3x^2 - 1)}{(x^2+1)^3} = 0 \text{ for } x = \pm\frac{1}{\sqrt{3}} \approx \pm 0.577$$

Slant asymptote: $y = x$

Points of inflection: $(-0.577, 2.423), (0.577, 3.577)$

Relative maximum: $(0.129, 4.064)$

Relative minimum: $(1.608, 2.724)$



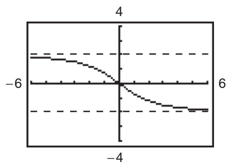
47. $f(x) = \dfrac{-2x}{\sqrt{x^2 + 7}}$

$f'(x) = \dfrac{-14}{\left(x^2 + 7\right)^{3/2}} < 0$

$f''(x) = \dfrac{42x}{\left(x^2 + 7\right)^{5/2}} = 0$ at $x = 0$

Horizontal asymptotes: $y = \pm 2$

Point of inflection: $(0, 0)$



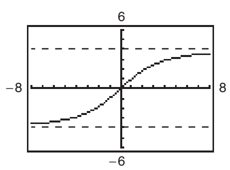
48. $f(x) = \dfrac{4x}{\sqrt{x^2 + 15}}$

$f'(x) = \dfrac{60}{\left(x^2 + 15\right)^{3/2}} > 0$

$f''(x) = \dfrac{-180x}{\left(x^2 + 15\right)^{5/2}} = 0$ at $x = 0$

Horizontal asymptotes: $y = \pm 4$

Point of inflection: $(0, 0)$



49. $y = \cos x - \dfrac{1}{4}\cos 2x,\ 0 \le x \le 2\pi$

$y = 0$ at $x \approx 1.797, 4.486$

$y = -\sin x + \dfrac{1}{2}\sin 2x = -\sin x + \sin x \cos x$

$= \sin x(\cos x - 1)$

$y'' = -\cos x + \cos 2x = -\cos x + 2\cos^2 x - 1$

$= (2\cos x + 1)(\cos x - 1)$

$y' = 0 \Rightarrow x = 0, \pi, 2\pi$

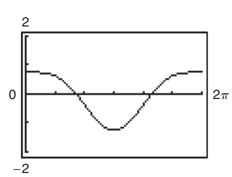
$y'' = 0 \Rightarrow x = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, 0, 2\pi$

Relative minimum:

$\left(\pi, -\dfrac{5}{4}\right)$

Points of inflection:

$\left(\dfrac{2\pi}{3}, -\dfrac{3}{8}\right), \left(\dfrac{4\pi}{3}, -\dfrac{3}{8}\right)$



50. $y = 2x - \tan x,\ -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$y' = 2 - \sec^2 x = 0$ when $x = \pm\dfrac{\pi}{4}$.

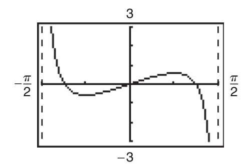
$y'' = -2\sec^2 x \tan x = 0$ when $x = 0$.

Vertical asymptotes: $x = \pm\dfrac{\pi}{2}$

Relative minimum: $\left(-\dfrac{\pi}{4}, 1 - \dfrac{\pi}{2}\right)$

Relative maximum: $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2} - 1\right)$

Point of inflection: $(0, 0)$

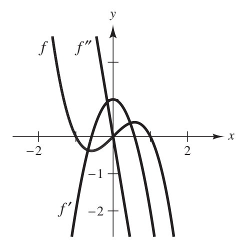


51. f is cubic.

f' is quadratic.

f'' is linear.

The zeros of f' correspond to the points where the graph of f has horizontal tangents. The zero of f'' corresponds to the point where the graph of f' has a horizontal tangent.

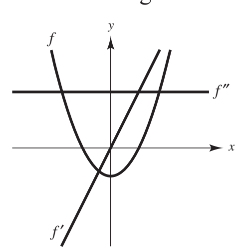


52. f'' is constant.

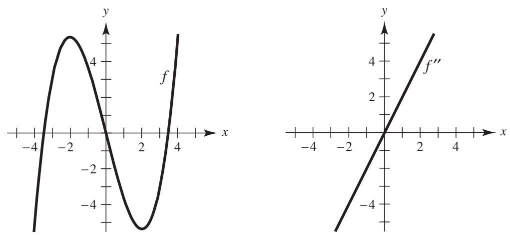
f' is linear.

f is quadratic.

The zero of f' corresponds to the points where the graph of f has a horizontal tangent. There are no zeros on of f'', which means the graph of f' has no horizontal tangent.

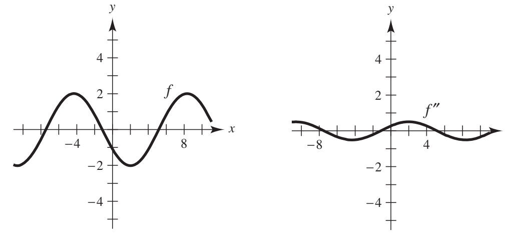


53.



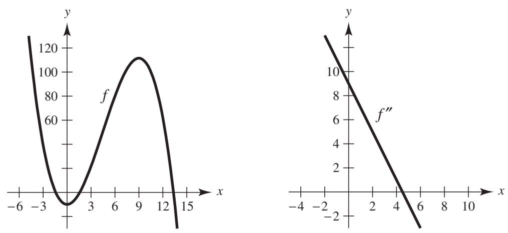
(or any vertical translation of f)

55.



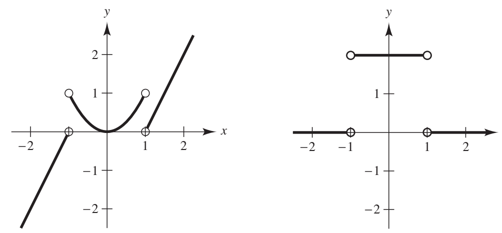
(or any vertical translation of f)

54.



(or any vertical translation of f)

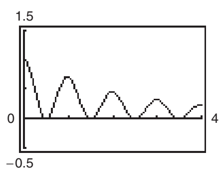
56.



(or any vertical translation of the 3 segments of f)

57. $f(x) = \dfrac{\cos^2 \pi x}{\sqrt{x^2 + 1}}, (0, 4)$

(a)



On $(0, 4)$ there seem to be 7 critical numbers: 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5

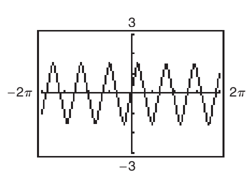
(b) $f'(x) = \dfrac{-\cos \pi x\left(x \cos \pi x + 2\pi\left(x^2 + 1\right) \sin \pi x\right)}{\left(x^2 + 1\right)^{3/2}} = 0$

Critical numbers $\approx \dfrac{1}{2}, 0.97, \dfrac{3}{2}, 1.98, \dfrac{5}{2}, 2.98, \dfrac{7}{2}.$

The critical numbers where maxima occur appear to be integers in part (a), but approximating them using f' shows that they are not integers.

58. $f(x) = \tan(\sin \pi x)$

(a)



(b) $f(-x) = \tan(\sin(-\pi x)) = \tan(-\sin \pi x) = -\tan(\sin \pi x) = -f(x)$

Symmetry with respect to the origin

(c) Periodic with period 2

(d) On $(-1, 1)$, there is a relative maximum at $\left(\frac{1}{2}, \tan 1\right)$ and a relative minimum at $\left(-\frac{1}{2}, -\tan 1\right)$.

(e) On $(0, 1)$, the graph of f is concave downward.

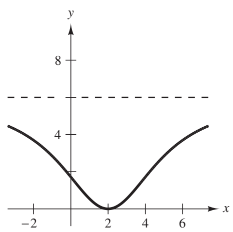
59. $x = 2$ is a critical number.

$f'(x) < 0$ for $x < 2$.

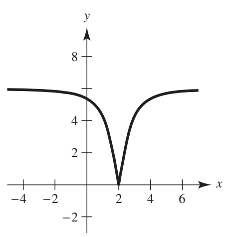
$f'(x) > 0$ for $x > 2$.

$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 6$

For example, let $f(x) = \dfrac{-6}{0.1(x-2)^2 + 1} + 6$.



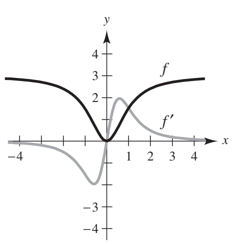
60. Yes. For example, let $f(x) = \dfrac{6|x-2|}{\sqrt{(x-2)^2 + 1}}$.



61. Because the slope is negative, the function is decreasing on $(2, 8)$, and so $f(3) > f(5)$.

62. If $f'(x) = 2$ in $[-5, 5]$, then $f(x) = 2x + 3$ and $f(2) = 7$ is the least possible value of $f(2)$. If $f'(x) = 4$ in $[-5, 5]$, then $f(x) = 4x + 3$ and $f(2) = 11$ is the greatest possible value of $f(2)$.

63. (a)



(b) $\lim_{x\to\infty} f(x) = 3$ $\qquad \lim_{x\to\infty} f'(x) = 0$

(c) Because $\lim_{x\to\infty} f(x) = 3$, the graph approaches that of a horizontal line, $\lim_{x\to\infty} f'(x) = 0$.

64. (a) $f'(x) = 0$ for $x = -2$ (relative maximum) and $x = 2$ (relative minimum).

f' is negative for $-2 < x < 2$ (decreasing).

f' is positive for $x > 2$ and $x < -2$ (increasing).

(b) $f''(x) = 0$ at $x = 0$ (point of inflection).

f'' is positive for $x > 0$ (concave upward).

f'' is negative for $x < 0$ (concave downward).

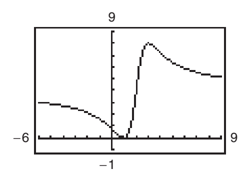
(c) f' is increasing on $(0, \infty)$. $(f'' > 0)$

(d) $f'(x)$ is minimum at $x = 0$. The rate of change of f at $x = 0$ is less than the rate of change of f for all other values of x.

65. $f(x) = \dfrac{4(x-1)^2}{x^2 - 4x + 5}$

Vertical asymptote: none

Horizontal asymptote: $y = 4$

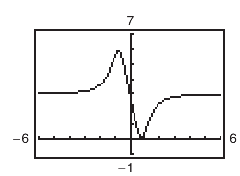


The graph crosses the horizontal asymptote $y = 4$. If a function has a vertical asymptote at $x = c$, the graph would not cross it because $f(c)$ is undefined.

66. $g(x) = \dfrac{3x^4 - 5x + 3}{x^4 + 1}$

Vertical asymptote: none

Horizontal asymptote: $y = 3$

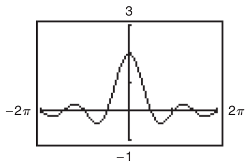


The graph crosses the horizontal asymptote $y = 3$. If a function has a vertical asymptote at $x = c$, the graph would not cross it because $f(c)$ is undefined.

67. $h(x) = \dfrac{\sin 2x}{x}$

Vertical asymptote: none

Horizontal asymptote: $y = 0$



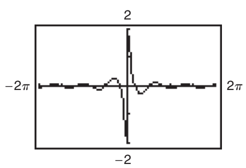
Yes, it is possible for a graph to cross its horizontal asymptote.

It is not possible to cross a vertical asymptote because the function is not continuous there.

68. $f(x) = \dfrac{\cos 3x}{4x}$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 0$



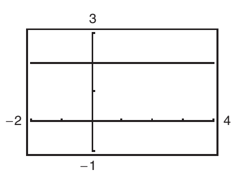
Yes, it is possible for a graph to cross its horizontal asymptote.

It is not possible to cross a vertical asymptote because the function is not continuous there.

69. $h(x) = \dfrac{6 - 2x}{3 - x}$

$$= \frac{2(3 - x)}{3 - x} = \begin{cases} 2, & \text{if } x \neq 3 \\ \text{Undefined}, & \text{if } x = 3 \end{cases}$$

The rational function is not reduced to lowest terms.

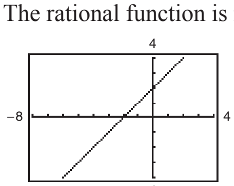


There is a hole at $(3, 2)$.

70. $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

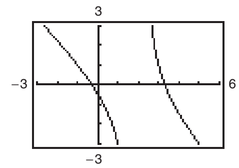
$$= \frac{(x + 2)(x - 1)}{x - 1} = \begin{cases} x + 2, & \text{if } x \neq 1 \\ \text{Undefined}, & \text{if } x = 1 \end{cases}$$

The rational function is not reduced to lowest terms.



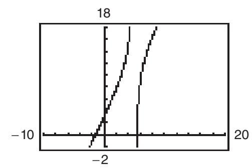
There is a hole at $(1, 3)$.

71. $x_n - \dfrac{F(x_n)}{F'(x_n)}$



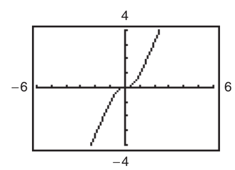
The graph appears to approach the slant asymptote $y = -x + 1$.

72. $g(x) = \dfrac{2x^2 - 8x - 15}{x - 5} = 2x + 2 - \dfrac{5}{x - 5}$



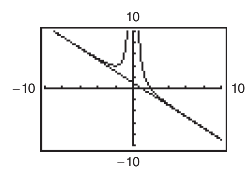
The graph appears to approach the slant asymptote $y = 2x + 2$.

73. $f(x) = \dfrac{2x^3}{x^2 + 1} = 2x - \dfrac{2x}{x^2 + 1}$



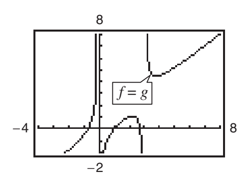
The graph appears to approach the slant asymptote $y = 2x$.

74. $h(x) = \dfrac{-x^3 + x^2 + 4}{x^2} = -x + 1 + \dfrac{4}{x^2}$



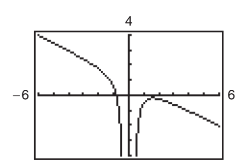
The graph appears to approach the slant asymptote $y = -x + 1$.

75. $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x - 3)} = x + \dfrac{2}{x(x - 3)}$



The graph appears to approach the slant asymptote $y = x$.

76. $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2} = -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2}$



The graph appears to approach the slant asymptote $y = -\dfrac{1}{2}x + 1$.

77. Tangent line at P: $y - y_0 = f'(x_0)(x - x_0)$

(a) Let $y = 0$: $-y_0 = f'(x_0)(x - x_0)$

$$f'(x_0)x = x_0 f'(x_0) - y_0$$

$$x = x_0 - \frac{y_0}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

x-intercept: $\left(x_0 - \dfrac{f(x_0)}{f'(x_0)}, 0\right)$

(b) Let $x = 0$: $y - y_0 = f'(x_0)(-x_0)$

$$y = y_0 - x_0 f'(x_0)$$

$$y = f(x_0) - x_0 f'(x_0)$$

y-intercept: $\left(0, f(x_0) - x_0 f'(x_0)\right)$

(c) Normal line: $y - y_0 = -\dfrac{1}{f'(x_0)}(x - x_0)$

Let $y = 0$: $-y_0 = -\dfrac{1}{f'(x_0)}(x - x_0)$

$$-y_0 f'(x_0) = -x + x_0$$

$$x = x_0 + y_0 f'(x_0) = x_0 + f(x_0)f'(x_0)$$

x-intercept: $\left(x_0 + f(x_0)f'(x_0), 0\right)$

(d) Let $x = 0$: $y - y_0 = \dfrac{-1}{f'(x_0)}(-x_0)$

$$y = y_0 + \frac{x_0}{f'(x_0)}$$

y-intercept: $\left(0, y_0 + \dfrac{x_0}{f'(x_0)}\right)$

(e) $|BC| = \left|x_0 - \dfrac{f(x_0)}{f'(x_0)} - x_0\right| = \left|\dfrac{f(x_0)}{f'(x_0)}\right|$

(f) $|PC|^2 = y_0^2 + \left(\dfrac{f(x_0)}{f'(x_0)}\right)^2 = \dfrac{f(x_0)^2 f'(x_0)^2 + f(x_0)^2}{f'(x_0)^2}$

$$|PC| = \left|\frac{f(x_0)\sqrt{1 + [f'(x_0)]^2}}{f'(x_0)}\right|$$

(g) $|AB| = \left|x_0 - \left(x_0 + f(x_0)f'(x_0)\right)\right| = \left|f(x_0)f'(x_0)\right|$

(h) $|AP|^2 = f(x_0)^2 f'(x_0)^2 + y_0^2$

$$|AP| = |f(x_0)|\sqrt{1 + [f'(x_0)]^2}$$

78. (a) $f'(x) = 0$ at x_0, x_2 and x_4 (horizontal tangent).

(b) $f''(x) = 0$ at x_2 and x_3 (point of inflection).

(c) $f'(x)$ does not exist at x_1 (sharp corner).

(d) f has a relative maximum at x_1.

(e) f has a point of inflection at x_2 and x_3 (change in concavity).

79. Vertical asymptote: $x = 3$

Horizontal asymptote: $y = 0$

$$y = \frac{1}{x - 3}$$

80. Vertical asymptote: $x = -5$

Horizontal asymptote: none

$$y = \frac{x^2}{x + 5}$$

81. Vertical asymptote: $x = 3$

Slant asymptote: $y = 3x + 2$

$$y = 3x + 2 + \frac{1}{x - 3} = \frac{3x^2 - 7x - 5}{x - 3}$$

82. Vertical asymptote: $x = 2$

Slant asymptote: $y = -x$

$$y = -x + \frac{1}{x - 2} = \frac{-x^2 + 2x + 1}{x - 2}$$

83. False. Let $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$, $f'(x) < 0$ for all real numbers.

84. False. Let $y_1 = \sqrt{x + 1}$, then $y_1(0) = 1$. So $y_1' = 1/\left(2\sqrt{x + 1}\right)$ and $y_1'(0) = 1/2$. Finally, $y_1'' = -\dfrac{1}{4(x + 1)^{3/2}}$ and $y_1''(0) = -\dfrac{1}{4}$. Let $p = ax^2 + bx + 1$, then $p(0) = 1$. So, $p' = 2ax + b$ and $p'(0) = \dfrac{1}{2} \Rightarrow b = \dfrac{1}{2}$. Finally, $p'' = 2a$ and $p''(0) = -\dfrac{1}{4} \Rightarrow a = -\dfrac{1}{8}$. Therefore,

$$f(x) = \begin{cases} (-1/8)x^2 + (1/2)x + 1, & x < 0 \\ \sqrt{x + 1}, & x \ge 0 \end{cases} \text{ and } f(0) = 1,$$

$$f'(x) = \begin{cases} (1/2) - (1/4)x, & x < 0 \\ 1/\left(2\sqrt{x + 1}\right), & x > 0 \end{cases} \text{ and } f'(0) = \frac{1}{2}, \text{ and}$$

$$f''(x) = \begin{cases} (-1/4), & x < 0 \\ -1/\left(4(x + 1)^{3/2}\right), & x > 0 \end{cases} \text{ and } f''(0) = -\frac{1}{4}.$$

$f''(x) < 0$ for all real x, but $f(x)$ *increases* without bound.

85. False. The graph of a rational function (having no common factors and whose denominator is of degree 1 or greater) has a slant asymptote only when the degree of the numerator exceeds the degree of the denominator by exactly 1.

86. False. $f(x) = x$ does not have an absolute maximum nor an absolute minimum.

87. (a) $f'(x) < 0$ when $-3 < x < 1$.

So, f is decreasing on the interval $(-3, 1)$.

(b) The slope of the graph of f' is negative for $-7 < x < -1$.

So, the graph of f is concave downward on the interval $(-7, -1)$.

(c) $f'(x) = 0$ when $x = -3$ and $x = 1$.

Because f is decreasing on the interval $(-3, 1)$, f has a relative maximum at $x = -3$ and a relative minimum at $x = 1$.

(d) The slope of the graph of f' is 0 when $x = -1$.

So, the graph of f has a point of inflection at $x = -1$.

88. (a) $f'(x) > 0$ when $-4 < x < -2$ and $-2 < x < 1$.

So, f is increasing on $(-4, -2)$ and $(-2, 1)$.

(b) The slope of the graph of f' is positive for $-2 < x < 0$.

So, the graph of f is concave upward on the interval $(-2, 0)$.

(c) $f'(x) = 0$ when $x = -2$ and $x = 1$. Because f is increasing on the interval $(-2, 1)$ and f' crosses the x-axis at $x = 1$, f has a relative maximum at $x = 1$.

(d) The slope of the graph of f' is 0 when $x = -2$ and $x = 0$. So, the graph of f has points of inflection at $x = -2$ and $x = 0$.

89. $f(x) = \dfrac{ax}{(x-b)^2}$

Answers will vary. *Sample answer*: The graph has a vertical asymptote at $x = b$. If a and b are both positive, or both negative, then the graph of f approaches ∞ as x approaches b, and the graph has a minimum at $x = -b$. If a and b have opposite signs, then the graph of f approaches $-\infty$ as x approaches b, and the graph has a maximum at $x = -b$.

90. $f(x) = \frac{1}{2}(ax)^2 - (ax) = \frac{1}{2}(ax)(ax - 2),\ a \neq 0$

$f'(x) = a^2x - a = a(ax - 1) = 0$ when $x = \dfrac{1}{a}$.

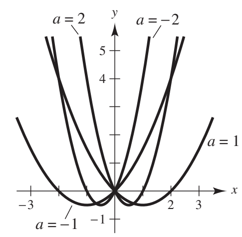
$f''(x) = a^2 > 0$ for all x.

(a) Intercepts: $(0, 0), \left(\dfrac{2}{a}, 0\right)$

Relative minimum: $\left(\dfrac{1}{a}, -\dfrac{1}{2}\right)$

Points of inflection: none

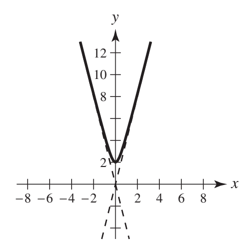
(b)



91. $y = \sqrt{4 + 16x^2}$

As $x \to \infty$, $y \to 4x$. As $x \to -\infty$, $y \to -4x$.

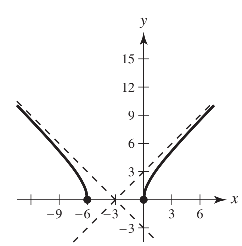
Slant asymptotes: $y = \pm 4x$



92. $y = \sqrt{x^2 + 6x} = \sqrt{(x+3)^2 - 9}$

$y \to x + 3$ as $x \to \infty$, and $y \to -x - 3$ as $x \to -\infty$.

Slant asymptotes:
$y = x + 3,\ y = -x - 3$



93. $f(x) = \dfrac{2x^n}{x^4 + 1}$

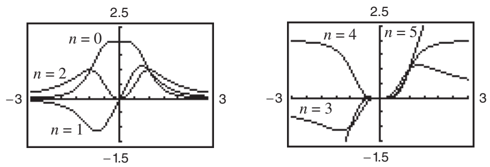
(a) For n even, f is symmetric about the y-axis. For n odd, f is symmetric about the origin.

(b) The x-axis will be the horizontal asymptote if the degree of the numerator is less than 4. That is, $n = 0, 1, 2, 3$.

(c) $n = 4$ gives $y = 2$ as the horizontal asymptote.

(d) There is a slant asymptote $y = 2x$ if $n = 5$: $\dfrac{2x^5}{x^4 + 1} = 2x - \dfrac{2x}{x^4 + 1}$.

