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WORLD'S #1 ACADEMIC OUTLINE

QUICK STUDY ACADEMIC

CALCULUS METHODS

BASIC CALCULUS FOR NON-SCIENCE MAJORS

LIMITS & CONTINUITY

- $\lim_{x \to a} f(x) = L$ if $f(x)$ is close to L for all x sufficiently close (but not equal) to a.
- $f(x)$ is continuous at $x = a$ if:

1. $f(a)$ is defined, 2. $\lim_{x \to a} f(x) = L$ exists, and 3. $L = f(a)$

INTEGRALS

THE DEFINITE INTEGRAL

- LET $f(x)$ BE CONTINUOUS ON $[a, b]$

1. **Riemann Sum Definition of Definite Integral**

a. Divide $[a,b]$ into n equal subintervals of length $h = \frac{b-a}{n}$

b. Let $x_0 = a, x_1, x_2, \ldots, x_n = b$ denote the endpoints of the subintervals

i. They are found by: $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, $x_3 = a + 3h, \ldots, x_n = a + nh = b$

c. Let $m_1, m_2, \ldots m_n$ denote the midpoints of the subintervals

i. They are found by: $m_1 = 0.5(x_0 + x_1)$, $m_2 = 0.5(x_1 + x_2)$, $m_3 = 0.5(x_2 + x_3), \ldots, m_n = 0.5(x_n - 1 + x_n)$

$\int_a^b f(x)dx = h[f(m_1) + f(m_2) + \ldots + f(m_n)]$

2. **MidpointRule:** $\int_a^b f(x)dx = \lim_{n \to \infty} h[f(x_1) + f(x_2) + \ldots + f(x_n)]$

3. **Trapezoid Rule:**

$\int_a^b f(x)dx \approx \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)]$

4. **Simpson's Rule:** $\int_a^b f(x)dx \approx \frac{h}{6}[f(x_0) + 4f(m_1) + 2f(x_1) + 4f(m_2) + 2f(x_2) + \ldots + 2f(x_{n-1}) + 4f(m_n) + f(x_n)]$

THE INDEFINITE INTEGRAL

- $F(x)$ IS CALLED AN ANTIDERIVATIVE OF $f(x)$, IF $F'(x) = f(x)$

1. The most general antiderivative is denoted $\int f(x)dx$
2. $\int f(x)dx$ is also called the Indefinite Integral of $f(x)$

- **fundamental theorem of calculus**

1. If $F'(x) = f(x)$ and $f(x)$ is continuous on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$

INTEGRATION FORMULAS

1. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$
2. $\int kf(x)dx = k\int f(x)dx$ if k is a constant
3. $\int u^n du = \frac{u^{n+1}}{n+1} + C$ 4. $\int \frac{1}{u} du = \ln|u| + C$ 5. $\int e^u du = e^u + C$
6. If $y = f(x) \geq 0$ on $[a, b]$, $\int_a^b f(x)dx$ gives the area under the curve.
7. If $f(x) \geq g(x)$ on $[a,b]$, $\int_a^b [f(x) - g(x)]\,dx$ gives the area between the 2 curves $y = f(x)$ and $y = g(x)$
8. Average value of $f(x)$ on $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$
9. Volume of the solid of revolution obtained by revolving about the x-axis the region under the curve $y = f(x)$ from $x = a$ to $x = b$ is $\int_a^b \pi[f(x)]^2 dx$

INTEGRATION BY PARTS

1. Factor the integrand into 2 parts: u and dv
2. Find du and $v = \int dv$ 3. Find $\int v\,du$
4. Set $\int u\,dv = uv - \int v\,du$

INTEGRATION BY SUBSTITUTION

- TO SOLVE $\int f(g(x))g'(x)dx$

1. Set $u = g(x)$, where $g(x)$ is chosen so as to simplify the integrand
2. Substitute $u = g(x)$ and $du = g'(x)dx$ into the integrand.
   a. This step usually requires multiplying or dividing by a constant.
3. Solve $\int f(u)du = F(u) + C$
4. Substitute $u = g(x)$ to get the answer: $F(g(x)) + C$