(e)



n	0	1	2	3	4	5
M	1	2	3	2	1	0
N	2	3	4	5	2	3

94. Let $\lambda = \dfrac{\dfrac{f(x) - f(a)}{x - a} - \dfrac{f(b) - f(a)}{b - a}}{x - b}$, $a < x < b$.

$$\lambda(x - b) = \frac{f(x) - f(a)}{x - a} - \frac{f(b) - f(a)}{b - a}$$

$$\lambda(x - b)(x - a) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

$$f(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) + \lambda(x - b)(x - a)$$

Let $h(t) = f(t) - \left\{f(a) + \dfrac{f(b) - f(a)}{b - a}(t - a) + \lambda(t - a)(t - b)\right\}$.

$h(a) = 0, h(b) = 0, h(x) = 0$

By Rolle's Theorem, there exist numbers α_1 and α_2 such that $a < \alpha_1 < x < \alpha_2 < b$ and $h'(\alpha_1) = h'(\alpha_2) = 0$.

By Rolle's Theorem, there exists β in (a, b) such that $h''(\beta) = 0$.

Finally,

$0 = h''(\beta) = f''(\beta) - \{2\lambda\} \Rightarrow \lambda = \frac{1}{2}f''(\beta)$.

Section 3.7 Optimization Problems

1. A primary equation is a formula for the quantity to be optimized. A secondary equation can be solved for a variable and then substituted into the primary equation to obtain a function of a single variable. A feasible domain is the set of input values that make sense in an optimization problem.

2. To solve applied minimum and maximum problems, first identify all given quantities and all quantities to be determined. It may be helpful to draw a sketch. Write a primary equation for the quantity to be maximized or minimized. Use secondary equations, if necessary, to reduce the primary equation to an equation having a single variable. Determine the values for which the stated problem makes sense. Use calculus techniques to determine the maximum or minimum value(s).

3. (a)

First Number, x	Second Number	Product, P
10	110 – 10	10(110 – 10) = 1000
20	110 – 20	20(110 – 20) = 1800
30	110 – 30	30(110 – 30) = 2400
40	110 – 40	40(110 – 40) = 2800
50	110 – 50	50(110 – 50) = 3000
60	110 – 60	60(110 – 60) = 3000

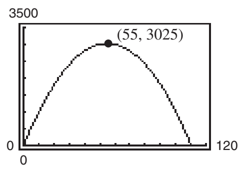
(b) $P = x(110 - x) = 110x - x^2$

(c) $\dfrac{dP}{dx} = 110 - 2x = 0$ when $x = 55$.

$\dfrac{d^2P}{dx^2} = -2 < 0$

P is a maximum when $x = 110 - x = 55$. The two numbers are 55 and 55.

(d)



The solution appears to be $x = 55$.

4. (a)

Height, x	Length & Width	Volume
1	24 – 2(1)	$1[24 - 2(1)]^2 = 484$
2	24 – 2(2)	$2[24 - 2(2)]^2 = 800$
3	24 – 2(3)	$3[24 - 2(3)]^2 = 972$
4	24 – 2(4)	$4[24 - 2(4)]^2 = 1024$
5	24 – 2(5)	$5[24 - 2(5)]^2 = 980$
6	24 – 2(6)	$6[24 - 2(6)]^2 = 864$

The maximum is attained near $x = 4$.

(b) $V = x(24 - 2x)^2, 0 < x < 12$

(c) $\dfrac{dV}{dx} = 2x(24 - 2x)(-2) + (24 - 2x)^2 = (24 - 2x)(24 - 6x)$

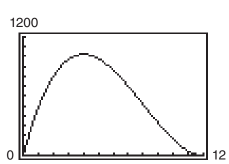
$= 12(12 - x)(4 - x) = 0$ when $x = 12, 4$ (12 is not in the domain).

$\dfrac{d^2V}{dx^2} = 12(2x - 16)$

$\dfrac{d^2V}{dx^2} < 0$ when $x = 4$.

When $x = 4, V = 1024$ is maximum.

(d)



The maximum volume seems to be 1024.

5. Let x and y be two positive numbers such that $x + y = S$.

$$P = xy = x(S - x) = Sx - x^2$$
$$\frac{dP}{dx} = S - 2x = 0 \text{ when } x = \frac{S}{2}.$$
$$\frac{d^2P}{dx^2} = -2 < 0 \text{ when } x = \frac{S}{2}.$$

For $x = \frac{S}{2}$, $\frac{S}{2} + y = S \Rightarrow y = \frac{S}{2}$.

P is a maximum when $x = y = S/2$.

6. Let x and y be two positive numbers such that $xy = 185$.

$$S = x + y = x + \frac{185}{x}$$
$$\frac{dS}{dx} = 1 - \frac{185}{x^2} = 0 \text{ when } x = \sqrt{185}.$$
$$\frac{d^2S}{dx^2} = \frac{370}{x^3} > 0 \text{ when } x = \sqrt{185}$$

For $x = \sqrt{185}$, $\sqrt{185}y = 185 \Rightarrow y = \sqrt{185}$.

S is a minimum when $x = y = \sqrt{185}$.

7. Let x and y be two positive numbers such that $xy = 147$.

$$S = x + 3y = \frac{147}{y} + 3y$$
$$\frac{dS}{dy} = 3 - \frac{147}{y^2} = 0 \text{ when } y = 7.$$
$$\frac{d^2S}{dy^2} = \frac{294}{y^3} > 0 \text{ when } y = 7.$$

For $y = 7$, $7x = 147 \Rightarrow x = 21$.

S is minimum when $y = 7$ and $x = 21$.

8. Let x and y be two positive numbers such that $x^2 + y = 54$.

$$P = xy = x(54 - x^2) = 54x - x^3$$
$$\frac{dP}{dx} = 54 - 3x^2 = 0 \text{ when } x = 3\sqrt{2}.$$
$$\frac{d^2P}{dx^2} = -6x < 0 \text{ when } x = 3\sqrt{2}.$$

For $x = 3\sqrt{2}$, $\left(3\sqrt{2}\right)^2 + y = 54 \Rightarrow y = 36$.

The product is a maximum when $x = 3\sqrt{2}$ and $y = 36$.

9. Let x and y be two positive numbers such that $x + 2y = 108$.

$$P = xy = y(108 - 2y) = 108y - 2y^2$$
$$\frac{dP}{dy} = 108 - 4y = 0 \text{ when } y = 27.$$
$$\frac{d^2P}{dy^2} = -4 < 0 \text{ when } y = 27.$$

For $y = 27$, $x + 2(27) = 108 \Rightarrow x = 54$.

P is a maximum when $x = 54$ and $y = 27$.

10. Let x and y be two positive numbers such that $x^3 + y = 500$.

$$P = xy = x(500 - x^3) = 500x - x^4$$
$$\frac{dP}{dx} = 500 - 4x^3 = 0 \text{ when } x^3 = 125 \text{ or } x = 5.$$
$$\frac{d^2P}{dx^2} = -12x^2 < 0 \text{ when } x = 5.$$

For $x = 5$, $y = 500 - x^3 = 375$.

P is a maximum when $x = 5$ and $y = 375$.

11. Let x be the length and y the width of the rectangle.

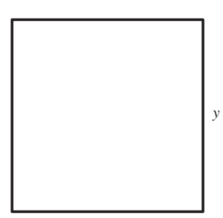
$$2x + 2y = 80$$
$$y = 40 - x$$
$$A = xy = x(40 - x) = 40x - x^2$$
$$\frac{dA}{dx} = 40 - 2x = 0 \text{ when } x = 20.$$
$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = 20.$$

A is maximum when $x = y = 20$ meters.

12. Let x be the length and y the width of the rectangle.

$$2x + 2y = P$$
$$y = \frac{P - 2x}{2} = \frac{P}{2} - x$$
$$A = xy = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2$$
$$\frac{dA}{dx} = \frac{P}{2} - 2x = 0 \text{ when } x = \frac{P}{4}.$$
$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = \frac{P}{4}.$$

A is maximum when $x = y = P/4$ units. (A square!)



13. Let x be the length and y the width of the rectangle.

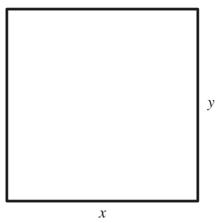
$xy = 49 \Rightarrow y = \dfrac{49}{x}$

$P = 2x + 2y = 2x + 2\left(\dfrac{49}{x}\right) = 2x + \dfrac{98}{x}$

$\dfrac{dP}{dx} = 2 - 98x^{-2} = 0$ when $x = 7$.

$\dfrac{d^2P}{dx^2} > 0$ when $x = 7$.

P is a minimum when $x = y = 7$ feet.



14. Let x be the length and y the width of the rectangle.

$xy = A$

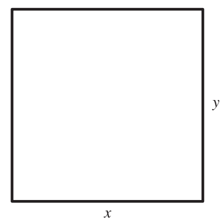
$y = \dfrac{A}{x}$

$P = 2x + 2y = 2x + 2\left(\dfrac{A}{x}\right) = 2x + \dfrac{2A}{x}$

$\dfrac{dP}{dx} = 2 - \dfrac{2A}{x^2} = 0$ when $x = \sqrt{A}$.

$\dfrac{d^2P}{dx^2} = \dfrac{4A}{x^3} > 0$ when $x = \sqrt{A}$.

P is minimum when $x = y = \sqrt{A}$ centimeters.
(A square!)



15. The distance from $(0, 3)$ to $y = x^2$ is $d = \sqrt{(x - 0)^2 + (y - 3)^2} = \sqrt{x^2 + (x^2 - 3)^2}$.

Because d is smallest when d^2 is smallest, find the critical numbers of $f(x) = x^2 + (x^2 - 3)^2 = x^4 - 5x^2 + 9$.

$f'(x) = 4x^3 - 10x = 2x(2x^2 - 5) = 0 \Rightarrow x = \pm\sqrt{\frac{5}{2}}$

By the First Derivative Test, the closest points are $\left(-\sqrt{\frac{5}{2}}, \frac{5}{2}\right)$ and $\left(\sqrt{\frac{5}{2}}, \frac{5}{2}\right)$.

16. The distance from $(0, -1)$ to $y = x^2 - 2$ is $d = \sqrt{(x - 0)^2 + (y + 1)^2} = \sqrt{x^2 + (x^2 - 2 + 1)^2} = \sqrt{x^2 + (x^2 - 1)^2}$.

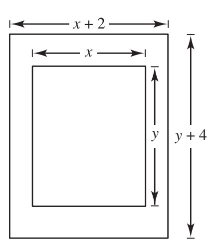
Because d is smallest when d^2 is smallest, find the critical numbers of $f(x) = x^2 + (x^2 - 1)^2 = x^4 - x^2 + 1$.

$f'(x) = 4x^3 - 2x = 2x(2x^2 - 1) = 0 \Rightarrow x = \pm\sqrt{\dfrac{1}{2}} = \pm\dfrac{\sqrt{2}}{2}$

By the First Derivative Test, the closest points are $\left(-\dfrac{\sqrt{2}}{2}, -\dfrac{3}{2}\right)$ and $\left(\dfrac{\sqrt{2}}{2}, -\dfrac{3}{2}\right)$.

17. $xy = 648 \Rightarrow y = \dfrac{648}{x}$

$$\begin{aligned} A &= (x + 2)(y + 4) \\ &= (x + 2)\left(\frac{648}{x} + 4\right) \\ &= 648 + \frac{1296}{x} + 4x + 8 \\ &= 4x + 1296x^{-1} + 656 \end{aligned}$$



$A'(x) = 4 - 1296x^{-2} = 4 - \dfrac{1296}{x^2} = 0 \Rightarrow 4x^2 = 1296 \Rightarrow x = 18$

So, $y = \dfrac{648}{18} = 36$. By the First Derivative Test, these values yield a minimum.

The dimensions of the poster are 20 inches $\times$ 40 inches.

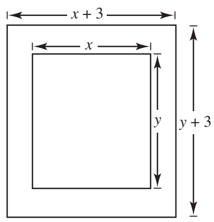
18. $xy = 36 \Rightarrow y = \dfrac{36}{x}$

$$A = (x+3)(y+3) = (x+3)\left(\frac{36}{x} + 3\right)$$

$$= 36 + \frac{108}{x} + 3x + 9$$

$$\frac{dA}{dx} = \frac{-108}{x^2} + 3 = 0 \Rightarrow 3x^2 = 108 \Rightarrow x = 6$$

So, $y = \dfrac{36}{6} = 6$. By the First Derivative Test, these values yield a minimum. The dimensions of the page are 9 inches $\times$ 9 inches.



19. $xy = 405{,}000 \Rightarrow y = \dfrac{405{,}000}{x}$

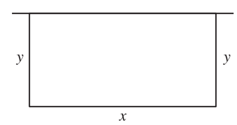
$$S = x + 2y = x + 2\left(\frac{405{,}000}{x}\right)$$

$$S'(x) = 1 - 810{,}000x^{-2} = 0$$

$$x^2 = 810{,}000$$

$$x = 900$$

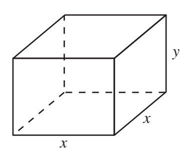
So, $y = \dfrac{405{,}000}{900} = 450$. By the First Derivative Test, these values yield a minimum. The dimensions of the fence are 900 meters $\times$ 450 meters.



20. $S = 2x^2 + 4xy = 337.5$

$$y = \frac{337.5 - 2x^2}{4x}$$

$$V = x^2y = x^2\left[\frac{337.5 - 2x^2}{4x}\right] = 84.375x - \frac{1}{2}x^3$$



$$\frac{dV}{dx} = 84.375 - \frac{3}{2}x^2 = 0 \Rightarrow x^2 = 56.25 \Rightarrow x = 7.5 \text{ and } y = 7.5.$$

$$\frac{d^2V}{dx^2} = -3x < 0 \text{ for } x = 7.5.$$

The maximum value occurs when $x = y = 7.5$ cm.

21. $16 = 2y + x + \pi\left(\dfrac{x}{2}\right)$

$$32 = 4y + 2x + \pi x$$

$$y = \frac{32 - 2x - \pi x}{4}$$

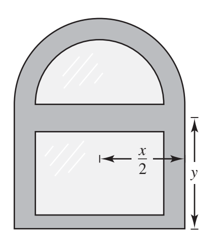
$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2 = \left(\frac{32 - 2x - \pi x}{4}\right)x + \frac{\pi x^2}{8} = 8x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2$$

$$\frac{dA}{dx} = 8 - x - \frac{\pi}{2}x + \frac{\pi}{4}x = 8 - x\left(1 + \frac{\pi}{4}\right) = 0 \text{ when } x = \frac{8}{1 + (\pi/4)} = \frac{32}{4 + \pi}.$$

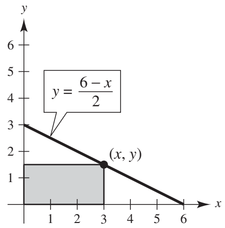
$$\frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{4}\right) < 0 \text{ when } x = \frac{32}{4 + \pi}.$$

$$y = \frac{32 - 2[32/(4+\pi)] - \pi[32/(4+\pi)]}{4} = \frac{16}{4 + \pi}$$

The area is maximum when $y = \dfrac{16}{4+\pi}$ ft and $x = \dfrac{32}{4+\pi}$ ft.



22. You can see from the figure that $A = xy$ and $y = \dfrac{6 - x}{2}$.



$$A = x\left(\frac{6 - x}{2}\right) = \frac{1}{2}\left(6x - x^2\right).$$

$$\frac{dA}{dx} = \frac{1}{2}(6 - 2x) = 0 \text{ when } x = 3.$$

$$\frac{d^2A}{dx^2} = -1 < 0 \text{ when } x = 3.$$

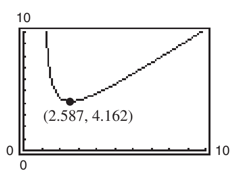
A is a maximum when $x = 3$ and $y = 3/2$.

23. (a) $\dfrac{y - 2}{0 - 1} = \dfrac{0 - 2}{x - 1}$

$$y = 2 + \frac{2}{x - 1}$$

$$L = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(2 + \frac{2}{x - 1}\right)^2} = \sqrt{x^2 + 4 + \frac{8}{x - 1} + \frac{4}{(x - 1)^2}}, \quad x > 1$$

(b)



L is minimum when $x \approx 2.587$ and $L \approx 4.162$.

(c) Area $= A(x) = \dfrac{1}{2}xy = \dfrac{1}{2}x\left(2 + \dfrac{2}{x - 1}\right) = x + \dfrac{x}{x - 1}$

$$A'(x) = 1 + \frac{(x - 1) - x}{(x - 1)^2} = 1 - \frac{1}{(x - 1)^2} = 0$$

$$(x - 1)^2 = 1$$

$$x - 1 = \pm 1$$

$$x = 0, 2 \text{ (select } x = 2)$$

They $y = 4$ and $A = 4$.

Vertices: $(0, 0), (2, 0), (0, 4)$

24. (a) $A = \frac{1}{2}\text{ base} \times \text{height} = \frac{1}{2}\left(2\sqrt{36 - h^2}\right)(6 + h) = \sqrt{36 - h^2}(6 + h)$

$$\frac{dA}{dh} = \frac{1}{2}\left(36 - h^2\right)^{-1/2}(-2h)(6 + h) + \left(36 - h^2\right)^{1/2}$$

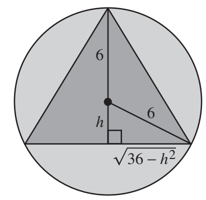
$$= \left(36 - h^2\right)^{-1/2}\left[-h(6 + h) + \left(36 - h^2\right)\right]$$

$$= \frac{-2\left(h^2 + 3h - 18\right)}{\sqrt{36 - h^2}}$$

$$= \frac{-2(h + 6)(h - 3)}{\sqrt{36 - h^2}}$$

$\frac{dA}{dh} = 0$ when $h = 3$, which is a maximum by the First Derivative Test.

So, the sides are $2\sqrt{36 - h^2} = 6\sqrt{3}$, an equilateral triangle. Area $= 27\sqrt{3}$ sq. units.



(b) $\cos\alpha = \frac{6 + h}{2\sqrt{3}\sqrt{6 + h}} = \frac{\sqrt{6 + h}}{2\sqrt{3}}$

$\tan\alpha = \frac{\sqrt{36 - h^2}}{6 + h}$

$\text{Area} = 2\left(\frac{1}{2}\right)\left(\sqrt{36 - h^2}\right)(6 + h) = (6 + h)^2 \tan\alpha = 144\cos^4\alpha\tan\alpha$

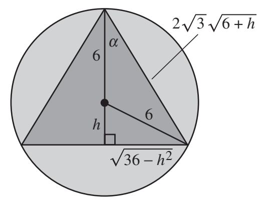
$A'(\alpha) = 144\left[\cos^4\alpha\sec^2\alpha + 4\cos^3(-\sin\alpha)\tan\alpha\right] = 0$

$\Rightarrow \cos^4\alpha\sec^2\alpha = 4\cos^3\alpha\sin\alpha\tan\alpha$

$1 = 4\cos\alpha\sin\alpha\tan\alpha$

$\frac{1}{4} = \sin^2\alpha$

$\sin\alpha = \frac{1}{2} \Rightarrow \alpha = 30°$ and $A = 27\sqrt{3}$.



(c) Equilateral triangle

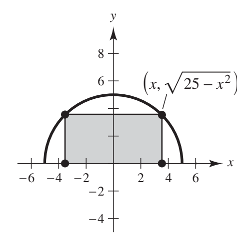
25. $A = 2xy = 2x\sqrt{25 - x^2}$ (see figure)

$$\frac{dA}{dx} = 2x\left(\frac{1}{2}\right)\left(\frac{-2x}{\sqrt{25 - x^2}}\right) + 2\sqrt{25 - x^2} = 2\left(\frac{25 - 2x^2}{\sqrt{25 - x^2}}\right) = 0 \text{ when } x = y = \frac{5\sqrt{2}}{2} \approx 3.54.$$

By the First Derivative Test, the inscribed rectangle of maximum area has vertices

$\left(\pm\frac{5\sqrt{2}}{2}, 0\right), \left(\pm\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right).$

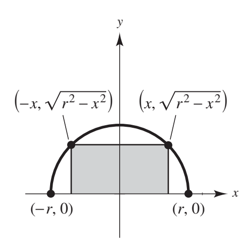
Width: $\frac{5\sqrt{2}}{2}$; Length: $5\sqrt{2}$



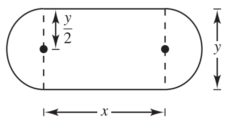
26. $A = 2xy = 2x\sqrt{r^2 - x^2}$ (see figure)

$$\frac{dA}{dx} = \frac{2\left(r^2 - 2x^2\right)}{\sqrt{r^2 - x^2}} = 0 \text{ when } x = \frac{\sqrt{2}r}{2}.$$

By the First Derivative Test, A is maximum when the rectangle has dimensions $\sqrt{2}r$ by $\left(\sqrt{2}r\right)/2$.



27. (a) $P = 2x + 2\pi r = 2x + 2\pi\left(\frac{y}{2}\right) = 2x + \pi y = 200 \Rightarrow y = \frac{200 - 2x}{\pi} = \frac{2}{\pi}(100 - x)$



(b)

Length, x	Width, y	Area, xy
10	$\frac{2}{\pi}(100 - 10)$	$(10)\frac{2}{\pi}(100 - 10) \approx 573$
20	$\frac{2}{\pi}(100 - 20)$	$(20)\frac{2}{\pi}(100 - 20) \approx 1019$
30	$\frac{2}{\pi}(100 - 30)$	$(30)\frac{2}{\pi}(100 - 30) \approx 1337$
40	$\frac{2}{\pi}(100 - 40)$	$(40)\frac{2}{\pi}(100 - 40) \approx 1528$
50	$\frac{2}{\pi}(100 - 50)$	$(50)\frac{2}{\pi}(100 - 50) \approx 1592$
60	$\frac{2}{\pi}(100 - 60)$	$(60)\frac{2}{\pi}(100 - 60) \approx 1528$

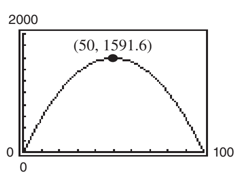
The maximum area of the rectangle is approximately 1592 m^2.

(c) $A = xy = x\frac{2}{\pi}(100 - x) = \frac{2}{\pi}(100x - x^2)$

(d) $A' = \frac{2}{\pi}(100 - 2x)$. $A' = 0$ when $x = 50$.

Maximum value is approximately 1592 when length $= 50$ m and width $= \frac{100}{\pi}$.

(e)



Maximum area is approximately 1591.55 m^2 $(x = 50 \text{ m})$.

28. $V = \pi r^2 h = 22$ cubic inches or $h = \frac{22}{\pi r^2}$

(a)

Radius, r	Height	Surface Area
0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \frac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \frac{22}{\pi(0.4)^2}\right] \approx 111.0$
0.6	$\frac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \frac{22}{\pi(0.6)^2}\right] \approx 75.6$
0.8	$\frac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \frac{22}{\pi(0.8)^2}\right] \approx 59.0$

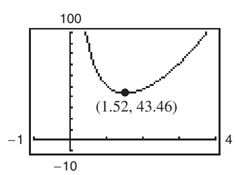
(b)

Radius, r	Height	Surface Area
0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \frac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \frac{22}{\pi(0.4)^2}\right] \approx 111.0$
0.6	$\frac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \frac{22}{\pi(0.6)^2}\right] \approx 75.6$
0.8	$\frac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \frac{22}{\pi(0.8)^2}\right] \approx 59.0$
1.0	$\frac{22}{\pi(1.0)^2}$	$2\pi(1.0)\left[1.0 + \frac{22}{\pi(1.0)^2}\right] \approx 50.3$
1.2	$\frac{22}{\pi(1.2)^2}$	$2\pi(1.2)\left[1.2 + \frac{22}{\pi(1.2)^2}\right] \approx 45.7$
1.4	$\frac{22}{\pi(1.4)^2}$	$2\pi(1.4)\left[1.4 + \frac{22}{\pi(1.4)^2}\right] \approx 43.7$
1.6	$\frac{22}{\pi(1.6)^2}$	$2\pi(1.6)\left[1.6 + \frac{22}{\pi(1.6)^2}\right] \approx 43.6$
1.8	$\frac{22}{\pi(1.8)^2}$	$2\pi(1.8)\left[1.8 + \frac{22}{\pi(1.8)^2}\right] \approx 44.8$
2.0	$\frac{22}{\pi(2.0)^2}$	$2\pi(2.0)\left[2.0 + \frac{22}{\pi(2.0)^2}\right] \approx 47.1$

The minimum seems to the about 43.6 for $r = 1.6$.

(c) $S = 2\pi r^2 + 2\pi rh = 2\pi r(r + h) = 2\pi r\left[r + \frac{22}{\pi r^2}\right] = 2\pi r^2 + \frac{44}{r}$

(d)



The minimum seems to be 43.46 for $r \approx 1.52$.

(e) $\frac{dS}{dr} = 4\pi r - \frac{44}{r^2} = 0$ when $r = \sqrt[3]{11/\pi} \approx 1.52$ in.

$h = \frac{22}{\pi r^2} \approx 3.04$ in.

Note: Notice that $h = \frac{22}{\pi r^2} = \frac{22}{\pi(11/\pi)^{2/3}} = 2\left(\frac{11^{1/3}}{\pi^{1/3}}\right) = 2r.$

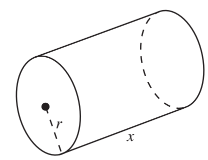
29. Let x be the sides of the square ends and y the length of the package.

$$P = 4x + y = 108 \Rightarrow y = 108 - 4x$$
$$V = x^2y = x^2(108 - 4x) = 108x^2 - 4x^3$$
$$\frac{dV}{dx} = 216x - 12x^2 = 12x(18 - x) = 0 \text{ when } x = 18.$$
$$\frac{d^2V}{dx^2} = 216 - 24x = -216 < 0 \text{ when } x = 18.$$

The volume is maximum when $x = 18$ in. and $y = 108 - 4(18) = 36$ in.

30.
$$V = \pi r^2 x$$
$$x + 2\pi r = 108 \Rightarrow x = 108 - 2\pi r \text{ (see figure)}$$
$$V = \pi r^2(108 - 2\pi r) = \pi(108r^2 - 2\pi r^3)$$
$$\frac{dV}{dr} = \pi(216r - 6\pi r^2) = 6\pi r(36 - \pi r) = 0 \text{ when } r = \frac{36}{\pi} \text{ and } x = 36.$$
$$\frac{d^2V}{dr^2} = \pi(216 - 12\pi r) < 0 \text{ when } r = \frac{36}{\pi}.$$

Volume is maximum when $x = 36$ in. and $r = 36/\pi \approx 11.459$ in.



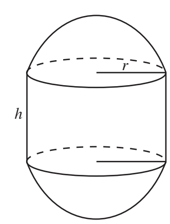
31. No. The volume will change because the shape of the container changes when squeezed.

32. No, there is no minimum area. If the sides are x and y, then $2x + 2y = 20 \Rightarrow y = 10 - x$. The area is $A(x) = x(10 - x) = 10x - x^2$. This can be made arbitrarily small by selecting $x \approx 0$.

33.
$$V = 14 = \frac{4}{3}\pi r^3 + \pi r^2 h$$
$$h = \frac{14 - (4/3)\pi r^3}{\pi r^2} = \frac{14}{\pi r^2} - \frac{4}{3}r$$
$$S = 4\pi r^2 + 2\pi rh = 4\pi r^2 + 2\pi r\left(\frac{14}{\pi r^2} - \frac{4}{3}r\right) = 4\pi r^2 + \frac{28}{r} - \frac{8}{3}\pi r^2 = \frac{4}{3}\pi r^2 + \frac{28}{r}$$
$$\frac{dS}{dr} = \frac{8}{3}\pi r - \frac{28}{r^2} = 0 \text{ when } r = \sqrt[3]{\frac{21}{2\pi}} \approx 1.495 \text{ cm.}$$
$$\frac{d^2S}{dr^2} = \frac{8}{3}\pi + \frac{56}{r^3} > 0 \text{ when } r = \sqrt[3]{\frac{21}{2\pi}}.$$

The surface area is minimum when $r = \sqrt[3]{\frac{21}{2\pi}}$ cm and $h = 0$.

The resulting solid is a sphere of radius $r \approx 1.495$ cm.



34.
$$V = 4000 = \frac{4}{3}\pi r^3 + \pi r^2 h$$
$$h = \frac{4000}{\pi r^2} - \frac{4}{3}r$$

Let k = cost per square foot of the surface area of the sides, then $2k$ = cost per square foot of the hemispherical ends.

$$C = 2k(4\pi r^2) + k(2\pi rh) = k\left[8\pi r^2 + 2\pi r\left(\frac{4000}{\pi r^2} - \frac{4}{3}r\right)\right] = k\left[\frac{16}{3}\pi r^2 + \frac{8000}{r}\right]$$
$$\frac{dC}{dr} = k\left[\frac{32}{3}\pi r - \frac{8000}{r^2}\right] = 0 \text{ when } r = \sqrt[3]{\frac{750}{\pi}} \approx 6.204 \text{ ft and } h \approx 24.814 \text{ ft.}$$

By the Second Derivative Test, you have $\frac{d^2C}{dr^2} = k\left[\frac{32}{3}\pi + \frac{12{,}000}{r^3}\right] > 0$ when $r = \sqrt[3]{\frac{750}{\pi}}$.

The cost is minimum when $r = \sqrt[3]{\frac{750}{\pi}}$ ft and $h \approx 24.814$ ft.

35. Let x be the length of a side of the square and y the length of a side of the triangle.

$$4x + 3y = 10$$

$$A = x^2 + \frac{1}{2}y\left(\frac{\sqrt{3}}{2}y\right) = \frac{(10-3y)^2}{16} + \frac{\sqrt{3}}{4}y^2$$

$$\frac{dA}{dy} = \frac{1}{8}(10-3y)(-3) + \frac{\sqrt{3}}{2}y = 0$$

$$-30 + 9y + 4\sqrt{3}y = 0$$

$$y = \frac{30}{9+4\sqrt{3}}$$

$$\frac{d^2A}{dy^2} = \frac{9+4\sqrt{3}}{8} > 0$$

A is minimum when $y = \dfrac{30}{9+4\sqrt{3}}$ and $x = \dfrac{10\sqrt{3}}{9+4\sqrt{3}}$.

36. (a) Let x be the side of the triangle and y the side of the square.

$$A = \frac{3}{4}\left(\cot\frac{\pi}{3}\right)x^2 + \frac{4}{4}\left(\cot\frac{\pi}{4}\right)y^2 \text{ where } 3x + 4y = 20 = \frac{\sqrt{3}}{4}x^2 + \left(5 - \frac{3}{4}x\right)^2, 0 \le x \le \frac{20}{3}.$$

$$A' = \frac{\sqrt{3}}{2}x + 2\left(5 - \frac{3}{4}x\right)\left(-\frac{3}{4}\right) = 0$$

$$x = \frac{60}{4\sqrt{3}+9}$$

When $x = 0$, $A = 25$, when $x = 60/\left(4\sqrt{3}+9\right)$, $A \approx 10.847$, and when $x = 20/3$, $A \approx 19.245$. Area is maximum when all 20 feet are used on the square.

(b) Let x be the side of the square and y the side of the pentagon.

$$A = \frac{4}{4}\left(\cot\frac{\pi}{4}\right)x^2 + \frac{5}{4}\left(\cot\frac{\pi}{5}\right)y^2 \text{ where } 4x + 5y = 20 = x^2 + 1.7204774\left(4 - \frac{4}{5}x\right)^2, 0 \le x \le 5.$$

$$A' = 2x - 2.75276384\left(4 - \frac{4}{5}x\right) = 0$$

$$x \approx 2.62$$

When $x = 0$, $A \approx 27.528$, when $x \approx 2.62$, $A \approx 13.102$, and when $x = 5$, $A \approx 25$. Area is maximum when all 20 feet are used on the pentagon.

(c) Let x be the side of the pentagon and y the side of the hexagon.

$$A = \frac{5}{4}\left(\cot\frac{\pi}{5}\right)x^2 + \frac{6}{4}\left(\cot\frac{\pi}{6}\right)y^2 \text{ where } 5x + 6y = 20 = \frac{5}{4}\left(\cot\frac{\pi}{5}\right)x^2 + \frac{3}{2}\left(\sqrt{3}\right)\left(\frac{20-5x}{6}\right)^2, 0 \le x \le 4.$$

$$A' = \frac{5}{2}\left(\cot\frac{\pi}{5}\right)x + 3\sqrt{3}\left(-\frac{5}{6}\right)\left(\frac{20-5x}{6}\right) = 0$$

$$x \approx 2.0475$$

When $x = 0$, $A \approx 28.868$, when $x \approx 2.0475$, $A \approx 14.091$, and when $x = 4$, $A \approx 27.528$. Area is maximum when all 20 feet are used on the hexagon

(d) Let x be the side of the hexagon and r the radius of the circle.

$$A = \frac{6}{4}\left(\cot\frac{\pi}{6}\right)x^2 + \pi r^2 \text{ where } 6x + 2\pi r = 20 = \frac{3\sqrt{3}}{2}x^2 + \pi\left(\frac{10}{\pi} - \frac{3x}{\pi}\right)^2, 0 \le x \le \frac{10}{3}.$$

$$A' = 3\sqrt{3} - 6\left(\frac{10}{\pi} - \frac{3x}{\pi}\right) = 0$$

$$x \approx 1.748$$

When $x = 0$, $A \approx 31.831$, when $x \approx 1.748$, $A \approx 15.138$, and when $x = 10/3$, $A \approx 28.868$. Area is maximum when all 20 feet are used on the circle.

In general, using all of the wire for the figure with more sides will enclose the most area.

37. Let S be the strength and k the constant of proportionality. Given $h^2 + w^2 = 20^2, h^2 = 20^2 - w^2$,

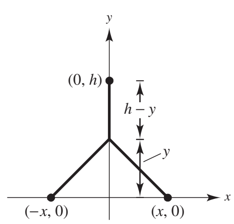
$$S = kwh^2$$
$$S = kw(400 - w^2) = k(400w - w^3)$$
$$\frac{dS}{dw} = k(400 - 3w^2) = 0 \text{ when } w = \frac{20\sqrt{3}}{3} \text{ in. and } h = \frac{20\sqrt{6}}{3} \text{ in.}$$
$$\frac{d^2S}{dw^2} = -6kw < 0 \text{ when } w = \frac{20\sqrt{3}}{3}.$$

These values yield a maximum.

38. Let A be the amount of the power line.

$$A = h - y + 2\sqrt{x^2 + y^2}$$
$$\frac{dA}{dy} = -1 + \frac{2y}{\sqrt{x^2 + y^2}} = 0 \text{ when } y = \frac{x}{\sqrt{3}}.$$
$$\frac{d^2A}{dy^2} = \frac{2x^2}{(x^2 + y^2)^{3/2}} > 0 \text{ for } y = \frac{x}{\sqrt{3}}.$$

The amount of power line is minimum when $y = x/\sqrt{3}$.

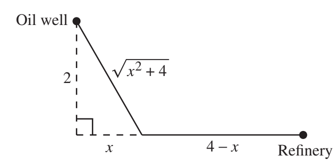


39. $C(x) = 2k\sqrt{x^2 + 4} + k(4 - x)$

$$C'(x) = \frac{2xk}{\sqrt{x^2 + 4}} - k = 0$$
$$2x = \sqrt{x^2 + 4}$$
$$4x^2 = x^2 + 4$$
$$3x^2 = 4$$
$$x = \frac{2}{\sqrt{3}}$$

Or, use Exercise 50(d): $\sin\theta = \frac{C_2}{C_1} = \frac{1}{2} \Rightarrow \theta = 30°$.

So, $x = \frac{2}{\sqrt{3}}$.



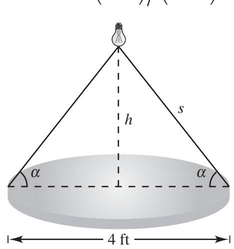
40. $\sin\alpha = \frac{h}{s} \Rightarrow s = \frac{h}{\sin\alpha}, 0 < \alpha < \frac{\pi}{2}$

$$\tan\alpha = \frac{h}{2} \Rightarrow h = 2\tan\alpha \Rightarrow s = \frac{2\tan\alpha}{\sin\alpha} = 2\sec\alpha$$
$$I = \frac{k\sin\alpha}{s^2} = \frac{k\sin\alpha}{4\sec^2\alpha} = \frac{k}{4}\sin\alpha\cos^2\alpha$$
$$\frac{dI}{d\alpha} = \frac{k}{4}\left[\sin\alpha(-2\sin\alpha\cos\alpha) + \cos^2\alpha(\cos\alpha)\right] = \frac{k}{4}\cos\alpha\left[\cos^2\alpha - 2\sin^2\alpha\right] = \frac{k}{4}\cos\alpha\left[1 - 3\sin^2\alpha\right] = 0$$

when $\alpha = \frac{\pi}{2}, \frac{3\pi}{2}$, or when $\sin\alpha = \pm\frac{1}{\sqrt{3}}$.

Because α is acute, you have $\sin\alpha = \frac{1}{\sqrt{3}} \Rightarrow h = 2\tan\alpha = 2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$ ft.

Because $(d^2I)/(d\alpha^2) = (k/4)\sin\alpha(9\sin^2\alpha - 7) < 0$ when $\sin\alpha = 1/\sqrt{3}$, this yields a maximum.



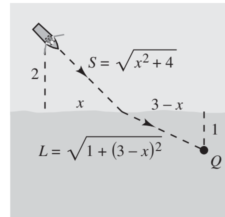
41. $$S = \sqrt{x^2 + 4},\ L = \sqrt{1 + (3 - x)^2}$$

$$\text{Time} = T = \frac{\sqrt{x^2 + 4}}{2} + \frac{\sqrt{x^2 - 6x + 10}}{4}$$

$$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2 + 4}} + \frac{x - 3}{4\sqrt{x^2 - 6x + 10}} = 0$$

$$\frac{x^2}{x^2 + 4} = \frac{9 - 6x + x^2}{4(x^2 - 6x + 10)}$$

$$x^4 - 6x^3 + 9x^2 + 8x - 12 = 0$$



You need to find the roots of this equation in the interval $[0, 3]$. By using a computer or graphing utility you can determine that this equation has only one root in this interval $(x = 1)$. Testing at this value and at the endpoints, you see that $x = 1$ yields the minimum time. So, the man should row to a point 1 mile from the nearest point on the coast.

42. $$T = \frac{\sqrt{x^2 + 4}}{v_1} + \frac{\sqrt{x^2 - 6x + 10}}{v_2}$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2 + 4}} + \frac{x - 3}{v_2\sqrt{x^2 - 6x + 10}} = 0$$

Because

$$\frac{x}{\sqrt{x^2 + 4}} = \sin\theta_1 \text{ and } \frac{x - 3}{\sqrt{x^2 - 6x + 10}} = -\sin\theta_2$$

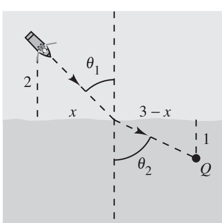
you have

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \Rightarrow \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

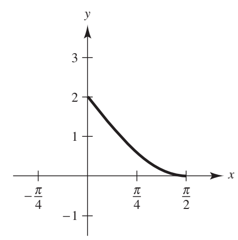
Because

$$\frac{d^2T}{dx^2} = \frac{4}{v_1(x^2 + 4)^{3/2}} + \frac{1}{v_2(x^2 - 6x + 10)^{3/2}} > 0$$

this condition yields a minimum time.

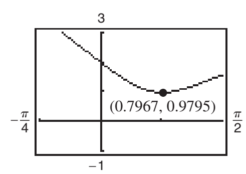


43. $f(x) = 2 - 2\sin x$



(a) Distance from origin to y-intercept is 2.
Distance from origin to x-intercept is $\pi/2 \approx 1.57$.

(b) $d = \sqrt{x^2 + y^2} = \sqrt{x^2 + (2 - 2\sin x)^2}$



Minimum distance $= 0.9795$ at $x = 0.7967$.

(c) Let $f(x) = d^2(x) = x^2 + (2 - 2\sin x)^2$.

$f'(x) = 2x + 2(2 - 2\sin x)(-2\cos x)$

Setting $f'(x) = 0$, you obtain $x \approx 0.7967$, which corresponds to $d = 0.9795$.

44. $$T = \frac{\sqrt{x^2 + d_1^2}}{v_1} + \frac{\sqrt{d_2^2 + (a - x)^2}}{v_2}$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2 + d_1^2}} + \frac{x - a}{v_2\sqrt{d_2^2 + (a - x)^2}} = 0$$

Because $\dfrac{x}{\sqrt{x^2 + d_1^2}} = \sin\theta_1$ and $\dfrac{x - a}{\sqrt{d_2^2 + (a - x)^2}} = -\sin\theta_2$

you have $\dfrac{\sin\theta_1}{v_1} - \dfrac{\sin\theta_2}{v_2} = 0 \Rightarrow \dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$.

Because $\dfrac{d^2T}{dx^2} = \dfrac{d_1^2}{v_1(x^2 + d_1^2)^{3/2}} + \dfrac{d_2^2}{v_2\left[d_2^2 + (a - x)^2\right]^{3/2}} > 0$ this condition yields a minimum time.

45. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\sqrt{144 - r^2}$

$$\frac{dV}{dr} = \frac{1}{3}\pi\left[r^2\left(\frac{1}{2}\right)\left(144 - r^2\right)^{-1/2}(-2r) + 2r\sqrt{144 - r^2}\right] = \frac{1}{3}\pi\left[\frac{288r - 3r^3}{\sqrt{144 - r^2}}\right] = \pi\left[\frac{r\left(96 - r^2\right)}{\sqrt{144 - r^2}}\right] = 0 \text{ when } r = 0, 4\sqrt{6}.$$

By the First Derivative Test, V is maximum when $r = 4\sqrt{6}$ and $h = 4\sqrt{3}$.

Area of circle: $A = \pi(12)^2 = 144\pi$

Lateral surface area of cone: $S = \pi\left(4\sqrt{6}\right)\sqrt{\left(4\sqrt{6}\right)^2 + \left(4\sqrt{3}\right)^2} = 48\sqrt{6}\pi$

Area of sector: $144\pi - 48\sqrt{6}\pi = \frac{1}{2}\theta r^2 = 72\theta$

$$\theta = \frac{144\pi - 48\sqrt{6}\pi}{72} = \frac{2\pi}{3}\left(3 - \sqrt{6}\right) \approx 1.153 \text{ radians or } 66°$$

46. (a)

Base 1	Base 2	Altitude	Area
8	$8 + 16\cos 10°$	$8\sin 10°$	≈ 22.1
8	$8 + 16\cos 20°$	$8\sin 20°$	≈ 42.5
8	$8 + 16\cos 30°$	$8\sin 30°$	≈ 59.7
8	$8 + 16\cos 40°$	$8\sin 40°$	≈ 72.7
8	$8 + 16\cos 50°$	$8\sin 50°$	≈ 80.5
8	$8 + 16\cos 60°$	$8\sin 60°$	≈ 83.1

(b)

Base 1	Base 2	Altitude	Area
8	$8 + 16\cos 10°$	$8\sin 10°$	≈ 22.1
8	$8 + 16\cos 20°$	$8\sin 20°$	≈ 42.5
8	$8 + 16\cos 30°$	$8\sin 30°$	≈ 59.7
8	$8 + 16\cos 40°$	$8\sin 40°$	≈ 72.7
8	$8 + 16\cos 50°$	$8\sin 50°$	≈ 80.5
8	$8 + 16\cos 60°$	$8\sin 60°$	≈ 83.1
8	$8 + 16\cos 70°$	$8\sin 70°$	≈ 80.7
8	$8 + 16\cos 80°$	$8\sin 80°$	≈ 74.0
8	$8 + 16\cos 90°$	$8\sin 90°$	≈ 64.0

The maximum cross-sectional area is approximately 83.1 ft^2.

(c) $A = (a + b)\frac{h}{2}$

$= \left[8 + (8 + 16\cos\theta)\right]\frac{8\sin\theta}{2}$

$= 64(1 + \cos\theta)\sin\theta, 0° < \theta < 90°$

(d) $\frac{dA}{d\theta} = 64(1 + \cos\theta)\cos\theta + (-64\sin\theta)\sin\theta$

$= 64\left(\cos\theta + \cos^2\theta - \sin^2\theta\right)$

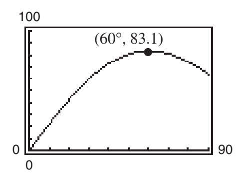
$= 64\left(2\cos^2\theta + \cos\theta - 1\right)$

$= 64(2\cos\theta - 1)(\cos\theta + 1)$

$= 0$ when $\theta = 60°, 180°, 300°$.

The maximum occurs when $\theta = 60°$.

(e)



47. Let d be the amount deposited in the bank, i be the interest rate paid by the bank, and P be the profit.

$P = (0.12)d - id$

$d = ki^2$ (because d is proportional to i^2)

$P = (0.12)\left(ki^2\right) - i\left(ki^2\right) = k\left(0.12i^2 - i^3\right)$

$\frac{dP}{di} = k\left(0.24i - 3i^2\right) = 0$ when $i = \frac{0.24}{3} = 0.08$.

$\frac{d^2P}{di^2} = k(0.24 - 6i) < 0$ when $i = 0.08$ (**Note:** $k > 0$).

The profit is a maximum when $i = 8\%$.

48. (a) The profit is increasing on $(0, 40)$.

(b) The profit is decreasing on $(40, 60)$.

(c) In order to yield a maximum profit, the company should spend about $40 thousand.

(d) The point of diminishing returns is the point where the concavity changes, which in this case is $x = 20$ thousand dollars.

49. $S_1 = (4m-1)^2 + (5m-6)^2 + (10m-3)^2$

$$\frac{dS_1}{dm} = 2(4m-1)(4) + 2(5m-6)(5) + 2(10m-3)(10) = 282m - 128 = 0 \text{ when } m = \frac{64}{141}.$$

Line: $y = \frac{64}{141}x$

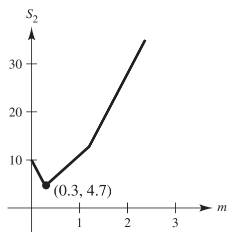
$$S = \left|4\left(\frac{64}{141}\right) - 1\right| + \left|5\left(\frac{64}{141}\right) - 6\right| + \left|10\left(\frac{64}{141}\right) - 3\right| = \left|\frac{256}{141} - 1\right| + \left|\frac{320}{141} - 6\right| + \left|\frac{640}{141} - 3\right| = \frac{858}{141} \approx 6.1 \text{ mi}$$

50. $S_2 = |4m-1| + |5m-6| + |10m-3|$

Using a graphing utility, you can see that the minimum occurs when $m = 0.3$.

Line $y = 0.3x$

$S_2 = |4(0.3)-1| + |5(0.3)-6| + |10(0.3)-3| = 4.7$ mi.

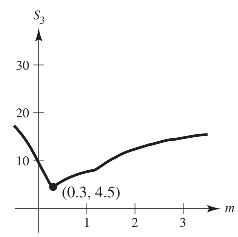


51. $S_3 = \frac{|4m-1|}{\sqrt{m^2+1}} + \frac{|5m-6|}{\sqrt{m^2+1}} + \frac{|10m-3|}{\sqrt{m^2+1}}$

Using a graphing utility, you can see that the minimum occurs when $x \approx 0.3$.