IMPROPER INTEGRALS

- INFINITE LIMITS OF INTEGRATION

1. $\int_a^{\infty} f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx$ 2. $\int_{-\infty}^b f(x)dx = \lim_{a \to -\infty} \int_a^b f(x)dx$

- IMPROPER AT THE LEFT OR RIGHT ENDPOINTS

1. If $f(x)$ is discontinuous at $x = b$, $\int_a^b f(x)dx = \lim_{h \to b^-} \int_a^h f(x)dx$
2. If $f(x)$ is discontinuous at $x = a$, $\int_a^b f(x)dx = \lim_{h \to \infty} \int_h^b f(x)dx$

DERIVATIVES & THEIR APPLICATIONS

DERIVATIVE BASICS

- DEFINITION OF DERIVATIVE

1. $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$
2. If $y = f(x)$, the derivative $f'(x)$ is also denoted $\frac{dy}{dx}$

- FORMULAS:

1. **Power Rule:** $\frac{d}{dx}(x^n) = nx^{n-1}$
2. $\frac{d}{dx}(e^{kx}) = ke^{kx}$ 3. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
4. **General Power Rule:** $\frac{d}{dx}([f(x)]^n) = n[f(x)]^{n-1} f'(x)$
5. $\frac{d}{dx}[e^{f(x)}] = e^{f(x)} f'(x)$ 6. $\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$
7. **Sum or Difference Rule:** $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
8. **Constant Multiple Rule:** $\frac{d}{dx}[kf(x)] = kf'(x)$
9. **Product Rule:** $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
10. **Quotient Rule:** $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
11. **ChainRule:** $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$, or $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
12. **Derivative of an inverse function:** $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$

IMPLICIT DIFFERENTIATION

- GIVEN AN EQUATION INVOLVING FUNCTION OF x AND y, TO FIND : $\frac{dy}{dx}$

1. Differentiate both sides of the equation with respect to x, treating y as a function of x and applying the chain rule to each term involving y (i.e. $\frac{d}{dx}[f(y)] = f'(y)\frac{dy}{dx}$).
2. Move all terms with $\frac{dy}{dx}$ to left side and all other terms to the right.
3. Solve for $\frac{dy}{dx}$.

CURVE SKETCHING

- STEPS TO FOLLOW IN SKETCHING THE CURVE $y = f(x)$:

1. Determine the domain of $f(x)$.
2. Analyze all points where $f(x)$ is discontinuous. Sketch the graph near all such points.
3. Test for vertical, horizontal and oblique asymptotes.
   a. $f(x)$ has a vertical asymptote at $x = a$ if:
   $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$
   b. $f = (x)$ has a horizontal asymptote $y = b$ if:
   $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$
   c. Sketch any asymptotes.
4. Find $f'(x)$ and $f''(x)$.
5. Find all critical points. These are points $x = a$ where $f'(a)$ does not exist or $f'a = 0$. Repeat steps **5.a.&b.** for each critical point $x = a$:
   a. If f(x) is continuous at $x = a$,
   i. $f(x)$ has a relative maximum at $x = a$ if:
   (a) $f'(a) = 0$ and $f''(a) < 0$, or
   (b) $f'(x) > 0$ to the left of a and $f'(x) < 0$ to the right of a.
   ii. $f(x)$ has a relative minimum at $x = a$ if:
   (a) $f'(a) = 0$ and $f''(a) > 0$, or
   (b) $f'(x) < 0$ to the left of a and $f'(x) > 0$ to the right of a.
   b. Sketch $f(x)$ near $(a, f(a))$.
6. Find all possible inflection points. These are points $x = a$ where $f''(x)$ does not exist or $f''(x) = 0$.
   Repeat steps 6. a. & b. for each such $x = a$:
   a. $f(x)$ has an inflection point at $x = a$ if $f(x)$ is continuous at $x = a$ and
   i. $f''(x) < 0$ to the left of a and $f''(x) > 0$ to the right of a, or
   ii. $f''(x) > 0$ to the left of a and $f''(x) < 0$ to the right of a.
   b. Sketch $f(x)$ near $(a, f(a))$.
7. If possible, plot the x__ and y__ intercepts.
8. Finish the sketch.