Line: $y \approx 0.3x$

$$S_3 = \frac{|4(0.3)-1| + |5(0.3)-6| + |10(0.3)-3|}{\sqrt{(0.3)^2+1}} \approx 4.5 \text{ mi.}$$



52. (a) Label the figure so that $r^2 = x^2 + h^2$.

Then, the area A is 8 times the area of the region given by $OPQR$:

$$A = 8\left[\frac{1}{2}h^2 + (x-h)h\right] = 8\left[\frac{1}{2}(r^2-x^2) + \left(x - \sqrt{r^2-x^2}\right)\sqrt{r^2-x^2}\right] = 8x\sqrt{r^2-x^2} + 4x^2 - 4r^2$$

$$A'(x) = 8\sqrt{r^2-x^2} - \frac{8x^2}{\sqrt{r^2-x^2}} + 8x = 0$$

$$\frac{8x^2}{\sqrt{r^2-x^2}} = 8x + 8\sqrt{r^2-x^2}$$

$$x^2 = x\sqrt{r^2-x^2} + (r^2-x^2)$$

$$2x^2 - r^2 = x\sqrt{r^2-x^2}$$

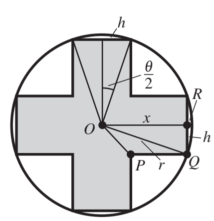
$$4x^4 - 4x^2r^2 + r^4 = x^2(r^2-x^2)$$

$$5x^4 - 5x^2r^2 + r^4 = 0 \quad \text{Quadratic in } x^2.$$

$$x^2 = \frac{5r^2 \pm \sqrt{25r^4 - 20r^4}}{10} = \frac{r^2}{10}\left[5 \pm \sqrt{5}\right].$$

Take positive value.

$$x = r\sqrt{\frac{5+\sqrt{5}}{10}} \approx 0.85065r \quad \text{Critical number}$$



(b) Note that $\sin\dfrac{\theta}{2} = \dfrac{h}{r}$ and $\cos\dfrac{\theta}{2} = \dfrac{x}{r}$. The area A of the cross equals the sum of two large rectangles minus the common square in the middle.

$$A = 2(2x)(2h) - 4h^2 = 8xh - 4h^2 = 8r^2\sin\frac{\theta}{2}\cos\frac{\theta}{2} - 4r^2\sin^2\frac{\theta}{2} = 4r^2\left(\sin\theta - \sin^2\frac{\theta}{2}\right)$$

$$A'(\theta) = 4r^2\left(\cos\theta - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = 0$$

$$\cos\theta = \sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{1}{2}\sin\theta$$

$$\tan\theta = 2$$

$$\theta = \arctan(2) \approx 1.10715 \quad \text{or} \quad 63.4°$$

(c) Note that $x^2 = \dfrac{r^2}{10}\left(5 + \sqrt{5}\right)$ and $r^2 - x^2 = \dfrac{r^2}{10}\left(5 - \sqrt{5}\right)$.

$$\begin{aligned}A(x) &= 8x\sqrt{r^2 - x^2} + 4x^2 - 4r^2\\ &= 8\left[\frac{r^2}{10}\left(5 + \sqrt{5}\right)\frac{r^2}{10}\left(5 - \sqrt{5}\right)\right]^{1/2} + 4\frac{r^2}{10}\left(5 + \sqrt{5}\right) - 4r^2\\ &= 8\left[\frac{r^4}{10}(20)\right]^{1/2} + 2r^2 + \frac{2}{5}\sqrt{5}r^2 - 4r^2\\ &= \frac{8}{5}r^2\sqrt{5} - 2r^2 + \frac{2\sqrt{5}}{5}r^2\\ &= 2r^2\left[\frac{4}{5}\sqrt{5} - 1 + \frac{\sqrt{5}}{5}\right] = 2r^2\left(\sqrt{5} - 1\right)\end{aligned}$$

Using the angle approach, note that $\tan\theta = 2$, $\sin\theta = \dfrac{2}{\sqrt{5}}$ and $\sin^2\left(\dfrac{\theta}{2}\right) = \dfrac{1}{2}(1 - \cos\theta) = \dfrac{1}{2}\left(1 - \dfrac{1}{\sqrt{5}}\right)$.

So, $A(\theta) = 4r^2\left(\sin\theta - \sin^2\dfrac{\theta}{2}\right) = 4r^2\left(\dfrac{2}{\sqrt{5}} - \dfrac{1}{2}\left(1 - \dfrac{1}{\sqrt{5}}\right)\right) = \dfrac{4r^2\left(\sqrt{5} - 1\right)}{2} = 2r^2\left(\sqrt{5} - 1\right)$

53. $16x = y^2$, $(6, 0)$

$$\begin{aligned}d &= \sqrt{(x - 6)^2 + (y - 0)^2}\\ &= \sqrt{\left(\frac{y^2}{16} - 6\right)^2 + y^2}\end{aligned}$$

Minimize d^2.

$$f(y) = \left(\frac{y^2}{16} - 6\right)^2 + y^2$$

$$\begin{aligned}f'(y) &= 2\left(\frac{y^2}{16} - 6\right)\left(\frac{y}{8}\right) + 2y\\ &= \frac{y^3}{64} + \frac{y}{2}\\ &= \frac{y}{64}\left(y^2 + 32\right)\end{aligned}$$

$f'(y) = 0$ when $y = 0$ and $x = 0$. By the First Derivative Test, the closest point is $(0, 0)$.

54. $x = \sqrt{10y}$, $(0, 4)$

$$d = \sqrt{(x - 0)^2 + (y - 4)^2} = \sqrt{10y + (y - 4)^2}$$

Minimize d^2.

$$f(y) = 10y + (y - 4)^2$$

$$f'(y) = 10 + 2(y - 4) = 2y + 2 = 0 \Rightarrow y = -1.$$

But, $y \ge 0$ and $x \ge 0$.

The minimum occurs at $(0, 0)$.

55. $f(x) = x^3 - 3x;\ x^4 + 36 \le 13x^2$

$$x^4 - 13x^2 + 36 = (x^2 - 9)(x^2 - 4)$$
$$= (x - 3)(x - 2)(x + 2)(x + 3) \le 0$$

So, $-3 \le x \le -2$ or $2 \le x \le 3$.

$f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1)$

f is increasing on $(-\infty, -1)$ and $(1, \infty)$.

So, f is increasing on $[-3, -2]$ and $[2, 3]$.

$f(-2) = -2,\ f(3) = 18$. The maximum value of f is 18.

56. Let $a = \left(x + \frac{1}{x}\right)^3$ and $b = x^3 + \frac{1}{x^3},\ x > 0$.

$$a^2 - b^2 = \left(x + \frac{1}{x}\right)^6 - \left(x^3 + \frac{1}{x^3}\right)^2$$
$$= \left(x + \frac{1}{x}\right)^6 - \left(x^6 + \frac{1}{x^6} + 2\right)$$

$$\text{Let } f(x) = \frac{(x + 1/x)^6 - (x^6 + 1/x^6 + 2)}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$
$$= \frac{a^2 - b^2}{a + b} = a - b$$
$$= \left(x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}\right) - \left(x^3 + \frac{1}{x^3}\right)$$
$$= 3x + \frac{3}{x} = 3\left(x + \frac{1}{x}\right).$$

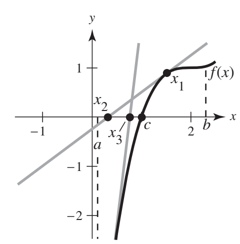
Let $g(x) = x + \frac{1}{x},\ g'(x) = 1 - \frac{1}{x^2} = 0 \Rightarrow x = 1$.

$g''(x) = \frac{2}{x^3}$ and $g''(1) = 2 > 0$. So g is a minimum at $x = 1$: $g(1) = 2$.

Finally, f is a minimum of $3(2) = 6$.

Section 3.8 Newton's Method

1. Answers will vary. *Sample answer:* If f is a function continuous on $[a, b]$ and differentiable on (a, b), where $c \in [a, b]$ and $f(c) = 0$, then Newton's Method uses tangent lines to approximate c. First, estimate an initial x_1 close to c. (See graph.) Then determine x_2 using $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$. Calculate a third estimate using $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$. Continue this process until $|x_n - x_{n+1}|$ is within the desired accuracy, and let x_{n+1} be the final approximation of c.



2. Because Newton's Method has $f'(x_n)$ in the denominator, the method will fail if $f'(x_n) = 0$. Graphically, this means that the tangent line of the graph of f at that point is horizontal.

In the solutions for Exercises 3–6, the values in the tables have been rounded for convenience. Because a calculator and a computer program calculate internally using more digits than they display, you may produce slightly different values from those shown in the tables.

3. $f(x) = x^2 - 5$

$f'(x) = 2x$

$x_1 = 2$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	2	–1	4	–0.25	2.25
2	2.25	0.0625	4.5	0.0139	2.2361

4. $f(x) = x^3 - 3$

$f'(x) = 3x^2$

$x_1 = 1.4$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.4000	–0.2560	5.8800	–0.0435	1.4435
2	1.4435	0.0080	6.2514	0.0013	1.4423

5. $f(x) = \cos x$

$f'(x) = -\sin x$

$x_1 = 1.6$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.6000	–0.0292	–0.9996	0.0292	1.5708
2	1.5708	0.0000	–1.0000	0.0000	1.5708

6. $f(x) = \tan x$

$f'(x) = \sec^2 x$

$x_1 = 0.1$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.1000	0.1003	1.0101	0.0993	0.0007
2	0.0007	0.0007	1.0000	0.0007	0.0000

7. $f(x) = x^3 + 4$

$f'(x) = 3x^2$

$x_1 = -2$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	–2.0000	–4.0000	12.0000	–0.3333	–1.6667
2	–1.6667	–0.6296	8.3333	–0.0756	–1.5911
3	–1.5911	–0.0281	7.5949	–0.0037	–1.5874
4	–1.5874	–0.0000	7.5596	0.0000	–1.5874

Approximation of the zero of f is –1.587.

8. $f(x) = 2 - x^3$

$f'(x) = -3x^2$

$x_1 = 1.0$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.0000	1.0000	–3.0000	–0.3333	1.3333
2	1.3333	–0.3704	–5.3333	0.0694	1.2639
3	1.2639	–0.0190	–4.7922	0.0040	1.2599
4	1.2599	0.0001	–4.7623	0.0000	1.2599

Approximation of the zero of f is 1.260.

9. $f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	–0.3750	1.7500	–0.2143	0.7143
2	0.7143	0.0788	2.5307	0.0311	0.6832
3	0.6832	0.0021	2.4003	0.0009	0.6823

Approximation of the zero of f is 0.682.

10. $f(x) = x^5 + x - 1$

$f'(x) = 5x^4 + 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	–0.4688	1.3125	–0.3571	0.8571
2	0.8571	0.3196	3.6983	0.0864	0.7707
3	0.7707	0.0426	2.7641	0.0154	0.7553
4	0.7553	0.0011	2.6272	0.0004	0.7549

Approximation of the zero of f is 0.755.

11. $f(x) = 5\sqrt{x-1} - 2x$

$f'(x) = \dfrac{5}{2\sqrt{x-1}} - 2$

From the graph you see that there are two zeros. Begin with $x = 1.2$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.2000	–0.1639	3.5902	–0.0457	1.2457
2	1.2457	–0.0131	3.0440	–0.0043	1.2500
3	1.2500	–0.0001	3.0003	–0.0003	1.2500

Approximation of the zero of f is 1.250.
Similarly, the other zero is approximately 5.000.
(**Note:** These answers are exact)

12. $f(x) = x - 2\sqrt{x+1}$

$f'(x) = 1 - \dfrac{1}{\sqrt{x+1}}$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	5.0000	0.1010	0.5918	0.1707	4.8293
2	4.8293	0.0005	0.5858	0.00085	4.8284

Approximation of the zero of f is 4.8284.

13. $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$

$f'(x) = 3x^2 - 7.8x + 4.79$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	–0.3360	1.6400	–0.2049	0.7049
2	0.7049	–0.0921	0.7824	–0.1177	0.8226
3	0.8226	–0.0231	0.4037	–0.0573	0.8799
4	0.8799	–0.0045	0.2495	–0.0181	0.8980
5	0.8980	–0.0004	0.2048	–0.0020	0.9000
6	0.9000	0.0000	0.2000	0.0000	0.9000

Approximation of the zero of f is 0.900.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.1	0.0000	–0.1600	–0.0000	1.1000

Approximation of the zero of f is 1.100.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.9	0.0000	0.8000	0.0000	1.9000

Approximation of the zero of f is 1.900.

14. $f(x) = -x^3 + 2.7x^2 + 3.55x - 2.422$

$f'(x) = -3x^2 + 5.4x + 3.55$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	–1.0000	–2.2720	–4.8500	0.4685	–1.4685
2	–1.4685	1.3542	–10.8494	–0.1248	–1.3437
3	–1.3437	0.1089	–9.12261	–0.0199	–1.3318
4	–1.3318	0.0013	–8.9628	–0.0001	–1.3317

Approximate zero: –1.3317

Similarly, you can approximate the other two zeros: 0.5176 and 3.5141.

15. $f(x) = 1 - x + \sin x$

$f'(x) = -1 + \cos x$

$x_1 = 2$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	2.0000	–0.0907	–1.4161	0.0640	1.9360
2	1.9360	–0.0019	–1.3571	0.0014	1.9346
3	1.9346	0.0000	–1.3558	0.0000	1.9346

Approximate zero: $x \approx 1.935$

16. $f(x) = x^3 - \cos x$

$f'(x) = 3x^2 + \sin x$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.9000	0.1074	3.2133	0.0334	0.8666
2	0.8666	0.0034	3.0151	0.0011	0.8655
3	0.8655	0.0000	3.0087	0.0000	0.8655

Approximation of the zero of f is 0.866.

17. $h(x) = f(x) - g(x) = 2x + 1 - \sqrt{x + 4}$

$h'(x) = 2 - \frac{1}{2\sqrt{x + 4}}$

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	0.6000	0.0552	1.7669	0.0313	0.5687
2	0.5687	0.0000	1.7661	0.0000	0.5687

Point of intersection of the graphs of f and g occurs when $x \approx 0.569$.

18. $h(x) = f(x) - g(x) = 3 - x - \dfrac{1}{x^2 + 1}$

$h'(x) = -1 + \dfrac{2x}{(x^2 + 1)^2}$

n	x_n	$h(x_n)$	$h'(x_n)$	$\dfrac{h(x_n)}{h'(x_n)}$	$x_n - \dfrac{h(x_n)}{h'(x_n)}$
1	2.9000	–0.0063	–0.9345	0.0067	2.8933
2	2.8933	0.0000	–0.9341	0.0000	2.8933

Point of intersection of the graphs of *f* and *g* occurs when $x \approx 2.893$.

19. $h(x) = f(x) - g(x) = x - \tan x$

$h'(x) = 1 - \sec^2 x$

n	x_n	$h(x_n)$	$h'(x_n)$	$\dfrac{h(x_n)}{h'(x_n)}$	$x_n - \dfrac{h(x_n)}{h'(x_n)}$
1	4.5000	–0.1373	–21.5048	0.0064	4.4936
2	4.4936	–0.0039	–20.2271	0.0002	4.4934

Point of intersection of the graphs of *f* and *g* occurs when $x \approx 4.493$.

Note: $f(x) = x$ and $g(x) = \tan x$ intersect infinitely often.

20. $h(x) = f(x) - g(x) = x^2 - \cos x$

$h'(x) = 2x + \sin x$

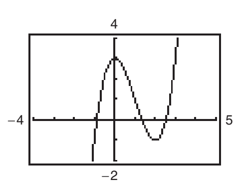
n	x_n	$h(x_n)$	$h'(x_n)$	$\dfrac{h(x_n)}{h'(x_n)}$	$x_n - \dfrac{h(x_n)}{h'(x_n)}$
1	0.8000	–0.0567	2.3174	–0.0245	0.8245
2	0.8245	0.0009	2.3832	0.0004	0.8241

One point of intersection of the graphs of *f* and *g* occurs when $x \approx 0.824$.

Because $f(x) = x^2$ and $g(x) = \cos x$ are both symmetric with respect to the *y*-axis, the other point of intersection occurs when $x \approx -0.824$.

21. $f(x) = x^3 - 3x^2 + 3$

(a)



(b) $f'(x) = 3x^2 - 6x$

$x_1 = 1$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} \approx 1.3333$

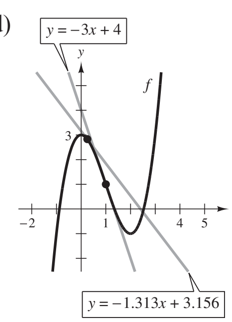
Continuing, you obtain the zero 1.3473.

(c) $x_1 = \dfrac{1}{4}$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} \approx 2.4048$

Continuing, you obtain the zero 2.5321.

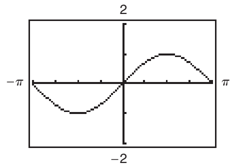
(d)



If the initial estimate $x = x_1$, is not sufficiently close to the desired zero of a function, then the *x*-intercept of the corresponding tangent line to the function may approximate a second zero of the function.

22. $f(x) = \sin x,\ f'(x) = \cos x$

(a)



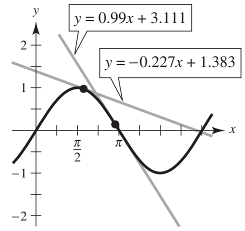
(b) $x_1 = 1.8$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 6.086$$

(c) $x_1 = 3$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 3.143$$

(d)



The x-intercept of $y = -0.227x + 1.383$ is approximately 6.086.
The x-intercept of $y = 0.99x + 3.111$ is approximately 3.143.

The x-intercepts correspond to the values resulting from the first iteration of Newton's Method.

(e) If the initial guess x_1 is not "close to" the desired zero of the function, the x-intercept of the tangent line may approximate another zero of the function.

23. $y = 2x^3 - 6x^2 + 6x - 1 = f(x)$

$y' = 6x^2 - 12x + 6 = f'(x)$

$x_1 = 1$

$f'(x) = 0$; therefore, the method fails.

n	x_n	$f(x_n)$	$f'(x_n)$
1	1	1	0

24. $y = x^3 - 2x - 2,\ x_1 = 0$

$y' = 3x^2 - 2$

$x_1 = 0$

$x_2 = -1$

$x_3 = 0$

$x_4 = -1$ and so on.

Fails to converge

25. Let $g(x) = f(x) - x = \cos x - x$

$g'(x) = -\sin x - 1.$

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	1.0000	–0.4597	–1.8415	0.2496	0.7504
2	0.7504	–0.0190	–1.6819	0.0113	0.7391
3	0.7391	0.0000	–1.6736	0.0000	0.7391

The fixed point is approximately 0.74.

26. Let $g(x) = f(x) - x = \cot x - x$

$g'(x) = -\csc^2 x - 1.$

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	1.0000	–0.3579	–2.4123	0.1484	0.8516
2	0.8516	0.0240	–2.7668	–0.0087	0.8603
3	0.8603	0.0001	–2.7403	0.0000	0.8603

The fixed point is approximately 0.86.

27. The future iterations will all be x_1 because $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n, n = 1, 2, 3, \ldots.$

28. Yes, because $f'(x_1) = 0$.

29. $y = f(x) = 4 - x^2, (1, 0)$

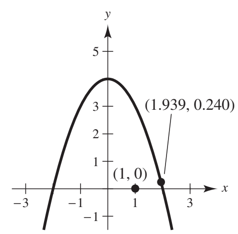
$$d = \sqrt{(x-1)^2 + (y-0)^2} = \sqrt{(x-1)^2 + (4-x^2)^2} = \sqrt{x^4 - 7x^2 - 2x + 17}$$

d is minimized when $D = x^4 - 7x^2 - 2x + 17$ is a minimum.

$g(x) = D' = 4x^3 - 14x - 2$

$g'(x) = 12x^2 - 14$

n	x_n	$g(x_n)$	$g'(x_n)$	$\dfrac{g(x_n)}{g'(x_n)}$	$x_n - \dfrac{g(x_n)}{g'(x_n)}$
1	2.0000	2.0000	34.0000	0.0588	1.9412
2	1.9412	0.0830	31.2191	0.0027	1.9385
3	1.9385	–0.0012	31.0934	0.0000	1.9385



$x \approx 1.939$

Point closest to $(1, 0)$ is $\approx (1.939, 0.240)$.

30. Maximize: $C = \dfrac{3t^2 + t}{50 + t^3}$

$$C' = \frac{-3t^4 - 2t^3 + 300t + 50}{(50 + t^3)^2} = 0$$

Let $f(x) = 3t^4 + 2t^3 - 300t - 50$

$f'(x) = 12t^3 + 6t^2 - 300.$

Because $f(4) = -354$ and $f(5) = 575$, the solution is in the interval $(4, 5)$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	4.5000	12.4375	915.0000	0.0136	4.4864
2	4.4864	0.0658	904.3822	0.0001	4.4863

Approximation: $t \approx 4.486$ hours

31. $$\text{Minimize: } T = \frac{\text{Distance rowed}}{\text{Rate rowed}} + \frac{\text{Distance walked}}{\text{Rate walked}}$$

$$T = \frac{\sqrt{x^2 + 4}}{3} + \frac{\sqrt{x^2 - 6x + 10}}{4}$$

$$T' = \frac{x}{3\sqrt{x^2 + 4}} + \frac{x - 3}{4\sqrt{x^2 - 6x + 10}} = 0$$

$$4x\sqrt{x^2 - 6x + 10} = -3(x - 3)\sqrt{x^2 + 4}$$

$$16x^2(x^2 - 6x + 10) = 9(x - 3)^2(x^2 + 4)$$

$$7x^4 - 42x^3 + 43x^2 + 216x - 324 = 0$$

Let $f(x) = 7x^4 - 42x^3 + 43x^2 + 216x - 324$ and $f'(x) = 28x^3 - 126x^2 + 86x + 216$.

Because $f(1) = -100$ and $f(2) = 56$, the solution is in the interval $(1, 2)$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	1.7000	19.5887	135.6240	0.1444	1.5556
2	1.5556	–1.0480	150.2780	–0.0070	1.5626
3	1.5626	0.0014	49.5591	0.0000	1.5626

Approximation: $x \approx 1.563$ mi

32. At $x = -3$ and $x = 2$, the tangent lines to the curve are horizontal. Hence, Newton's Method will not converge for these initial approximations.

33. (a) $f(x) = x^2 - a, a > 0$

$f'(x) = 2x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

(b) $\sqrt{5}$: $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{5}{x_n}\right), x_1 = 2$

n	1	2	3	4
x_n	2	2.25	2.2361	2.2361

For example, given $x_1 = 2$,

$$x_2 = \frac{1}{2}\left(2 + \frac{5}{2}\right) = \frac{9}{4} = 2.25.$$

$\sqrt{5} \approx 2.236$

$\sqrt{7}$: $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{7}{x_n}\right), x_1 = 2$

n	1	2	3	4	5
x_n	2	2.75	2.6477	2.6458	2.6458

$\sqrt{7} \approx 2.646$

34. (a) $f(x) = x^n - a, a > 0$

$f'(x) = nx^{n-1}$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^n - a}{nx_i^{n-1}} = \frac{(n-1)x_i^n + a}{nx_i^{n-1}}$$

(b) $\sqrt[4]{6}$: $x_{i+1} = \dfrac{3x_i^4 + 6}{4x_i^3}, x_1 = 1.5$

i	1	2	3	4
x_i	1.5	1.5694	1.5651	1.5651

$\sqrt[4]{6} \approx 1.565$

$\sqrt[3]{15}$: $x_{i+1} = \dfrac{2x_i^3 + 15}{3x_i^2}, x_1 = 2.5$

i	1	2	3	4
x_i	2.5	2.4667	2.4662	2.4662

$\sqrt[3]{15} \approx 2.466$

35. $f(x) = \frac{1}{x} - a = 0$

$f'(x) = -\frac{1}{x^2}$

$$x_{n+1} = x_n - \frac{(1/x_n) - a}{-1/x_n^2} = x_n + x_n^2\left(\frac{1}{x_n} - a\right) = x_n + x_n - x_n^2 a = 2x_n - x_n^2 a = x_n(2 - ax_n)$$

36. (a) $x_{n+1} = x_n(2 - 3x_n)$

i	1	2	3	4
x_i	0.3000	0.3300	0.3333	0.3333

$\frac{1}{3} \approx 0.333$

(b) $x_{n+1} = x_n(2 - 11x_n)$

i	1	2	3	4
x_i	0.1000	0.0900	0.0909	0.0909

$\frac{1}{11} \approx 0.091$

37. False. Let $f(x) = (x^2 - 1)/(x - 1)$. $x = 1$ is a discontinuity. It is not a zero of $f(x)$. This statement would be true if $f(x) = p(x)/q(x)$ was given in **reduced** form.

38. True

39. True

40. True because $f'(x_1) = 0$.

41. $f(x) = -\sin x$

$f'(x) = -\cos x$

Let $(x_0, y_1) = (x_0, -\sin(x_0))$ be a point on the graph of f. If (x_0, y_0) is a point of tangency, then

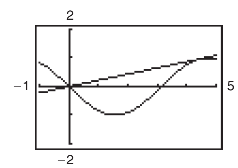
$$-\cos(x_0) = \frac{y_0 - 0}{x_0 - 0} = \frac{y_0}{x_0} = \frac{-\sin(x_0)}{x_0}.$$

So, $x_0 = \tan(x_0)$.

$x_0 \approx 4.4934$

Slope $= -\cos(x_0) \approx 0.217$

You can verify this answer by graphing $y_1 = -\sin x$ and the tangent line $y_2 = 0.217x$.



42. Let (x_1, y_1) be the point of tangency.

$f(x) = \cos x$, $f'(x) = -\sin x$, $f'(x_1) = -\sin(x_1)$.

At the point of tangency,

$$f'(x_1) = \frac{y_1 - 0}{x_1 - 0}$$

$-\sin(x_1) = \cos(x_1)/x_1$

$\cos(x_1) + x_1 \sin(x_1) = 0$

Using Newton's method with initial guess 3, you obtain $x_1 \approx 2.798$ and $y_1 \approx -0.942$.

Section 3.9 Differentials

1. The equation of the tangent line approximation to the graph of f at $(c, f(c))$ is $y = f(c) + f'(c)(x - c)$.

2. $dx = \Delta x$ is a small change in x.

dy is a small change in y: $dy = f'(x)\,dx$.

3. Propagated error $= f(x + \Delta x) - f(x)$,

relative error $= \left|\frac{dy}{y}\right|$, and the percent error

$= \left|\frac{dy}{y}\right| \times 100$.

4. Given a function $y = f(x)$, $dy = f'(x)\,dx$.

5. $f(x) = x^2$

$f'(x) = 2x$

Tangent line at (2, 4): $y - f(2) = f'(2)(x - 2)$

$$y - 4 = 4(x - 2)$$

$$y = 4x - 4$$

x	1.9	1.99	2	2.01	2.1
$f(x) = x^2$	3.6100	3.9601	4	4.0401	4.4100
$T(x) = 4x - 4$	3.6000	3.9600	4	4.0400	4.4000

6. $f(x) = \dfrac{6}{x^2} = 6x^{-2}$

$f'(x) = -12x^{-3} = \dfrac{-12}{x^3}$

Tangent line at $\left(2, \dfrac{3}{2}\right)$:

$$y - \frac{3}{2} = \frac{-12}{8}(x - 2) = \frac{-3}{2}(x - 2)$$

$$y = -\frac{3}{2}x + \frac{9}{2}$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \dfrac{6}{x^2}$	1.6620	1.5151	1.5	1.4851	1.3605
$T(x) = -\dfrac{3}{2}x + \dfrac{9}{2}$	1.65	1.515	1.5	1.485	1.35

7. $f(x) = x^5$

$f'(x) = 5x^4$

Tangent line at $(2, 32)$:

$$y - f(2) = f'(2)(x - 2)$$

$$y - 32 = 80(x - 2)$$

$$y = 80x - 128$$

x	1.9	1.99	2	2.01	2.1
$f(x) = x^5$	24.7610	31.2080	32	32.8080	40.8410
$T(x) = 80x - 128$	24.0000	31.2000	32	32.8000	40.0000

8. $f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

Tangent line at $(2, \sqrt{2})$:

$$y - f(2) = f'(2)(x - 2)$$

$$y - \sqrt{2} = \frac{1}{2\sqrt{2}}(x - 2)$$

$$y = \frac{x}{2\sqrt{2}} + \frac{1}{\sqrt{2}}$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \sqrt{x}$	1.3784	1.4107	1.4142	1.4177	1.4491
$T(x) = \dfrac{x}{2\sqrt{2}} + \dfrac{1}{\sqrt{2}}$	1.3789	1.4107	1.4142	1.4177	1.4496

9. $f(x) = \sin x$

$f'(x) = \cos x$

Tangent line at $(2, \sin 2)$:

$$y - f(2) = f'(2)(x - 2)$$

$$y - \sin 2 = (\cos 2)(x - 2)$$

$$y = (\cos 2)(x - 2) + \sin 2$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \sin x$	0.9463	0.9134	0.9093	0.9051	0.8632
$T(x) = (\cos 2)(x - 2) + \sin 2$	0.9509	0.9135	0.9093	0.9051	0.8677

10. $f(x) = \csc x$

$f'(x) = -\csc x \cot x$

Tangent line at $(2, \csc 2)$:

$$y - f(2) = f'(2)(x - 2)$$

$$y - \csc 2 = (-\csc 2 \cot 2)(x - 2)$$

$$y = (-\csc 2 \cot 2)(x - 2) + \csc 2$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \csc x$	1.0567	1.0948	1.0998	1.1049	1.1585
$T(x) = (-\csc 2 \cot 2)(x - 2) + \csc 2$	1.0494	1.0947	1.0998	1.1048	1.1501

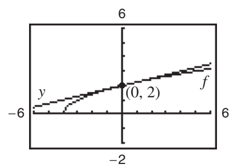
11. $f(x) = \sqrt{x+4}$

$f'(x) = \dfrac{1}{2\sqrt{x+4}}$

At $(0, 2)$, $f(0) = 2$, $f'(0) = \dfrac{1}{4}$

Tangent line: $y - 2 = \dfrac{1}{4}(x - 0)$

$y = \dfrac{1}{4}x + 2$



12. $f(x) = \tan x$

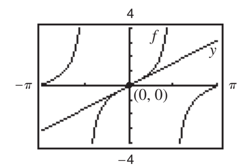
$f'(x) = \sec^2 x$

$f(0) = 0$

$f'(0) = 1$

Tangent line at $(0, 0)$: $y - 0 = (x - 0)$

$y = x$



13. $y = f(x) = 0.5x^3, f'(x) = 1.5x^2, x = 1, \Delta x = dx = 0.1$

$\Delta y = f(x + \Delta x) - f(x)$
$= f(1.1) - f(1)$
$= 0.1655$

$dy = f'(x)\,dx$
$= 1.5x^2\,dx$
$= 1.5(1)^2(0.1)$
$= 0.15$

14. $y = f(x) = 6 - 2x^2, f'(x) = -4x, x = -2, \Delta x = dx = 0.1$

$\Delta y = f(x + \Delta x) - f(x)$
$= f(-1.9) - f(-2)$
$= 6 - 2(-1.9)^2 - \left(6 - 2(-2)^2\right)$
$= -1.22 - (-2) = 0.78$

$dy = f'(x)\,dx$
$= -4(-2)(0.1)$
$= 0.8$

15. $y = f(x) = x^4 + 1, f'(x) = 4x^3, x = -1, \Delta x = dx = 0.01$

$\Delta y = f(x + \Delta x) - f(x)$
$= f(-0.99) - f(-1)$
$= \left[(-0.99)^4 + 1\right] - \left[(-1)^4 + 1\right] \approx -0.0394$

$dy = f'(x)\,dx$
$= f'(-1)(0.01)$
$= (-4)(0.01) = -0.04$

16. $y = f(x) = 2 - x^4, f'(x) = -4x^3, x = 2, \Delta x = dx = 0.01$

$\Delta y = f(x + \Delta x) - f(x)$
$= f(2.01) - f(2)$
$\approx -14.3224 - (-14) = -0.3224$

$dy = f'(x)\,dx$
$= \left(-4x^3\right)dx$
$= -4(2)^3(0.01)$
$= -0.32$

17. $y = f(x) = x - 2x^3, f'(x) = 1 - 6x^2, x = 3, \Delta x = dx = 0.001$

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x) \\ &= f(3.001) - f(3) \\ &= -51.05302 - (-51) \\ &= -0.05302\end{aligned} \qquad \begin{aligned}dy &= f'(x)\,dx \\ &= (1 - 6x^2)\,dx \\ &= \left[1 - 6(3)^2\right](0.001) \\ &= -0.053\end{aligned}$$

18. $y = f(x) = 7x^2 - 5x, f'(x) = 14x - 5, x = -4, \Delta x = dx = 0.001$

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x) \\ &= f(-4 + 0.01) - f(-4) \\ &\approx 131.93901 - 132 \\ &\approx -0.06099\end{aligned} \qquad \begin{aligned}dy &= f'(x)\,dx \\ &= (14x - 5)\,dx \\ &= \left[14(-4) - 5\right](0.001) \\ &= -0.061\end{aligned}$$

19. $y = 3x^2 - 4$

$dy = 6x\,dx$

20. $y = 3x^{2/3}$

$dy = 2x^{-1/3}\,dx = \dfrac{2}{x^{1/3}}\,dx$

21. $y = x \tan x$

$dy = (x \sec^2 x + \tan x)dx$

22. $y = \csc 2x$

$dy = (-2 \csc 2x \cot 2x)dx$

23. $y = \dfrac{x + 1}{2x - 1}$

$dy = \dfrac{-3}{(2x - 1)^2}\,dx$

24. $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

$dy = \left(\dfrac{1}{2\sqrt{x}} - \dfrac{1}{2x\sqrt{x}}\right)dx = \dfrac{x - 1}{2x\sqrt{x}}\,dx$

25. $y = \sqrt{9 - x^2}$

$dy = \dfrac{1}{2}(9 - x^2)^{-1/2}(-2x)\,dx = \dfrac{-x}{\sqrt{9 - x^2}}\,dx$

26. $y = x\sqrt{1 - x^2}$

$dy = \left(x\dfrac{-x}{\sqrt{1 - x^2}} + \sqrt{1 - x^2}\right)dx = \dfrac{1 - 2x^2}{\sqrt{1 - x^2}}\,dx$

27. $y = 3x - \sin^2 x$

$dy = (3 - 2 \sin x \cos x)\,dx = (3 - \sin 2x)\,dx$

28. $y = \dfrac{\sec^2 x}{x^2 + 1}$

$$\begin{aligned}dy &= \left[\frac{(x^2 + 1)2 \sec^2 x \tan x - \sec^2 x(2x)}{(x^2 + 1)^2}\right]dx \\ &= \left[\frac{2 \sec^2 x(x^2 \tan x + \tan x - x)}{(x^2 + 1)^2}\right]dx\end{aligned}$$

29. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$\approx 1 + (1)(-0.1) = 0.9$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$\approx 1 + (1)(0.04) = 1.04$

30. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$\approx 1 + \left(-\frac{1}{2}\right)(-0.1) = 1.05$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$\approx 1 + \left(-\frac{1}{2}\right)(0.04) = 0.98$

31. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$\approx 8 + \left(-\frac{1}{2}\right)(-0.07) = 8.035$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$\approx 8 + \left(-\frac{1}{2}\right)(0.1) = 7.95$

32. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$\approx 8 + (3)(-0.07) = 7.79$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$\approx 8 + (3)(0.1) = 8.3$

33. $x = 10$ in., $\Delta x = dx = \pm\frac{1}{32}$ in.

(a) $A = x^2$

$dA = 2x\,dx$

$\Delta A \approx dA = 2(10)\left(\pm\frac{1}{32}\right) = \pm\frac{5}{8}$ in.^{2}

(b) Percent error:

$$\frac{dA}{A} = \frac{5/8}{100} = \frac{5}{800} = \frac{1}{160} = 0.00625 = 0.625\%$$

34. $b = 36$ cm, $h = 50$ cm,
$\Delta b = \Delta h = db = dh = \pm 0.25$ cm

(a) $A = \frac{1}{2}bh$

$dA = \frac{1}{2}b\,dh + \frac{1}{2}h\,db$

$\Delta A \approx dA = \frac{1}{2}(36)(\pm 0.25) + \frac{1}{2}(50)(\pm 0.25)$

$= \pm 10.75$ cm^2

(b) Percent error:

$$\frac{dA}{A} = \frac{10.75}{\frac{1}{2}(36)(50)} \approx 0.011944 = 1.19\%$$

35. $x = 15$ in., $\Delta x = dx = \pm 0.03$ in.

(a) $V = x^3$

$dV = 3x^2 dx$

$\Delta V \approx dV = 3(15)^2(\pm 0.03) = \pm 20.25$ in.^{3}

(b) $S = 6x^2$

$dS = 12x\,dx$

$\Delta S \approx dS = 12(15)(\pm 0.03) = \pm 5.4$ in.^{2}

(c) Percent error of volume:

$$\frac{dV}{V} = \frac{20.25}{15^3} = 0.006 \text{ or } 0.6\%$$

Percent error of surface area:

$$\frac{dS}{S} = \frac{5.4}{6(15)^2} = 0.004 \text{ or } 0.4\%$$

36. $r = 8$ in., $dr = \Delta r = \pm 0.02$ in.

(a) $V = \frac{4}{3}\pi r^3$

$dV = 4\pi r^2 dr$

$\Delta V \approx dV = 4\pi(8)^2(\pm 0.02) = \pm 5.12\pi$ in.^{3}

(b) $S = 4\pi r^2$

$dS = 8\pi r\,dr$

$\Delta S \approx dS = 8\pi(8)(\pm 0.02) = \pm 1.28\pi$ in.^{2}

(c) Percent error of volume:

$$\frac{dV}{V} = \frac{5.12\pi}{\frac{4}{3}\pi(8)^2} = 0.0075 \text{ or } 0.75\%$$

Percent error of surface area:

$$\frac{dS}{S} = \frac{1.28\pi}{4\pi(8)^2} = 0.005 \text{ or } 0.5\%$$

37. $T = 2.5x + 0.5x^2$, $\Delta x = dx = 26 - 25 = 1$, $x = 25$

$dT = (2.5 + x)dx = (2.5 + 25)(1) = 27.5$ mi

Percentage change $= \frac{dT}{T} = \frac{27.5}{375} \approx 7.3\%$

38. Because the slope of the tangent line is greater at $x = 900$ than at $x = 400$, the change in profit is greater at $x = 900$ units.

39. (a) $T = 2\pi\sqrt{L/g}$

$dT = \frac{\pi}{g\sqrt{L/g}}dL$

Relative error:

$$\frac{dT}{T} = \frac{(\pi\,dL)/\left(g\sqrt{L/g}\right)}{2\pi\sqrt{L/g}}$$

$$= \frac{dL}{2L}$$

$$= \frac{1}{2}(\text{relative error in } L)$$

$$= \frac{1}{2}(0.005) = 0.0025$$

Percentage error: $\frac{dT}{T}(100) = 0.25\% = \frac{1}{4}\%$

(b) $(0.0025)(3600)(24) = 216$ sec $= 3.6$ min

40. $E = IR$

$$R = \frac{E}{I}$$

$$dR = -\frac{E}{I^2}dI$$

$$\frac{dR}{R} = \frac{-\left(E/I^2\right)dI}{E/I} = -\frac{dI}{I}$$

$$\left|\frac{dR}{R}\right| = \left|-\frac{dI}{I}\right| = \left|\frac{dI}{I}\right|$$

41. $R = \frac{v_0^{\,2}}{32}(\sin 2\theta)$

$v_0 = 2500$ ft/sec

θ changes from 10° to 11°.

$$dR = \frac{v_0^{\,2}}{32}(2)(\cos 2\theta)d\theta = \frac{(2500)^2}{16}\cos 2\theta\, d\theta$$

$$\theta = 10\left(\frac{\pi}{180}\right)$$

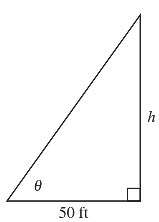
$$d\theta = (11 - 10)\frac{\pi}{180}$$

$$\Delta R \approx dR$$

$$= \frac{(2500)^2}{16}\cos\left(\frac{20\pi}{180}\right)\left(\frac{\pi}{180}\right)$$

$$\approx 6407 \text{ ft}$$

42. $h = 50 \tan \theta$



$$\theta = 71.5° = 1.2479 \text{ radians}$$

$$dh = 50 \sec^2\theta \cdot d\theta$$

$$\left|\frac{dh}{h}\right| = \left|\frac{50 \sec^2(1.2479)}{50 \tan(1.2479)}d\theta\right| \le 0.06$$

$$\left|\frac{9.9316}{2.9886}d\theta\right| \le 0.06$$

$$|d\theta| \le 0.018$$

43. Let $f(x) = \sqrt{x}$, $x = 100$, $dx = -0.6$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx$$

$$= \sqrt{x} + \frac{1}{2\sqrt{x}}dx$$

$$f(x + \Delta x) = \sqrt{99.4}$$

$$\approx \sqrt{100} + \frac{1}{2\sqrt{100}}(-0.6) = 9.97$$

Using a calculator: $\sqrt{99.4} \approx 9.96995$

44. Let $f(x) = \sqrt[3]{x}$, $x = 27$, $dx = -1$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = \sqrt[3]{x} + \frac{1}{3\sqrt[3]{x^2}}dx$$

$$\sqrt[3]{26} \approx \sqrt[3]{27} + \frac{1}{3\sqrt[3]{27^2}}(-1) = 3 - \frac{1}{27} \approx 2.9630$$

Using a calculator, $\sqrt[3]{26} \approx 2.9625$

45. Let $f(x) = \sqrt[4]{x}$, $x = 625$, $dx = -1$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = \sqrt[4]{x} + \frac{1}{4\sqrt[4]{x^3}}dx$$

$$f(x + \Delta x) = \sqrt[4]{624} \approx \sqrt[4]{625} + \frac{1}{4\left(\sqrt[4]{625}\right)^3}(-1)$$

$$= 5 - \frac{1}{500} = 4.998$$

Using a calculator, $\sqrt[4]{624} \approx 4.9980$.

46. Let $f(x) = x^3$, $x = 3$, $dx = -0.01$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = x^3 + 3x^2dx$$

$$f(x + \Delta x) = (2.99)^3 \approx 3^3 + 3(3)^2(-0.01)$$

$$= 27 - 0.27 = 26.73$$

Using a calculator: $(2.99)^3 \approx 26.7309$

47. In general, the value of dy becomes closer to the value of Δy as Δx approaches 0. Graphs will vary.

48. (a) Let $f(x) = \sqrt{x}$, $x = 4$, $dx = 0.02$,

$f'(x) = 1/\left(2\sqrt{x}\right)$.

Then $f(4.02) \approx f(4) + f'(4)\,dx$

$$\sqrt{4.02} \approx \sqrt{4} + \frac{1}{2\sqrt{4}}(0.02) = 2 + \frac{1}{4}(0.02).$$

(b) Let $f(x) = \tan x$, $x = 0$, $dx = 0.05$,

$f'(x) = \sec^2 x$.

Then $f(0.05) \approx f(0) + f'(0)\,dx$

$$\tan 0.05 \approx \tan 0 + \sec^2 0(0.05) = 0 + 1(0.05).$$

49. True

50. True, $\dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx} = a$

51. True

52. False

Let $f(x) = \sqrt{x}$, $x = 1$, and $\Delta x = dx = 3$. Then

$\Delta y = f(x + \Delta x) - f(x) = f(4) - f(1) = 1$

and $dy = f'(x)\,dx = \dfrac{1}{2\sqrt{1}}(3) = \dfrac{3}{2}$.

So, $dy > \Delta y$ in this example.

53. True

Review Exercises for Chapter 3

1. $f(x) = x^2 + 5x, \quad [-4, 0]$

$f'(x) = 2x + 5 = 0$ when $x = -5/2$

Critical number: $x = -5/2$

Left endpoint: $(-4, -4)$	
Critical number: $(-5/2, -25/4)$	Minimum
Right endpoint: $(0, 0)$	Maximum

2. $f(x) = x^3 + 6x^2, [-6, 1]$

$f'(x) = 3x^2 + 12x = 3x(x + 4) = 0$ when $x = 0, -4$

Critical numbers: $x = 0, -4$

Left endpoint: $(-6, 0)$	Minimum
Critical number: $(0, 0)$	Minimum
Critical number: $(-4, 32)$	Maximum
Right endpoint: $(1, 7)$	

3. $f(x) = \sqrt{x} - 2, [0, 4]$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

No critical numbers on $(0, 4)$

Left endpoint: $(0, -2)$	Minimum
Right endpoint: $(4, 0)$	Maximum

4. $h(x) = x - 3\sqrt{x}, [0, 9]$

$h'(x) = 1 - \dfrac{3}{2\sqrt{x}} = 0 \Rightarrow 2\sqrt{x} = 3 \Rightarrow x = \dfrac{9}{4}$

Critical number: $x = \dfrac{9}{4}$

Left endpoint: $(0, 0)$	Maximum
Critical number: $\left(\dfrac{9}{4}, -\dfrac{9}{4}\right)$	Minimum
Right endpoint: $(9, 0)$	Maximum

5. $f(x) = \dfrac{4x}{x^2 + 9}, [-4, 4]$

$f'(x) = \dfrac{(x^2 + 9)4 - 4x(2x)}{(x^2 + 9)^2} = \dfrac{36 - 4x^2}{(x^2 + 9)^2}$

$= 0 \Rightarrow 36 - 4x^2 = 0 \Rightarrow x = \pm 3$

Critical numbers: $x = \pm 3$

Left endpoint: $\left(-4, -\frac{16}{25}\right)$	
Critical number: $\left(-3, -\frac{2}{3}\right)$	Minimum
Critical number: $\left(3, \frac{2}{3}\right)$	Maximum
Right endpoint: $\left(4, \frac{16}{25}\right)$	

6. $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}, [0, 2]$

$f'(x) = x\left[-\dfrac{1}{2}(x^2 + 1)^{-3/2}(2x)\right] + (x^2 + 1)^{-1/2}$

$= \dfrac{1}{(x^2 + 1)^{3/2}}$

No critical numbers

Left endpoint: $(0, 0)$	Minimum
Right endpoint: $\left(2, 2/\sqrt{5}\right)$	Maximum

7. $g(x) = 2x + 5\cos x, [0, 2\pi]$

$g'(x) = 2 - 5\sin x = 0$ when $\sin x = \frac{2}{5}$.