OPTIMIZATION PROBLEMS

- TO OPTIMIZE SOME QUANTITY SUBJECT TO SOME CONSTRAINT:

1. Identify and label quantity to be maximized or minimized.
2. Identify and label all other quantities.
3. Write quantity to be optimized as a function of the other variables. This is called the objective function (or objective equation).
4. If the objective function is a function of more than 1 variable, find a constraint equation relating the other variables.
5. Use the constraint equation to write the objective function as a function of only 1 variable.
6. Using the curve sketching techniques, locate the maximum or minimum of the objective function.

APPROXIMATIONS AND DIFFERENTIALS

- LET $y = f(x)$ AND ASSUME $f'(a)$ EXISTS

1. The Equation of the Tangent Line to $y = f(x)$ at the point $(a, f(a))$ is $y - f(a) = f'(a)(x-a)$
2. The differential of y is $dy = f'(x)dx$
3. Linear Approximation, or Approximation by Differentials.
   Set $dx = \Delta x = x - a$, $\Delta y = f(x) - f(a)$
   The equation of the tangent line becomes:
   $\Delta y = f'(a)\Delta x = f'(a)dx$. If Δx is small, then $\Delta y \approx dy$.
   That is, $f(x) \approx f(a) + f'(a)(x-a)$.
4. The n-th Taylor polynomial of $f(x)$ centered at $x = a$ is
   $p_n(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \ldots + \frac{f^{(n)}(a)(x-a)^n}{n!}$

MOTION

- FORMULA

If $s = s(t)$ represents the position of an object at time t relative to some fixed point, then $v(t) = s'(t)$ = velocity at time t and $a(t) = v'(t) = s''(t)$ = acceleration at time t.

APPLICATIONS TO BUSINESS AND ECONOMICS

COST, REVENUE AND PROFIT

1. $C(x)$ = cost of producing x units of a product
2. $p = p(x)$ = price per unit; ($p = p(x)$ is also called the demand equation)
3. $R(x) = xp$ = revenue made by producing x units
4. $P(x) = R(x) - C(x)$ = profit made by producing x units
5. $C'(x)$ = marginal cost
6. $R'(x)$ = marginal revenue
7. $P'(x)$ = marginal profit

COMPOUNDING INTEREST

- STARTING WITH A PRINCIPAL P_o

1. If the interest is compounded for t years with m periods per year at the interest rate of r per annum, the compounded amount is:

$$P = P_o(1 + \frac{r}{m})^{mt}$$

2. If interest is continuously compounded, $m \to \infty$ and the formula becomes:

$$P = \lim_{m \to \infty} P_o (1 + \frac{r}{m})^{mt} = P_o e^{rt}$$

3. The formula $P = P_o e^{rt}$ gives the value at the end of t years, assuming continuously compounded interest. P_o is called the present value of P to be received in t years and is given by the formula

$$P_o = Pe^{-rt}$$

CONTINUED ON REVERSE SIDE

APPLICATIONS CONTINUED

QuickStudy

ELASTICITY OF DEMAND

• SOLVING FOR x IN THE DEMAND EQUATION $p=p(x)$ GIVES $x=f(p)$

1. Demand function which gives the quantity demanded x as a function of the price p.
2. The elasticity of demand is $E(p) = \frac{-pf'(p)}{f(p)}$

- **DEMAND IS ELASTIC AT $p=p0$ IF $E(p0)>1$**
  In this case an increase in price corresponds to a decrease in revenue.
- **DEMAND IS INELASTIC AT $p=p0$ IF $E(p0)<1$**
  In this case an increase in price corresponds to an increase in revenue.

CONSUMERS' SURPLUS

• IF A COMMODITY HAS DEMAND EQUATION $p=p(x)$
Consumers' Surplus is given by $\int_0^a [p(x) - p(a)]dx$ where a is the quantity demanded and $p(a)$ is the corresponding price.