Critical numbers: $x \approx 0.41, x \approx 2.73$

Left endpoint: $(0, 5)$	
Critical number: $(0.41, 5.41)$	
Critical number: $(2.73, 0.88)$	Minimum
Right endpoint: $(2\pi, 17.57)$	Maximum

8. $f(x) = \sin 2x, [0, 2\pi]$

$f'(x) = 2\cos 2x = 0$ when $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Left endpoint: $(0, 0)$

Critical number: $\left(\frac{\pi}{4}, 1\right)$ Maximum

Critical number: $\left(\frac{3\pi}{4}, -1\right)$ Minimum

Critical number: $\left(\frac{5\pi}{4}, 1\right)$ Maximum

Critical number: $\left(\frac{7\pi}{4}, -1\right)$ Minimum

Right endpoint: $(2\pi, 0)$

9. $f(x) = x^3 - 3x - 6, [-1, 2]$

$f(-1) = (-1)^3 - 3(-1) - 6 = -4$

$f(2) = 2^3 - 3(2) - 6 = -4$

f is continuous on $[-1, 2]$. f is differentiable on $(-1, 2)$.

Rolle's Theorem applies.

$f'(x) = 3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = 1$

c-value: 1

10. $f(x) = (x - 2)(x + 3)^2, [-3, 2]$

$f(-3) = (-3 - 2)(-3 + 3)^2 = 0$

$f(2) = (2 - 2)(2 + 3)^2 = 0$

f is continuous on $[-3, 2]$. f is differentiable on $(-3, 2)$.

Rolle's Theorem applies.

$f'(x) = (x + 3)(3x - 1) = 0 \Rightarrow x = \frac{1}{3}$

c-value: $\frac{1}{3}$

11. $f(x) = \dfrac{x^2}{1 - x^2}, [-2, 2]$

$f(-2) = \dfrac{(-2)^2}{1 - (-2)^2} = -\dfrac{4}{3}$

$f(2) = \dfrac{2^2}{1 - 2^2} = -\dfrac{4}{3}$

f is not continuous on $[-2, 2]$ because $f(-1)$ does not exist.

Rolle's Theorem does not apply.

12. $f(x) = \sin 2x, [-\pi, \pi]$

$f(-\pi) = f(\pi) = 0$. f is continuous on $[-\pi, \pi]$ and differentiable on $(-\pi, \pi)$.

Rolle's Theorem applies.

$f'(x) = 2\cos 2x = 0$ for $x = \pm\frac{3\pi}{4}, \pm\frac{\pi}{4}$.

c-values: $\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$

13. $f(x) = x^{2/3}, 1 \le x \le 8$

$f'(x) = \frac{2}{3}x^{-1/3}$

$\dfrac{f(b) - f(a)}{b - a} = \dfrac{4 - 1}{8 - 1} = \dfrac{3}{7}$

$f'(c) = \frac{2}{3}c^{-1/3} = \frac{3}{7}$

$c = \left(\frac{14}{9}\right)^3 = \frac{2744}{729} \approx 3.764$

14. $f(x) = \frac{1}{x}, 1 \le x \le 4$

$f'(x) = -\frac{1}{x^2}$

$\dfrac{f(b) - f(a)}{b - a} = \dfrac{(1/4) - 1}{4 - 1} = \dfrac{-3/4}{3} = -\dfrac{1}{4}$

$f'(c) = \frac{-1}{c^2} = -\frac{1}{4}$

$c = 2$

15. The Mean Value Theorem cannot be applied. f is not differentiable at $x = 5$ in $[2, 6]$.

16. The Mean Value Theorem cannot be applied. f is not defined for $x < 0$.

17. $f(x) = x - \cos x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$f'(x) = 1 + \sin x$

$\dfrac{f(b) - f(a)}{b - a} = \dfrac{(\pi/2) - (-\pi/2)}{(\pi/2) - (-\pi/2)} = 1$

$f'(c) = 1 + \sin c = 1$

$c = 0$

18. $f(x) = \sqrt{x} - 2x, 0 \le x \le 4$

$f'(x) = \dfrac{1}{2\sqrt{x}} - 2$

$\dfrac{f(b) - f(a)}{b - a} = \dfrac{-6 - 0}{4 - 0} = -\dfrac{3}{2}$

$f'(c) = \dfrac{1}{2\sqrt{c}} - 2 = -\dfrac{3}{2}$

$c = 1$

19. No; the function is discontinuous at $x = 0$ which is in the interval $[-2, 1]$.

20. (a) $f(x) = Ax^2 + Bx + C$

$f'(x) = 2Ax + B$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{A(x_2^2 - x_1^2) + B(x_2 - x_1)}{x_2 - x_1} = A(x_1 + x_2) + B$$

$f'(c) = 2Ac + B = A(x_1 + x_2) + B$

$2Ac = A(x_1 + x_2)$

$$c = \frac{x_1 + x_2}{2} = \text{Midpoint of } [x_1, x_2]$$

(b) $f(x) = 2x^2 - 3x + 1$

$f'(x) = 4x - 3$

$$\frac{f(b) - f(a)}{b - a} = \frac{21 - 1}{4 - 0} = 5$$

$f'(c) = 4c - 3 = 5$

$c = 2 = \text{Midpoint of } [0, 4]$

21. $f(x) = x^2 + 3x - 12$

$f'(x) = 2x + 3$

Critical number: $x = -\frac{3}{2}$

Intervals:	$-\infty < x < -\frac{3}{2}$	$-\frac{3}{2} < x < \infty$
Sign of $f'(x)$:	$f'(x) < 0$	$f'(x) > 0$
Conclusion:	Decreasing	Increasing

22. $h(x) = (x + 2)^{1/3} + 8$

$$h'(x) = \frac{1}{3}(x + 2)^{-2/3} = \frac{1}{3(x + 2)^{2/3}}$$

Critical number: $x = -2$

Intervals:	$(-\infty, -2)$	$(-2, \infty)$
Sign of $h'(x)$:	$h'(x) > 0$	$h'(x) > 0$
Conclusion:	Increasing	Increasing

h is increasing on $(-\infty, \infty)$.

23. $f(x) = (x - 1)^2(2x - 5)$

$f'(x) = 6(x - 2)(x - 1)$

Critical numbers: $x = 1, 2$

Intervals:	$-\infty < x < 1$	$1 < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion:	Increasing	Decreasing	Increasing

24. $g(x) = (x + 1)^3$

$g'(x) = 3(x + 1)^2$

Critical number: $x = -1$

Intervals:	$-\infty < x < -1$	$-1 < x < \infty$
Sign of $g'(x)$:	$g'(x) > 0$	$g'(x) > 0$
Conclusion:	Increasing	Increasing

25. $h(x) = \sqrt{x}(x - 3) = x^{3/2} - 3x^{1/2}$

Domain: $(0, \infty)$

$$h'(x) = \frac{3}{2}x^{3/2} - \frac{3}{2}x^{-1/2} = \frac{3}{2}x^{-1/2}(x - 1) = \frac{3(x - 1)}{2\sqrt{x}}$$

Critical number: $x = 1$

Intervals:	$0 < x < 1$	$1 < x < \infty$
Sign of $h'(x)$:	$h'(x) < 0$	$h'(x) > 0$
Conclusion:	Decreasing	Increasing

26. $f(x) = \sin x + \cos x, \quad 0 < x < 2\pi$

$f'(x) = \cos x - \sin x$

Critical numbers: $x = \frac{\pi}{4}, \frac{5\pi}{4}$

Intervals:	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < 2\pi$
Sign of $f'(x)$:	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion:	Increasing	Decreasing	Increasing

27. (a) $f(x) = x^2 - 6x + 5$

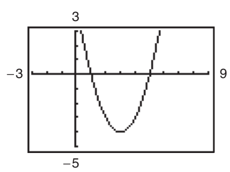
$f'(x) = 2x - 6 = 0$ when $x = 3$.

(b)

Intervals:	$-\infty < x < 3$	$3 < x < \infty$
Sign of $f'(x)$:	$f'(x) < 0$	$f'(x) > 0$
Conclusion:	Decreasing	Increasing

(c) Relative minimum: $(3, -4)$

(d)



28. (a) $f(x) = 4x^3 - 5x$

$f'(x) = 12x^2 - 5 = 0$ when $x = \pm\sqrt{\frac{5}{12}} = \pm\frac{\sqrt{15}}{6}$.

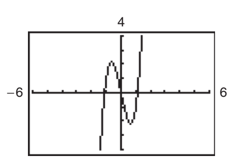
(b)

Intervals:	$-\infty < x < \frac{\sqrt{15}}{6}$	$-\frac{\sqrt{15}}{6} < x < \frac{\sqrt{15}}{6}$	$\frac{\sqrt{15}}{6} < x < \infty$
Sign of $f'(x)$:	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion:	Increasing	Decreasing	Increasing

(c) Relative maximum: $\left(-\frac{\sqrt{15}}{6}, \frac{5\sqrt{15}}{9}\right)$

Relative minimum: $\left(\frac{\sqrt{15}}{6}, -\frac{5\sqrt{15}}{9}\right)$

(d)



29. (a) $f(t) = \frac{1}{4}t^4 - 8t$

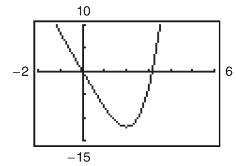
$f'(t) = t^3 - 8 = 0$ when $t = 2$.

(b)

Intervals:	$-\infty < t < 2$	$2 < t < \infty$
Sign of $f'(t)$:	$f'(t) < 0$	$f'(t) > 0$
Conclusion:	Decreasing	Increasing

(c) Relative minimum: $(2, -12)$

(d)



30. (a) $f(x) = \frac{1}{4}(x^3 - 8x)$

$f'(x) = \frac{3}{4}x^2 - 2 = 0 \Rightarrow x^2 = \frac{8}{3} \Rightarrow x = \pm\frac{2\sqrt{6}}{3}$

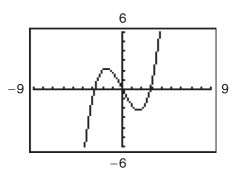
(b)

Intervals:	$-\infty < x < -\frac{2\sqrt{6}}{3}$	$-\frac{2\sqrt{6}}{3} < x < \frac{2\sqrt{6}}{3}$	$\frac{2\sqrt{6}}{3} < x < \infty$
Sign of $f'(x)$:	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion:	Increasing	Decreasing	Increasing

(c) Relative maximum: $\left(-\frac{2\sqrt{6}}{3}, \frac{8\sqrt{6}}{9}\right)$

Relative minimum: $\left(\frac{2\sqrt{6}}{3}, -\frac{8\sqrt{6}}{9}\right)$

(d)



31. (a) $f(x) = \frac{x + 4}{x^2}$

$f'(x) = \frac{x^2(1) - (x + 4)(2x)}{x^4} = -\frac{x^2 + 8x}{x^4} = -\frac{x + 8}{x^3}$

$f'(x) = 0$ when $x = -8$.

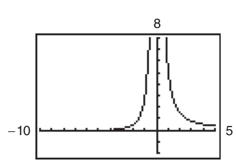
Discontinuity at: $x = 0$

(b)

Intervals:	$-\infty < x < -8$	$-8 < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f'(x) < 0$	$f'(x) > 0$	$f'(x) < 0$
Conclusion:	Decreasing	Increasing	Decreasing

(c) Relative minimum: $\left(-8, -\frac{1}{16}\right)$

(d)



32. (a) $f(x) = \dfrac{x^2 - 3x - 4}{x - 2}$

$$f'(x) = \frac{(x-2)(2x-3) - (x^2 - 3x - 4)(1)}{(x-2)^2}$$

$$= \frac{2x^2 - 7x + 6 - x^2 + 3x + 4}{(x-2)^2}$$

$$= \frac{x^2 - 4x + 10}{(x-2)^2}$$

$f'(x) \neq 0$ since $x^2 - 4x + 10 = 0$ has no real roots.

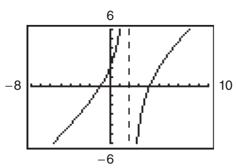
Discountinuity at: $x = 2$

(b)

Intervals:	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f'(x) > 0$	$f'(x) > 0$
Conclusion:	Increasing	Increasing

(c) No relative extrema

(d)



33. (a) $f(x) = \cos x - \sin x, (0, 2\pi)$

$f'(x) = -\sin x - \cos x = 0 \Rightarrow -\cos x = \sin x \Rightarrow \tan x = -1$

Critical numbers: $x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

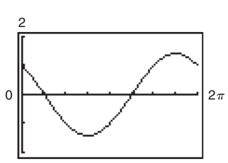
(b)

Intervals:	$0 < x < \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \frac{7\pi}{4}$	$\frac{7\pi}{4} < x < 2\pi$
Sign of $f'(x)$:	$f'(x) < 0$	$f'(x) > 0$	$f'(x) < 0$
Conclusion:	Decreasing	Increasing	Decreasing

(c) Relative minimum: $\left(\dfrac{3\pi}{4}, -\sqrt{2}\right)$

Relative maximum: $\left(\dfrac{7\pi}{4}, \sqrt{2}\right)$

(d)



34. (a) $f(x) = \frac{3}{2}\sin\left(\frac{\pi x}{2} - 1\right), \; [0, 4]$

$f'(x) = \frac{3}{2}\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi x}{2} - 1\right) = 0$ when $x = 1 + \frac{2}{\pi}, 3 + \frac{2}{\pi}.$

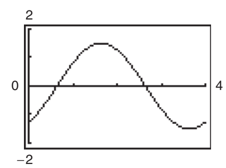
(b)

Intervals:	$0 < x < 1 + \frac{2}{\pi}$	$1 + \frac{2}{\pi} < x < 3 + \frac{2}{\pi}$	$3 + \frac{2}{\pi} < x < 4$
Sign of $f'(x)$:	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion:	Increasing	Decreasing	Increasing

(c) Relative maximum: $\left(1 + \frac{2}{\pi}, \frac{3}{2}\right)$

Relative minimum: $\left(3 + \frac{2}{\pi}, -\frac{3}{2}\right)$

(d)



35. (a) $s(t) = 3t - 2t^2, t \ge 0$

$v(t) = s'(t) = 3 - 4t$

(b) $v(t) = 0$ when $t = \frac{3}{4}$

$v(t) > 0$ for $0 \le t < \frac{3}{4} \Rightarrow$ positive direction

(c) $v(t) < 0$ for $t > \frac{3}{4} \Rightarrow$ negative direction

(d) Changes direction at $t = \frac{3}{4}$

36. (a) $s(t) = 6t^3 - 8t + 3, t \ge 0$

$v(t) = s'(t) = 18t^2 - 8$

(b) $v(t) = 0$ when $t^2 = \frac{8}{18} = \frac{4}{9} \Rightarrow t = \frac{2}{3}$

$v(t) > 0$ for $t > \frac{2}{3} \Rightarrow$ positive direction

(c) $v(t) < 0$ for $0 \le t < \frac{2}{3} \Rightarrow$ negative direction

(d) Changes direction at $t = \frac{2}{3}$

37. $f(x) = x^3 - 9x^2$

$f'(x) = 3x^2 - 18x$

$f''(x) = 6x - 18 = 0$ when $x = 3.$

Intervals:	$-\infty < x < 3$	$3 < x < \infty$
Sign of $f''(x)$:	$f''(x) < 0$	$f''(x) > 0$
Conclusion:	Concave downward	Concave upward

Point of inflection: $(3, -54)$

38. $f(x) = 6x^4 - x^2$

$f'(x) = 24x^3 - 2x$

$f''(x) = 72x^2 - 2 = 0 \Rightarrow x^2 = \frac{1}{36} \Rightarrow x = \pm\frac{1}{6}$

Intervals:	$-\infty < x < -\frac{1}{6}$	$-\frac{1}{6} < x < \frac{1}{6}$	$\frac{1}{6} < x < \infty$
Sign of $f''(x)$:	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Points of inflection: $\left(-\frac{1}{6}, -\frac{5}{216}\right), \left(\frac{1}{6}, -\frac{5}{216}\right)$

39. $g(x) = x\sqrt{x+5}$, Domain: $x \geq -5$

$$g'(x) = x\left(\frac{1}{2}\right)(x+5)^{-1/2} + (x+5)^{1/2} = \frac{1}{2}(x+5)^{-1/2}(x + 2(x+5)) = \frac{3x+10}{2\sqrt{x+5}}$$

$$g''(x) = \frac{2\sqrt{x+5}(3) - (3x+10)(x+5)^{-1/2}}{4(x+5)} = \frac{6(x+5) - (3x+10)}{4(x+5)^{3/2}} = \frac{3x+20}{4(x+5)^{3/2}} > 0 \text{ on } (-5, \infty).$$

Concave upward on $(-5, \infty)$

No point of inflection

40. $f(x) = 3x - 5x^3$

$f'(x) = 3 - 15x^2$

$f''(x) = -30x = 0$ when $x = 0$.

Intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f''(x)$:	$f''(x) > 0$	$f''(x) < 0$
Conclusion:	Concave upward	Concave downward

Point of inflection: $(0, 0)$

41. $f(x) = x + \cos x, 0 \leq x \leq 2\pi$

$f'(x) = 1 - \sin x$

$f''(x) = -\cos x = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f''(x)$:	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

Points of inflection: $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

42. $f(x) = \tan\frac{x}{4}, (0, 2\pi)$

$f'(x) = \frac{1}{4}\sec^2\frac{x}{4}$

$f''(x) = \frac{1}{4}(2)\left(\sec^2\frac{x}{4}\tan\frac{x}{4}\right)\left(\frac{1}{4}\right)$

$= \frac{1}{8}\sec^2\frac{x}{4}\tan\frac{x}{4} > 0 \text{ on} (0, 2\pi).$

Concave upward on $(0, 2\pi)$

No point of inflection

43. $f(x) = (x+9)^2$

$f'(x) = 2(x+9) = 0 \Rightarrow x = -9$

$f''(x) = 2 > 0 \Rightarrow (-9, 0)$ is a relative minimum.

44. $f(x) = x^4 - 2x^2 + 6$

$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x + 1)(x - 1)$

Critical numbers: $x = -1, 0, 1$

$f''(x) = 12x^2 - 4$

$f''(-1) = 12 - 4 = 8 > 0 \Rightarrow (-1, 5)$ is a relative minimum.

$f''(0) = -4 < 0 \Rightarrow (0, 6)$ is a relative maximum.

$f''(1) = 12 - 4 = 8 > 0 \Rightarrow (1, 5)$ is a relative minimum.

45. $g(x) = 2x^2(1 - x^2)$

$g'(x) = -4x(2x^2 - 1) = 0 \Rightarrow x = 0, \pm\frac{1}{\sqrt{2}}$

$g''(x) = 4 - 24x^2$

$g''(0) = 4 > 0 \quad (0, 0)$ is a relative minimum.

$g''\left(\pm\frac{1}{\sqrt{2}}\right) = -8 < 0 \left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ are relative maxima.

46. $h(t) = t - 4\sqrt{t + 1}$, Domain: $[-1, \infty]$

$h'(t) = 1 - \frac{2}{\sqrt{t + 1}} = 0 \Rightarrow t = 3$

$h''(t) = \frac{1}{(t + 1)^{3/2}}$

$h''(3) = \frac{1}{8} > 0 \quad (3, -5)$ is a relative minimum.

47. $f(x) = 2x + \frac{18}{x}$

$f'(x) = 2 - \frac{18}{x^2} = 0 \Rightarrow 2x^2 = 18 \Rightarrow x = \pm 3$

Critical numbers: $x = \pm 3$

$f''(x) = \frac{36}{x^3}$

$f''(-3) < 0 \Rightarrow (-3, -12)$ is a relative maximum.

$f''(3) > 0 \Rightarrow (3, 12)$ is a relative minimum.

48. $h(x) = x - 2\cos x, \quad [0, 4\pi]$

$h'(x) = 1 + 2\sin x = 0 \Rightarrow \sin x = -\frac{1}{2}$

Critical numbers: $x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$

$h''(x) = 2\cos x$

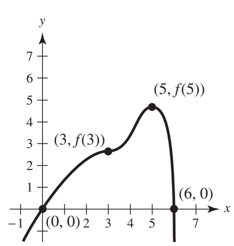
$h''\left(\frac{7\pi}{6}\right) = -\sqrt{3} < 0 \Rightarrow \left(\frac{7\pi}{6}, \frac{7\pi}{6} + \sqrt{3}\right) \approx (3.665, 5.397)$ is a relative maximum

$h''\left(\frac{11\pi}{6}\right) = \sqrt{3} > 0 \Rightarrow \left(\frac{11\pi}{6}, \frac{11\pi}{6} - \sqrt{3}\right) \approx (5.760, 4.028)$ is a relative minimum.

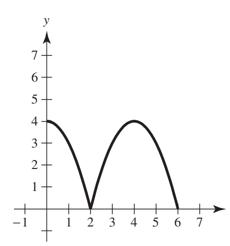
$h''\left(\frac{19\pi}{6}\right) = -\sqrt{3} < 0 \Rightarrow \left(\frac{19\pi}{6}, \frac{19\pi}{6} + \sqrt{3}\right) \approx (9.948, 11.680)$ is a relative maximum

$h''\left(\frac{23\pi}{6}\right) = \sqrt{3} > 0 \Rightarrow \left(\frac{23\pi}{6}, \frac{23\pi}{6} - \sqrt{3}\right) \approx (12.043, 10.311)$ is a relative minimum.

49.



50.

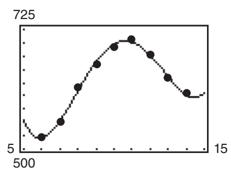


51. The first derivative is positive and the second derivative is negative. The graph is increasing and is concave down.

52.
$$C = \left(\frac{Q}{x}\right)s + \left(\frac{x}{2}\right)r$$
$$\frac{dC}{dx} = -\frac{Qs}{x^2} + \frac{r}{2} = 0$$
$$\frac{Qs}{x^2} = \frac{r}{2}$$
$$x^2 = \frac{2Qs}{r}$$
$$x = \sqrt{\frac{2Qs}{r}}$$

53. (a) $D = 0.41489t^4 - 17.1307t^3 + 249.888t^2 - 1499.45t + 3684.8$

(b)



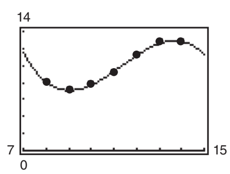
(c) The maximum occurred in 2011 $(t = 11)$.

The minimum occurred in 2006 $(t = 6)$.

(d) The outlays for national defense were increasing at the greatest rate in 2008 $(t = 8)$.

54. (a) $S = -0.1167t^3 + 3.980t^2 - 43.36t + 160.0$

(b)



(c) $S'(t) = -0.3501t^2 + 7.96t - 43.36$

$S''(t) = -0.7002t + 7.96 = 0$ when $t \approx 11$

Sales were increasing at the greatest rate in 2011.

(d) No, because the cubic equation has a negative leading coefficient and decreases for $t \geq 11$.

55. $\displaystyle\lim_{x\to\infty}\left(8 + \frac{1}{x}\right) = 8 + 0 = 8$

56. $\displaystyle\lim_{x\to-\infty}\frac{1 - 4x}{x + 1} = \lim_{x\to-\infty}\frac{1/x - 4}{1 + 4x} = -4$

57. $\displaystyle\lim_{x\to-\infty}\frac{x^2}{1 - 8x^2} = \lim_{x\to-\infty}\frac{1}{(1/x^2) - 8} = -\frac{1}{8}$

58. $\displaystyle\lim_{x\to-\infty}\frac{9x^3 + 5}{7x^4} = \lim_{x\to-\infty}\frac{9/x + 5/x^4}{7} = 0$

59. $\displaystyle\lim_{x\to-\infty}\frac{3x^2}{x + 5} = -\infty$

60. $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{x^2 + x}}{-2x} = 1/2$

61. $\displaystyle\lim_{x\to\infty}\frac{5\cos x}{x} = 0$, because $|5\cos x| \leq 5$.

62. $\displaystyle\lim_{x\to\infty}\frac{x^3}{\sqrt{x^2 + 2}} = \lim_{x\to\infty}\frac{x^3}{x\sqrt{1 + 2/x^2}}$

$\displaystyle= \lim_{x\to\infty}\frac{x^2}{\sqrt{1 + 2/x^2}} = \infty$

Limit does not exist.

63. $\displaystyle\lim_{x\to-\infty}\frac{6x}{x + \cos x} = 6$

64. $\displaystyle\lim_{x\to-\infty}\frac{x}{2\sin x}$ does not exist.

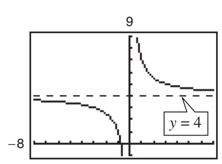
65. $f(x) = \dfrac{3}{x} + 4$

Discontinuity: $x = 0$

$\displaystyle\lim_{x\to\infty}\left(\frac{3}{x} + 4\right) = 4$

Vertical asymptote: $x = 0$

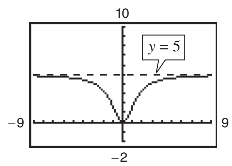
Horizontal asymptote: $y = 4$



66. $g(x) = \dfrac{5x^2}{x^2 + 2}$

$\lim_{x\to\infty} \dfrac{5x^2}{x^2 + 2} = \lim_{x\to\infty} \dfrac{5}{1 + (2/x^2)} = 5$

Horizontal asymptote: $y = 5$

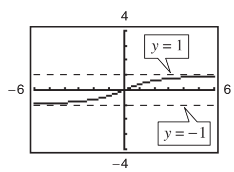


67. $f(x) = \dfrac{x}{\sqrt{x^2 + 6}}$

$\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2 + 6}} = \lim_{x\to\infty} \dfrac{1}{\sqrt{1 + 6/x^2}} = 1$

$\lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2 + 6}} = \lim_{x\to-\infty} \dfrac{-1}{\sqrt{1 + 6/x^2}} = -1$

Horizontal asymptotes: $y = \pm 1$

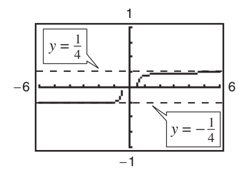


68. $f(x) = \dfrac{\sqrt{4x^2 - 1}}{8x + 1}$

$\lim_{x\to\infty} \dfrac{\sqrt{4x^2 + 1}}{8x + 1} = \lim_{x\to\infty} \dfrac{\sqrt{4 + 1/x^2}}{8 + 1/x} = \dfrac{1}{4}$

$\lim_{x\to-\infty} \dfrac{\sqrt{4x^2 + 1}}{8x + 1} = \lim_{x\to-\infty} \dfrac{\sqrt{4 + 1/x^2}}{-8 - 1/x} = -\dfrac{1}{4}$

Horizontal asymptotes: $y = \pm\dfrac{1}{4}$



69. $f(x) = 4x - x^2 = x(4 - x)$

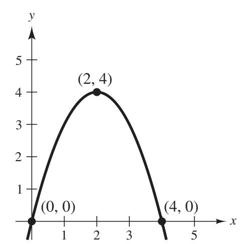
Domain: $(-\infty, \infty)$; Range: $(-\infty, 4]$

$f'(x) = 4 - 2x = 0$ when $x = 2$.

$f''(x) = -2$

Therefore, $(2, 4)$ is a relative maximum.

Intercepts: $(0, 0), (4, 0)$



70. $f(x) = x^4 - 2x^2 + 6$

Domain: $(-\infty, \infty)$; Range: $[5, \infty)$

$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1) = 0$ when $x = 0, 1, -1$.

$f''(x) = 12x^2 - 4 = 4(3x^2 - 1) = 0$ when $x = \pm\dfrac{\sqrt{3}}{3}$.

$f''(0) < 0$

Therefore, $(0, 6)$ is a relative maximum.

$f''(\pm 1) > 0$

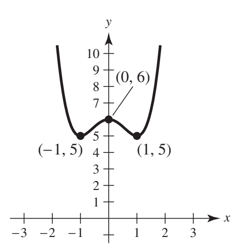
Therefore, $(\pm 1, 5)$ are relative minima.

y-intercept: $(0, 6)$

No x-intercepts

Symmetry with respect to y-axis

Points of inflection: $\left(\dfrac{1}{\sqrt{3}}, \pm\dfrac{49}{9}\right)$



71. $f(x) = x\sqrt{16 - x^2}$

Domain: $[-4, 4]$; Range: $[-8, 8]$

$f'(x) = \dfrac{16 - 2x^2}{\sqrt{16 - x^2}} = 0$ when $x = \pm 2\sqrt{2}$ and undefined when $x = \pm 4$.

$f''(x) = \dfrac{2x(x^2 - 24)}{(16 - x^2)^{3/2}}$

$f''(-2\sqrt{2}) > 0$

Therefore, $(-2\sqrt{2}, -8)$ is a relative minimum.

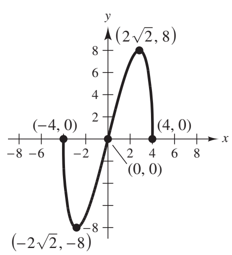
$f''(2\sqrt{2}) < 0$

Therefore, $(2\sqrt{2}, 8)$ is a relative maximum.

Point of inflection: $(0, 0)$

Intercepts: $(-4, 0), (0, 0), (4, 0)$

Symmetry with respect to origin



72. $f(x) = (x^2 - 4)^2$

Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$f'(x) = 4x(x^2 - 4) = 0$ when $x = 0, \pm 2$.

$f''(x) = 4(3x^2 - 4) = 0$ when $x = \pm\dfrac{2\sqrt{3}}{3}$.

$f''(0) < 0$

Therefore, $(0, 16)$ is a relative maximum.

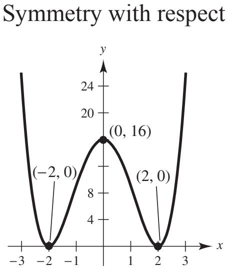
$f''(\pm 2) > 0$

Therefore, $(\pm 2, 0)$ are relative minima.

Points of inflection: $(\pm 2\sqrt{3}/3, 64/9)$

Intercepts: $(-2, 0), (0, 16), (2, 0)$

Symmetry with respect to y-axis



73. $f(x) = x^{1/3}(x + 3)^{2/3}$

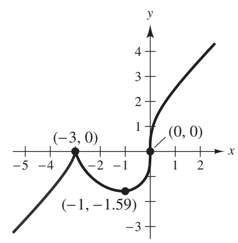
Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

$f'(x) = \dfrac{x + 1}{(x + 3)^{1/3}x^{2/3}} = 0$ when $x = -1$ and undefined when $x = -3, 0$.

$f''(x) = \dfrac{-2}{x^{5/3}(x + 3)^{4/3}}$ is undefined when $x = 0, -3$.

By the First Derivative Test $(-3, 0)$ is a relative maximum and $\left(-1, -\sqrt[3]{4}\right)$ is a relative minimum. $(0, 0)$ is a point of inflection.

Intercepts: $(-3, 0), (0, 0)$



74. $f(x) = (x - 3)(x + 2)^3$

Domain: $(-\infty, \infty)$; Range: $\left[-\frac{16,875}{256}, \infty\right)$

$f'(x) = (x - 3)(3)(x + 2)^2 + (x + 2)^3$

$= (4x - 7)(x + 2)^2 = 0$ when $x = -2, \frac{7}{4}$.

$f''(x) = (4x - 7)(2)(x + 2) + (x + 2)^2(4)$

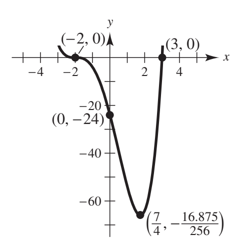
$= 6(2x - 1)(x + 2) = 0$ when $x = -2, \frac{1}{2}$.

$f''\left(\frac{7}{4}\right) > 0$

Therefore, $\left(\frac{7}{4}, -\frac{16,875}{256}\right)$ is a relative minimum.

Points of inflection: $(-2, 0), \left(\frac{1}{2}, -\frac{625}{16}\right)$

Intercepts: $(-2, 0), (0, -24,) (3, 0)$



75. $f(x) = \dfrac{5 - 3x}{x - 2}$

$f'(x) = \dfrac{1}{(x - 2)^2} > 0$ for all $x \neq 2$

$f''(x) = \dfrac{-2}{(x - 2)^3}$

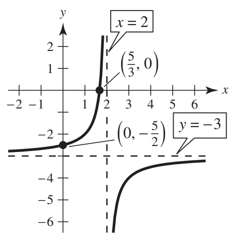
Concave upward on $(-\infty, 2)$

Concave downward on $(2, \infty)$

Vertical asymptote: $x = 2$

Horizontal asymptote: $y = -3$

Intercepts: $\left(\dfrac{5}{3}, 0\right), \left(0, -\dfrac{5}{2}\right)$



76. $f(x) = \dfrac{2x}{1 + x^2}$

Domain: $(-\infty, \infty)$; Range: $[-1, 1]$

$f'(x) = \dfrac{2(1 - x)(1 + x)}{(1 + x^2)^2} = 0$ when $x = \pm 1$.

$f''(x) = \dfrac{-4x(3 - x^2)}{(1 + x^2)^3} = 0$ when $x = 0, \pm\sqrt{3}$.

$f''(1) < 0$

Therefore, $(1, 1)$ is a relative maximum.

$f''(-1) > 0$

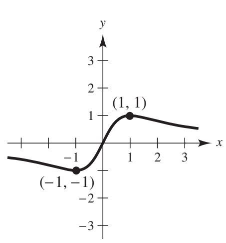
Therefore, $(-1, -1)$ is a relative minimum.

Points of inflection:
$\left(-\sqrt{3}, -\sqrt{3}/2\right), (0, 0), \left(\sqrt{3}, \sqrt{3}/2\right)$

Intercept: $(0, 0)$

Symmetric with respect to the origin

Horizontal asymptote: $y = 0$



77. $f(x) = x^3 + x + \dfrac{4}{x}$

Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, -6], [6, \infty)$

$f'(x) = 3x^2 + 1 - \dfrac{4}{x^2}$

$= \dfrac{3x^4 + x^2 - 4}{x^2} = \dfrac{(3x^2 + 4)(x^2 - 1)}{x^2} = 0$

when $x = \pm 1$.

$f''(x) = 6x + \dfrac{8}{x^3} = \dfrac{6x^4 + 8}{x^3} \neq 0$

$f''(-1) < 0$

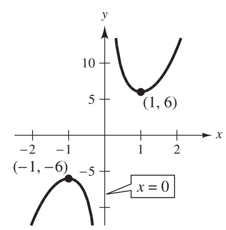
Therefore, $(-1, -6)$ is a relative maximum.

$f''(1) > 0$

Therefore, $(1, 6)$ is a relative minimum.

Vertical asymptote: $x = 0$

Symmetric with respect to origin



78. $f(x) = x^2 + \dfrac{1}{x} = \dfrac{x^3 + 1}{x}$

Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, \infty)$

$f'(x) = 2x - \dfrac{1}{x^2} = \dfrac{2x^3 - 1}{x^2} = 0$ when $x = \dfrac{1}{\sqrt[3]{2}}$.

$f''(x) = 2 + \dfrac{2}{x^3} = \dfrac{2(x^3 + 1)}{x^3} = 0$ when $x = -1$.

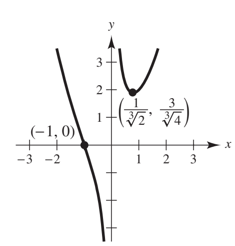
$f''\left(\dfrac{1}{\sqrt[3]{2}}\right) > 0$

Therefore, $\left(\dfrac{1}{\sqrt[3]{2}}, \dfrac{3}{\sqrt[3]{4}}\right)$ is a relative minimum.

Point of inflection: $(-1, 0)$

Intercept: $(-1, 0)$

Vertical asymptote: $x = 0$



79. Let x and y be two positive numbers such that $2x + 3y = 216$.

$$P = xy = x\left(\frac{216 - 2x}{3}\right) = \frac{1}{3}x(216 - 2x) = 72x - \frac{2}{3}x^2$$

$$\frac{dP}{dx} = 72 - \frac{4}{3}x = 0 \text{ when } x = 54.$$

$$\frac{d^2P}{dx^2} = -\frac{4}{3} < 0 \text{ when } x = 54.$$

For $x = 54, y = \dfrac{216 - 2(54)}{3} = 36.$

P is a maximum when $x = 54$ and $y = 36$.

80. The distance from $(6, 0)$ to $f(x) = \sqrt{x}$ is

$$d = \sqrt{(x - 6)^2 + (y - 0)^2} = \sqrt{(x - 6)^2 + x}.$$

Because d is smallest when d^2 is smallest, find the critical numbers of $f(x) = (x - 6)^2 + x$.

$$f'(x) = 2(x - 6) + 1 = 2x - 11 = 0 \Rightarrow x = \tfrac{11}{2}, y = \sqrt{\tfrac{11}{2}}$$

By the First Derivative Test, the closest point is $\left(\tfrac{11}{2}, \sqrt{\tfrac{11}{2}}\right)$.

81. $4x + 3y = 400$ is the perimeter.

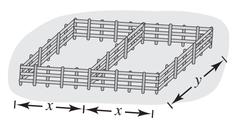
$$A = 2xy = 2x\left(\frac{400 - 4x}{3}\right) = \frac{8}{3}\left(100x - x^2\right)$$

$$\frac{dA}{dx} = \frac{8}{3}(100 - 2x) = 0 \text{ when } x = 50.$$

$$\frac{d^2A}{dx^2} = -\frac{16}{3} < 0 \text{ when } x = 50.$$

A is a maximum when

$x = 50$ ft and $y = \dfrac{200}{3}$ ft.



82. Ellipse: $\dfrac{x^2}{144} + \dfrac{y^2}{16} = 1, \; y = \dfrac{1}{3}\sqrt{144 - x^2}$

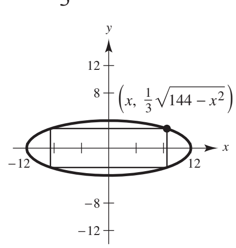
$$A = (2x)\left(\frac{2}{3}\sqrt{144 - x^2}\right) = \frac{4}{3}x\sqrt{144 - x^2}$$

$$\frac{dA}{dx} = \frac{4}{3}\left[\frac{-x^2}{\sqrt{144 - x^2}} + \sqrt{144 - x^2}\right]$$

$$= \frac{4}{3}\left[\frac{144 - 2x^2}{\sqrt{144 - x^2}}\right] = 0 \text{ when } x = \sqrt{72} = 6\sqrt{2}.$$

The dimensions of the rectangle are $2x = 12\sqrt{2}$ by

$y = \dfrac{2}{3}\sqrt{144 - 72} = 4\sqrt{2}.$



83. You have points $(0, y)$, $(x, 0)$, and $(1, 8)$. So,

$$m = \frac{y - 8}{0 - 1} = \frac{0 - 8}{x - 1} \text{ or } y = \frac{8x}{x - 1}.$$

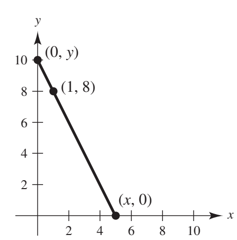
Let $f(x) = L^2 = x^2 + \left(\dfrac{8x}{x - 1}\right)^2.$

$$f'(x) = 2x + 128\left(\frac{x}{x - 1}\right)\left[\frac{(x - 1) - x}{(x - 1)^2}\right] = 0$$

$$x - \frac{64x}{(x - 1)^3} = 0$$

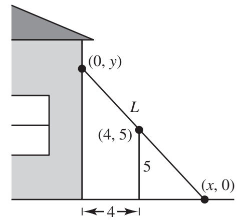
$$x\left[(x - 1)^3 - 64\right] = 0 \text{ when } x = 0, 5 \text{ (minimum)}.$$

Vertices of triangle: $(0, 0)$, $(5, 0)$, $(0, 10)$



84. You have points $(0, y)$, $(x, 0)$, and $(4, 5)$. So,

$$m = \frac{y - 5}{0 - 4} = \frac{5 - 0}{4 - x} \text{ or } y = \frac{5x}{x - 4}.$$



Let $f(x) = L^2 = x^2 + \left(\frac{5x}{x - 4}\right)^2$

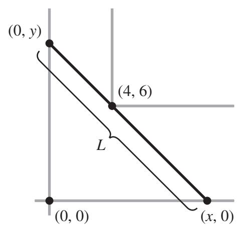
$$f'(x) = 2x + 50\left(\frac{x}{x - 4}\right)\left[\frac{x - 4 - x}{(x - 4)^2}\right] = 0$$

$$x - \frac{100x}{(x - 4)^3} = 0$$

$$x\left[(x - 4)^3 - 100\right] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{100}.$$

$$L = \sqrt{x^2 + \frac{25x^2}{(x - 4)^2}} = \frac{x}{x - 4}\sqrt{(x - 4)^2 + 25} = \frac{\sqrt[3]{100} + 4}{\sqrt[3]{100}}\sqrt{100^{2/3} + 25} \approx 12.7 \text{ ft}$$

85. You can form a right triangle with vertices $(0, 0)$, $(x, 0)$ and $(0, y)$. Assume that the hypotenuse of length L passes through $(4, 6)$.



$$m = \frac{y - 6}{0 - 4} = \frac{6 - 0}{4 - x} \text{ or } y = \frac{6x}{x - 4}$$

Let $f(x) = L^2 = x^2 + y^2 = x^2 + \left(\frac{6x}{x - 4}\right)^2$.

$$f'(x) = 2x + 72\left(\frac{x}{x - 4}\right)\left[\frac{-4}{(x - 4)^2}\right] = 0$$

$$x\left[(x - 4)^3 - 144\right] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{144}.$$

$$L \approx 14.05 \text{ ft}$$

86. $\csc \theta = \frac{L_1}{6}$ or $L_1 = 6 \csc \theta$ (see figure)

$$\sec \theta = \frac{L_2}{9} \text{ or } L_2 = 9 \sec \theta$$

$$L = L_1 + L_2 = 6 \csc \theta + 9 \sec \theta$$

$$\frac{dL}{d\theta} = -6 \csc \theta \cot \theta + 9 \sec \theta \tan \theta = 0$$

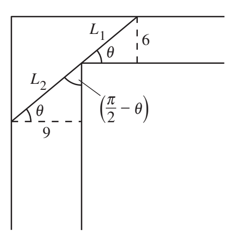
$$\tan^3 \theta = \frac{2}{3} \Rightarrow \tan \theta = \frac{\sqrt[3]{2}}{\sqrt[3]{3}}$$

$$\sec \theta = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + \left(\frac{2}{3}\right)^{2/3}} = \frac{\sqrt{3^{2/3} + 2^{2/3}}}{3^{1/3}}$$

$$\csc \theta = \frac{\sec \theta}{\tan \theta} = \frac{\sqrt{3^{2/3} + 2^{2/3}}}{2^{1/3}}$$

$$L = 6\frac{\left(3^{2/3} + 2^{2/3}\right)^{1/2}}{2^{1/3}} + 9\frac{\left(3^{2/3} + 2^{2/3}\right)^{1/2}}{3^{1/3}}$$

$$= 3\left(3^{2/3} + 2^{2/3}\right)^{3/2} \text{ ft} \approx 21.07 \text{ ft}$$



87. $V = \frac{1}{3}\pi x^2 h = \frac{1}{3}\pi x^2\left(r + \sqrt{r^2 - x^2}\right)$ (see figure)

$$\frac{dV}{dx} = \frac{1}{3}\pi\left[\frac{-x^3}{\sqrt{r^2 - x^2}} + 2x\left(r + \sqrt{r^2 - x^2}\right)\right] = \frac{\pi x}{3\sqrt{r^2 - x^2}}\left(2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2\right) = 0$$

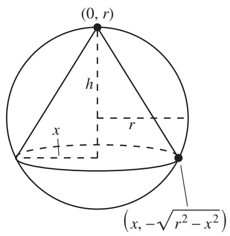
$$2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2 = 0$$

$$2r\sqrt{r^2 - x^2} = 3x^2 - 2r^2$$

$$4r^2\left(r^2 - x^2\right) = 9x^4 - 12x^2r^2 + 4r^4$$

$$0 = 9x^4 - 8x^2r^2 = x^2\left(9x^2 - 8r^2\right)$$

$$x = 0, \frac{2\sqrt{2}r}{3}$$

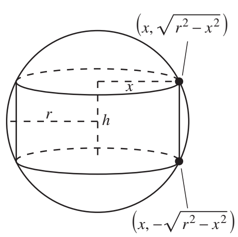


By the First Derivative Test, the volume is a maximum when $x = \frac{2\sqrt{2}r}{3}$ and $h = r + \sqrt{r^2 - x^2} = \frac{4r}{3}$.

Thus, the maximum volume is $V = \frac{1}{3}\pi\left(\frac{8r^2}{9}\right)\left(\frac{4r}{3}\right) = \frac{32\pi r^3}{81}$ cubic units.

88. $V = \pi x^2 h = \pi x^2\left(2\sqrt{r^2 - x^2}\right) = 2\pi x^2\sqrt{r^2 - x^2}$ (see figure)

$$\frac{dV}{dx} = 2\pi\left[x^2\left(\frac{1}{2}\right)\left(r^2 - x^2\right)^{-1/2}(-2x) + 2x\sqrt{r^2 - x^2}\right] = \frac{2\pi x}{\sqrt{r^2 - x^2}}\left(2r^2 - 3x^2\right) = 0 \text{ when } x = 0 \text{ and } x^2 = \frac{2r^2}{3} \Rightarrow x = \frac{\sqrt{6}r}{3}.$$



By the First Derivative Test, the volume is a maximum when $x = \frac{\sqrt{6}r}{3}$ and $h = \frac{2r}{\sqrt{3}}$.

Thus, the maximum volume is $V = \pi\left(\frac{2}{3}r^2\right)\left(\frac{2r}{\sqrt{3}}\right) = \frac{4\pi r^3}{3\sqrt{3}}$.

89. $f(x) = x^3 - 3x - 1$

From the graph you can see that $f(x)$ has three real zeros.

$f'(x) = 3x^2 - 3$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	–1.5000	0.1250	3.7500	0.0333	–1.5333
2	–1.5333	–0.0049	4.0530	–0.0012	–1.5321

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	–0.5000	0.3750	–2.2500	–0.1667	–0.3333
2	–0.3333	–0.0371	–2.6667	0.0139	–0.3472
3	–0.3472	–0.0003	–2.6384	0.0001	–0.3473

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.9000	0.1590	7.8300	0.0203	1.8797
2	1.8797	0.0024	7.5998	0.0003	1.8794

The three real zeros of $f(x)$ are $x \approx -1.532$, $x \approx -0.347$, and $x \approx 1.879$.

90. $f(x) = x^3 + 2x + 1$

From the graph, you can see that $f(x)$ has one real zero.

$f'(x) = 3x^2 + 2$

f changes sign in $[-1, 0]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	–0.5000	–0.1250	2.7500	–0.0455	–0.4545
2	–0.4545	–0.0029	2.6197	–0.0011	–0.4534

The real zero of $f(x)$ is: $x \approx -0.453$.

91. $f(x) = x^4 + x^3 - 3x^2 + 2$

From the graph you can see that $f(x)$ has two real zeros.