EXPONENTIAL MODELS

EXPONENTIAL GROWTH AND DECAY

• EXPONENTIAL GROWTH: $y=P_0e^{kt}$
1. Satisfies the differential equation $y'=ky$
2. P_0 is the initial size, $k>0$ is called the growth constant
3. The time it takes for the size to double is given by: $\frac{\ln 2}{k}$

• EXPONENTIAL DECAY: $y = P_0e^{-\lambda t}$
1. Satisfies the differential equation $y' = -\lambda y$
2. P_0 is the initial size, $\lambda > 0$ is called the decay constant
3. The half life $t_{1/2}$ is the time it takes for y to become $P_0/2$

It is found by $t_{1/2} = \frac{\ln .5}{-\lambda} = \frac{\ln 2}{\lambda}$

OTHER GROWTH CURVES

• THE LEARNING CURVE: $y=M(1-e^{-kt})$
Satisfies the differential equation $y' = k(M-y)$, $y(0)=0$ where M and k are positive constants.

• THE LOGISTIC GROWTH CURVE: $y = \frac{M}{1 + Be^{-Mkt}}$
Satisfies the differential equation $y' = ky(M-y)$ where B, M and k are positive constants.

PROBABILITY

DEFINITIONS

• PROBABILITY DENSITY FUNCTION
For the continuous random variable X is a function $p(x)$ satisfying: and $p(x) \geq 0$ if $A \leq x \leq B$ and $\int_A^B p(x)dx=1$, where we assume the values of X lie in $[A, B]$.

• THE PROBABILITY THAT
$$a \leq X \leq B \text{ is } P[a \leq X \leq b] = \int_a^b p(x)dx$$

• EXPECTED VALUE, OR MEAN OF X
Given by $m = E(X) = \int_A^B xp(x)dx$

• VARIANCE OF X: Given by
$$\sigma^2 = var(X) = \int_A^B (x-\mu)^2 p(x)dx = \int_A^B x^2 p(x)dx - \mu^2$$

COMMON PROBABILITY DENSITY FUNCTIONS

• UNIFORM DISTRIBUTION FUNCTION:
$$p(x)=\frac{1}{B-A}, \mu = E(X)= \frac{B+A}{2}, var(X)= \frac{(B-A)^2}{12}$$

• EXPONENTIAL DENSITY FUNCTION:
$p(x)=\lambda e^{-\lambda x}$. *In this case,* $A=0$, $B = \infty$, $\mu = E(X) = 1/\lambda$, $var(X) = 1/\lambda^2$

• NORMAL DENSITY FUNCTION:
with $E(X) = \mu$ and $var(X) = \sigma 2$ is:
$$p(x) \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi\sigma}} exp\left[-\frac{(x-\mu)^2}{2s^2}\right]$$

NOTE TO STUDENT: Due to its condensed format, use this *QUICK STUDY®* chart as a Calculus guide, but not as a replacement for assigned class work.

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CALCULUS OF FUNCTIONS OF 2 VARIABLES

PARTIAL DERIVATIVES

• WHERE $f(x,y)$ IS A FUNCTION OF 2 VARIABLES x AND y

1. $\frac{\partial f}{\partial x}$ is the derivative of $f(x,y)$ with respect to x, treating y as a constant.
2. $\frac{\partial f}{\partial y}$ is the derivative of $f(x,y)$ with respect to y, treating x as a constant.
3. $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial y} \cdot \frac{\partial f}{\partial x}$ is the second partial derivative of $f(x,y)$ with respect to x twice, keeping y constant each time
4. $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}$ is the second partial derivative of $f(x,y)$ first with respect to x keeping y constant then with respect to y keeping x constant.
5. Other notation for partial derivatives:
$$f_x(x,y)= \frac{\partial f}{\partial x},\ f_{xx}(x,y)= \frac{\partial^2 f}{\partial x^2},\ f_{xy}(x,y)= \frac{\partial^2 f}{\partial y \partial x}$$

DIFFERENTIALS

• If $f = f(x,y)$

1. $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = f_x(x,y)dx + f_y(x,y)dy$
2. Setting $dx = \Delta x = x - a$, $dy = \Delta y = y-b$ and $\Delta f = f(x,y) - f(a,b)$, if Δx and Δy are both small, then $\Delta f \approx df$. That is: $f(x,y) \approx f(a,b) + f_x(a,b)\Delta x + f_y(a,b)\Delta y$.