$f'(x) = 4x^3 + 3x^2 - 6x$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	−2.0	−2.0	−8.0	0.25	−2.25
2	−2.25	1.0508	−16.875	−0.0623	−2.1877
3	−2.1877	0.0776	−14.3973	−0.0054	−2.1823
4	−2.1823	0.0004	−14.3911	−0.00003	−2.1873

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	−1.0	−1.0	5.0	−0.2	−0.8
2	−0.8	−0.0224	4.6720	−0.0048	−0.7952
3	−0.7952	−0.00001	4.6569	−0.0000	−0.7952

The two zeros of $f(x)$ are $x \approx -2.1823$ and $x \approx -0.7952$.

92. $f(x) = 3\sqrt{x - 1} - x$

From the graph you can see that $f(x)$ has two real zeros.

$$f'(x) = \frac{3}{2\sqrt{x - 1}} - 1$$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.1	−0.1513	3.7434	−0.0404	1.1404
2	1.1404	−0.0163	3.0032	−0.0054	1.1458
3	1.1458	−0.0003	2.9284	−0.0000	1.1459

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	8.0	−0.0627	−0.4331	0.1449	7.8551
2	7.8551	−0.0004	−0.4271	0.0010	7.8541

The two zeros of $f(x)$ are $x \approx 1.1459$ and $x \approx 7.8541$.

93. $h(x) = f(x) - g(x)$

$= (1 - x) - (x^5 + 2)$

$= -x^5 - x - 1$

$h'(x) = -5x^4 - 1$

From the graph you can see that there is one point of intersection. That is, $h(x)$ has one real zero.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	−1.0	1.0	−6.0	−0.1667	−0.8333
2	−0.8333	0.2351	−3.4109	−0.0689	−0.7644
3	−0.7644	0.0254	−2.7071	−0.0094	−0.7550
4	−0.7550	0.0003	−2.6246	−0.0001	−0.7549

The point of intersection is $x \approx -0.7549$.

94. $h(x) = f(x) - g(x)$

$= \sin x - (x^2 - 2x + 1)$

$= \sin x - x^2 + 2x - 1$

From the graph you can see that there are two points of intersection. That is, $h(x)$ has two real zeros.

$h'(x) = \cos x - 2x + 2$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.3	−0.1945	2.3553	−0.0826	0.3826
2	0.3826	−0.0078	2.1625	−0.0036	0.3862

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	2.0	−0.0907	−2.4161	0.0375	1.9625
2	1.9625	−0.0021	−2.3068	0.0009	1.9616

The two points of intersection are $x \approx 0.3862$ and $x \approx 1.9616$.

95. $y = f(x) = 4x^3, f'(x) = 12x^2, x = 2,$

$\Delta x = dx = 0.1$

$\Delta y = f(x + \Delta x) - f(x)$

$= f(2.1) - f(2)$

$= 37.044 - 32$

$= 5.044$

$dy = f'(x)\,dx$

$= (12x^2)\,dx$

$= 12(2)^2(0.1)$

$= 4.8$

96. $y = f(x) = x^2 - 5x, f'(x) = 2x - 5, x = -3,$

$\Delta x = dx = 0.01$

$\Delta y = f(x + \Delta x) - f(x)$

$= f(-3 + 0.01) - f(-3)$

$= 23.8901 - 24$

$= -0.1099$

$dy = f'(x)\,dx$

$= (2x - 5)\,dx$

$= [2(-3) - 5](0.1)$

$= -0.11$

97. $y = x(1 - \cos x) = x - x\cos x$

$$\frac{dy}{dx} = 1 + x\sin x - \cos x$$

$$dy = (1 + x\sin x - \cos x)dx$$

98. $y = \sqrt{36 - x^2}$

$$\frac{dy}{dx} = \frac{1}{2}(36 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{36 - x^2}}$$

$$dy = \frac{-x}{\sqrt{36 - x^2}}dx$$

99. Let $f(x) = \sqrt{x}, x = 64, dx = -0.1.$

$$f(x + \Delta x) = f(x) + f'(x)\,dx = \sqrt{x} + \frac{1}{2\sqrt{x}}\,dx$$

$$f(x + \Delta x) = \sqrt{63.9} \approx \sqrt{64} + \frac{1}{2\sqrt{64}}(-0.1) = 8 - \frac{1}{16}(0.1) = 7.99375$$

Using a calculator: $\sqrt{63.9} \approx 7.9937476$

100. Let $f(x) = x^4, x = 2, dx = 0.02.$

$$f(x + \Delta x) = f(x) + f'(x)\,dx = x^4 + 4x^3\,dx$$

$$f(x + \Delta x) = (2.02)^4 \approx 2^4 + 4(2^3)(0.02) = 16.64$$

Using a calculator: $(2.02)^4 \approx 16.64966$

101. $r = 9$ cm, $dr = \Delta r = \pm 0.025$

(a) $V = \frac{4}{3}\pi r^3$

$dV = 4\pi r^2\,dr$

$\Delta V \approx dV = 4\pi(9)^2(\pm 0.025) = \pm 8.1\pi \text{ cm}^3$

(b) $S = 4\pi r^2$

$dS = 8\pi r\,dr$

$\Delta S \approx dS = 8\pi(9)(\pm 0.025) = \pm 1.8\pi \text{ cm}^2$

(c) Percent error of volume:

$$\frac{dV}{V} = \frac{8.1\pi}{\frac{4}{3}\pi(9)^3} = 0.0083, \text{ or } 0.83\%$$

Percent error of surface area:

$$\frac{dS}{S} = \frac{1.8\pi}{4\pi(9)^2} = 0.0056, \text{ or } 0.56\%$$

Problem Solving for Chapter 3

1. $p(x) = x^4 + ax^2 + 1$

(a) $p'(x) = 4x^3 + 2ax = 2x(2x^2 + a)$

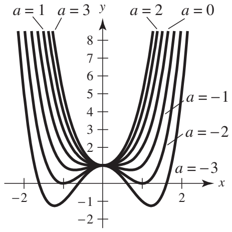
$p''(x) = 12x^2 + 2a$

For $a \geq 0$, there is one relative minimum at $(0, 1)$.

(b) For $a < 0$, there is a relative maximum at $(0, 1)$.

(c) For $a < 0$, there are two relative minima at $x = \pm\sqrt{-\frac{a}{2}}$.

(d) If $a < 0$, there are three critical points; if $a > 0$, there is only one critical point.



2. (a) For $a = -3, -2, -1, 0,$ p has a relative maximum at $(0, 0)$.

For $a = 1, 2, 3,$ p has a relative maximum at $(0, 0)$ and 2 relative minima.

(b) $p'(x) = 4ax^3 - 12x = 4x(ax^2 - 3) = 0 \Rightarrow x = 0, \pm\sqrt{\frac{3}{a}}$

$p''(x) = 12ax^2 - 12 = 12(ax^2 - 1)$

For $x = 0$, $p''(0) = -12 < 0 \Rightarrow$ p has a relative maximum at $(0, 0)$.

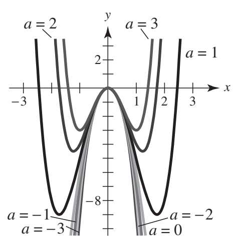
(c) If $a > 0$, $x = \pm\sqrt{\frac{3}{a}}$ are the remaining critical numbers.

$p''\left(\pm\sqrt{\frac{3}{a}}\right) = 12a\left(\frac{3}{a}\right) - 12 = 24 > 0 \Rightarrow$ p has relative minima for $a > 0$.

(d) $(0, 0)$ lies on $y = -3x^2$.

Let $x = \pm\sqrt{\frac{3}{a}}$. Then $p(x) = a\left(\frac{3}{a}\right)^2 - 6\left(\frac{3}{a}\right) = \frac{9}{a} - \frac{18}{a} = -\frac{9}{a}$.

So, $y = -\frac{9}{a} = -3\left(\pm\sqrt{\frac{3}{a}}\right)^2 = -3x^2$ is satisfied by all the relative extrema of p.



3. $f(x) = \frac{c}{x} + x^2$

$f'(x) = -\frac{c}{x^2} + 2x = 0 \Rightarrow \frac{c}{x^2} = 2x \Rightarrow x^3 = \frac{c}{2} \Rightarrow x = \sqrt[3]{\frac{c}{2}}$

$f''(x) = \frac{2c}{x^3} + 2$

If $c = 0$, $f(x) = x^2$ has a relative minimum, but no relative maximum.

If $c > 0$, $x = \sqrt[3]{\frac{c}{2}}$ is a relative minimum, because $f''\left(\sqrt[3]{\frac{c}{2}}\right) > 0$.

If $c < 0$, $x = \sqrt[3]{\frac{c}{2}}$ is a relative minimum, too.

Answer: All c.

4. (a) $f(x) = ax^2 + bx + c, a \neq 0$

$f'(x) = 2ax$

$f''(x) = 2a \neq 0$

No point of inflection

(b) $f(x) = ax^3 + bx^2 + cx + d, a \neq 0$

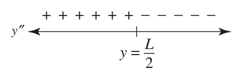
$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b = 0 \Rightarrow x = \dfrac{-b}{3a}$

One point of inflection

(c) $y' = ky\left(1 - \dfrac{y}{L}\right) = ky - \dfrac{k}{L}y^2$

$y'' = ky' - \dfrac{2k}{L}yy' = ky'\left(1 - \dfrac{2}{L}y\right)$



If $y = \dfrac{L}{2}$, then $y'' = 0$, and this is a point of inflection because of the analysis above.

5. Set $\dfrac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2} = k.$

Define $F(x) = f(x) - f(a) - f'(a)(x - a) - k(x - a)^2.$

$F(a) = 0, F(b) = f(b) - f(a) - f'(a)(b - a) - k(b - a)^2 = 0$

F is continuous on $[a, b]$ and differentiable on (a, b).

There exists $c_1, a < c_1 < b$, satisfying $F'(c_1) = 0.$

$F'(x) = f'(x) - f'(a) - 2k(x - a)$ satisfies the hypothesis of Rolle's Theorem on $[a, c_1]$:

$F'(a) = 0, F'(c_1) = 0.$

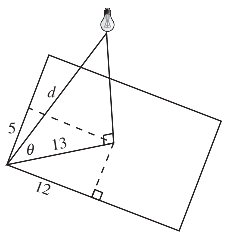
There exists $c_2, a < c_2 < c_1$ satisfying $F''(c_2) = 0.$

Finally, $F''(x) = f''(x) - 2k$ and $F''(c_2) = 0$ implies that

$k = \dfrac{f''(c_2)}{2}.$

So, $k = \dfrac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2} = \dfrac{f''(c_2)}{2} \Rightarrow f(b) = f(a) + f'(a)(b - a) + \dfrac{1}{2}f''(c_2)(b - a)^2.$

6.



$d = \sqrt{13^2 + x^2}$, $\sin\theta = \dfrac{x}{d}$.

Let A be the amount of illumination at one of the corners, as indicated in the figure. Then

$$A = \frac{kI}{\left(13^2 + x^2\right)}\sin\theta = \frac{kIx}{\left(13^2 + x^2\right)^{3/2}}$$

$$A'(x) = kI\frac{\left(x^2 + 169\right)^{3/2}(1) - x\left(\frac{3}{2}\right)\left(x^2 + 169\right)^{1/2}(2x)}{\left(169 + x^2\right)^3} = 0$$

$$\Rightarrow \left(x^2 + 169\right)^{3/2} = 3x^2\left(x^2 + 169\right)^{1/2}$$

$$x^2 + 169 = 3x^2$$

$$2x^2 = 169$$

$$x = \frac{13}{\sqrt{2}} \approx 9.19 \text{ ft}$$

By the First Derivative Test, this is a maximum.

7. Distance $= \sqrt{4^2 + x^2} + \sqrt{(4 - x)^2 + 4^2} = f(x)$

$$f'(x) = \frac{x}{\sqrt{4^2 + x^2}} - \frac{4 - x}{\sqrt{(4 - x)^2 + 4^2}} = 0$$

$$x\sqrt{(4 - x)^2 + 4^2} = -(x - 4)\sqrt{4^2 + x^2}$$

$$x^2\left[16 - 8x + x^2 + 16\right] = \left(x^2 - 8x + 16\right)\left(16 + x^2\right)$$

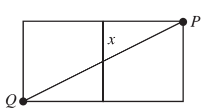
$$32x^2 - 8x^3 + x^4 = x^4 - 8x^3 + 32x^2 - 128x + 256$$

$$128x = 256$$

$$x = 2$$

The bug should head towards the midpoint of the opposite side.

Without Calculus: Imagine opening up the cube:



The shortest distance is the line PQ, passing through the midpoint.

8. Let T be the intersection of PQ and RS. Let MN be the perpendicular to SQ and PR passing through T.

Let $TM = x$ and $TN = b - x$.

$$\frac{SN}{b-x} = \frac{MR}{x} \Rightarrow SN = \frac{b-x}{x}MR$$

$$\frac{NQ}{b-x} = \frac{PM}{x} \Rightarrow NQ = \frac{b-x}{x}PM$$

$$SQ = \frac{b-x}{x}(MR + PM) = \frac{b-x}{x}d$$

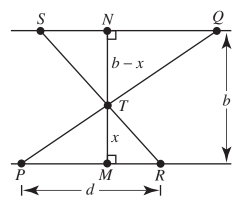
$$A(x) = \text{Area} = \frac{1}{2}dx + \frac{1}{2}\left(\frac{b-x}{x}d\right)(b-x) = \frac{1}{2}d\left[x + \frac{(b-x)^2}{x}\right] = \frac{1}{2}d\left[\frac{2x^2 - 2bx + b^2}{x}\right]$$

$$A'(x) = \frac{1}{2}d\left[\frac{x(4x-2b) - (2x^2 - 2bx + b^2)}{x^2}\right]$$

$$A'(x) = 0 \Rightarrow 4x^2 - 2xb = 2x^2 - 2bx + b^2$$

$$2x^2 = b^2$$

$$x = \frac{b}{\sqrt{2}}$$



So, you have $SQ = \dfrac{b-x}{x}d = \dfrac{b - (b/\sqrt{2})}{b/\sqrt{2}}d = (\sqrt{2} - 1)d.$

Using the Second Derivative Test, this is a minimum. There is no maximum.

9. f continuous at $x = 0$: $1 = b$

f continuous at $x = 1$: $a + 1 = 5 + c$

f differentiable at $x = 1$: $a = 2 + 4 = 6$. So, $c = 2$.

$$f(x) = \begin{cases} 1, & x = 0 \\ 6x + 1, & 0 < x \le 1 \\ x^2 + 4x + 2, & 1 < x \le 3 \end{cases}$$

$$= \begin{cases} 6x + 1, & 0 \le x \le 1 \\ x^2 + 4x + 2, & 1 < x \le 3 \end{cases}$$

10. f continuous at $x = -1$: $a = 2$

f continuous at $x = 0$: $2 = c$

f continuous at $x = 1$: $b + 2 = d + 4 \Rightarrow b = d + 2$

f differentiable at $x = 0$: $0 = 0$

f differentiable at $x = 1$: $2b = d$

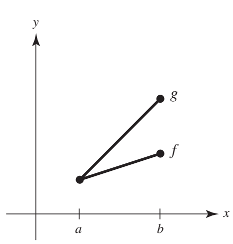
So, $b = -2$ and $d = -4$.

11. Let $h(x) = g(x) - f(x)$, which is continuous on $[a, b]$ and differentiable on (a, b). $h(a) = 0$ and $h(b) = g(b) - f(b)$.

By the Mean Value Theorem, there exists c in (a, b) such that

$$h'(c) = \frac{h(b) - h(a)}{b - a} = \frac{g(b) - f(b)}{b - a}.$$

Because $h'(c) = g'(c) - f'(c) > 0$ and $b - a > 0$, $g(b) - f(b) > 0 \Rightarrow g(b) > f(b)$.



12. (a) Let $M > 0$ be given. Take $N = \sqrt{M}$. Then whenever $x > N = \sqrt{M}$, you have $f(x) = x^2 > M$.

(b) Let $\varepsilon > 0$ be given. Let $M = \sqrt{\frac{1}{\varepsilon}}$. Then whenever $x > M = \sqrt{\frac{1}{\varepsilon}}$, you have $x^2 > \frac{1}{\varepsilon} \Rightarrow \frac{1}{x^2} < \varepsilon \Rightarrow \left|\frac{1}{x^2} - 0\right| < \varepsilon$.

(c) Let $\varepsilon > 0$ be given. There exists $N > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > N$.

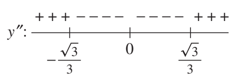
Let $\delta = \frac{1}{N}$. Let $x = \frac{1}{y}$.

If $0 < y < \delta = \frac{1}{N}$, then $\frac{1}{x} < \frac{1}{N} \Rightarrow x > N$ and $|f(x) - L| = \left|f\left(\frac{1}{y}\right) - L\right| < \varepsilon$.

13. $y = (1 + x^2)^{-1}$

$$y' = \frac{-2x}{(1 + x^2)^2}$$

$$y'' = \frac{2(3x^2 - 1)}{(x^2 + 1)^3} = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$$



The tangent line has greatest slope at $\left(-\frac{\sqrt{3}}{3}, \frac{3}{4}\right)$ and least slope at $\left(\frac{\sqrt{3}}{3}, \frac{3}{4}\right)$.

14. (a) $s = \dfrac{v\frac{\text{km}}{\text{h}}\left(1000\frac{\text{m}}{\text{km}}\right)}{\left(3600\frac{\text{sec}}{\text{h}}\right)} = \frac{5}{18}v$

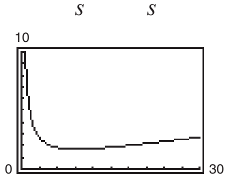
v	20	40	60	80	100
s	5.56	11.11	16.67	22.22	27.78
d	5.1	13.7	27.2	44.2	66.4

$d(s) = 0.071s^2 + 0.389s + 0.727$

(b) The distance between the back of the first vehicle and the front of the second vehicle is $d(s)$, the safe stopping distance. The first vehicle passes the given point in 5.5/s seconds, and the second vehicle takes $d(s)/s$ more seconds. So,

$T = \frac{d(s)}{s} + \frac{5.5}{s}$.

(c)



$T = \frac{1}{s}(0.071s^2 + 0.389s + 0.727) + \frac{5.5}{s}$

The minimum is attained when $s \approx 9.365$ m/sec.

(d) $T(s) = 0.071s + 0.389 + \frac{6.227}{s}$

$T'(s) = 0.071 - \frac{6.227}{s^2} \Rightarrow s^2 = \frac{6.227}{0.071} \Rightarrow s \approx 9.365$ m/sec

$T(9.365) \approx 1.719$ seconds

$9.365 \text{ m/sec} \cdot \frac{3600}{1000} = 33.7$ km/h

(e) $d(9.365) = 10.597$ m

15. Assume $y_1 < d < y_2$. Let $g(x) = f(x) - d(x - a)$. g is continuous on $[a, b]$ and therefore has a minimum $(c, g(c))$ on $[a, b]$. The point c cannot be an endpoint of $[a, b]$ because

$$g'(a) = f'(a) - d = y_1 - d < 0$$
$$g'(b) = f'(b) - d = y_2 - d > 0.$$

So, $a < c < b$ and $g'(c) = 0 \Rightarrow f'(c) = d$.

16. The line has equation $\frac{x}{3} + \frac{y}{4} = 1$ or $y = -\frac{4}{3}x + 4$.

Rectangle:

$$\text{Area} = A = xy = x\left(-\frac{4}{3}x + 4\right) = -\frac{4}{3}x^2 + 4x.$$

$$A'(x) = -\frac{8}{3}x + 4 = 0 \Rightarrow \frac{8}{3}x = 4 \Rightarrow x = \frac{3}{2}$$

Dimensions: $\frac{3}{2} \times 2$ Calculus was helpful.

Circle: The distance from the center (r, r) to the line $\frac{x}{3} + \frac{y}{4} - 1 = 0$ must be r:

$$r = \frac{\left|\frac{r}{3} + \frac{r}{4} - 1\right|}{\sqrt{\frac{1}{9} + \frac{1}{16}}} = \frac{12}{5}\left|\frac{7r - 12}{12}\right| = \frac{|7r - 12|}{5}$$

$$5r = |7r - 12| \Rightarrow r = 1 \text{ or } r = 6.$$

Clearly, $r = 1$.

Semicircle: The center lies on the line $\frac{x}{3} + \frac{y}{4} = 1$ and satisfies $x = y = r$.

So $\frac{r}{3} + \frac{r}{4} = 1 \Rightarrow \frac{7}{12}r = 1 \Rightarrow r = \frac{12}{7}$. No calculus necessary.

17.

$$p(x) = ax^3 + bx^2 + cx + d$$
$$p'(x) = 3ax^2 + 2bx + c$$
$$p''(x) = 6ax + 2b$$
$$6ax + 2b = 0$$
$$x = -\frac{b}{3a}$$

The sign of $p''(x)$ changes at $x = -b/3a$. Therefore, $(-b/3a, p(-b/3a))$ is a point of inflection.

$$p\left(-\frac{b}{3a}\right) = a\left(-\frac{b^3}{27a^3}\right) + b\left(\frac{b^2}{9a^2}\right) + c\left(-\frac{b}{3a}\right) + d = \frac{2b^3}{27a^2} - \frac{bc}{3a} + d$$

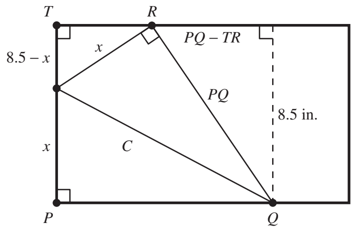
When $p(x) = x^3 - 3x^2 + 2$, $a = 1$, $b = -3$, $c = 0$, and $d = 2$.

$$x_0 = \frac{-(-3)}{3(1)} = 1$$

$$y_0 = \frac{2(-3)^3}{27(1)^2} - \frac{(-3)(0)}{3(1)} + 2 = -2 - 0 + 2 = 0$$

The point of inflection of $p(x) = x^3 - 3x^2 + 2$ is $(x_0, y_0) = (1, 0)$.

18. (a)



$$x^2 + PQ^2 = C^2 \Rightarrow PQ^2 = C^2 - x^2$$

$$TR^2 + (8.5 - x)^2 = x^2 \Rightarrow TR^2 = 17x - 8.5^2$$

$$(PQ - TR)^2 + 8.5^2 = PQ^2 \Rightarrow 2(PQ)(TR) = TR^2 + 8.5^2$$

So, $2(PQ)(TR) = 17x - 8.5^2 + 8.5^2$.

$$8.5x = (PQ)(TR) = \sqrt{C^2 - x^2}\sqrt{17x - 8.5^2}$$

$$\frac{(8.5x)^2}{17x - 8.5^2} = C^2 - x^2$$

$$C^2 = x^2 + \frac{(8.5x)^2}{17x - 8.5^2} = \frac{17x^3}{17x - 8.5^2}$$

$$C^2 = \frac{2x^3}{2x - 8.5}$$

(b) Domain: $4.25 < x < 8.5$

(c) To minimize C, minimize $f(x) = C^2$:

$$f'(x) = \frac{(2x - 8.5)(6x^2) - 2x^3(2)}{(2x - 8.5)^2} = \frac{8x^3 - 51x^2}{(2x - 8.5)^2} = 0$$

$$x = \frac{51}{8} = 6.375$$

By the First Derivative Test, $x = 6.375$ is a minimum.

(d) For $x = 6.375$, $C \approx 11.0418$ in.

19. (a) $f(x) = \dfrac{x}{x + 1}$, $f'(x) = \dfrac{1}{(x + 1)^2}$, $f''(x) = \dfrac{-2}{(x + 1)^3}$

$P(0) = f(0)$: $c_0 = 0$

$P'(0) = f'(0)$: $c_1 = 1$

$P''(0) = f''(0)$: $2c_2 = -2 \Rightarrow c_2 = -1$

$P(x) = x - x^2$

(b)