RELATIVE EXTREMA TEST

• TO LOCATE RELATIVE MAXIMA, RELATIVE MINIMA AND SADDLE POINTS ON THE GRAPH OF $z = f(x,y)$.

1. Solve simultaneously: $\frac{\partial f}{\partial x} = 0$ *and* $\frac{\partial f}{\partial y} = 0$. For each ordered pair (a,b) such that $\frac{\partial f}{\partial x}(a,b)=0$ *and* $\frac{\partial f}{\partial y}(a,b)=0$, apply the following test.
2. Set $A= \frac{\partial^2 f}{\partial x^2}(a,b)$, $B = \frac{\partial^2 f}{\partial y^2}(a,b)$, $C = \frac{\partial^2 f}{\partial x \partial y}(a,b)$ *and* $D = AB - C^2$
   a. If $D>0$ and $A>0$, then $f(x,y)$ has a relative minimum at (a,b).
   b. If $D>0$ and $A<0$, then $f(x,y)$ has a relative maximum at (a,b).
   c. If $D<0$, then $f(x,y)$ has a saddle point at (a,b).
   d. If $D=0$, then the test fails. $f(x,y)$ may or may not have an extremum or saddle point at (a,b).

THE METHOD OF LAGRANGE MULTIPLIERS

• SOLVES CONSTRAINED OPTIMIZATION PROBLEMS. TO MAXIMIZE OR MINIMIZE $f(x,y)$ SUBJECT TO THE CONSTRAINT $g(x,y)=0$,

1. Define the new function $F(x,y,\lambda) = f(x,y) + \lambda\, g(x,y)$.
2. Solve the system of 3 equations:

a. $\frac{\partial F}{\partial x} = 0$ *b.* $\frac{\partial F}{\partial y} = 0$, *and c.* $\frac{\partial F}{\partial \lambda} = 0$ simultaneously.

This is usually accomplished in 4 steps:

Step 1: Solve (a) and (b) for λ and equate the solutions.
Step 2: Solve the resulting equation for one of the variables, x or y.
Step 3: Substitute this expression for x or y into equation (c) and solve the resulting equation of one variable for the other variable.
Step 4: Substitute the value found in Step 3 into the equation found in Step 2. Use 1 of the equations from Step 1 to find λ. This gives the value of x and y.

DOUBLE INTEGRALS

1. If R is the region in the plane bounded by the 2 curves $y=g(x)$, $y=h(x)$ and the 2 vertical lines $x=a$, $x=b$, then the double integral $\iint_R f(x,y)\, dxdy$ is equal to the iterated integral $\int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)dy \right) dx$.
2. To evaluate the iterated integral
$$I = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)dy \right) dx.$$
   a. find an antiderivative $F(x,y)$ for $f(x,y)$ with respect to y keeping x constant. That is: $\frac{\partial F}{\partial y} = f(x,y)$
   b. Set: $I = \int_a^b [F(x,h(x)) - F(x,g(x))]\, dx$
   c. Solve this integral. The integrand is a function of one variable.

DIFFERENTIAL EQUATIONS

- **A DIFFERENTIAL EQUATION IS: any equation involving a derivative.**
  For example, it could be an equation involving $\frac{dy}{dx}$ (*or* y', or $y'(x)$), y and x.
- **A SOLUTION IS:** a function $y=y(x)$ such that $\frac{dy}{dx}$, y and x satisfy the original equation.
- **AN INITIAL VALUE PROBLEM** also specifies the value of the solution $y(a)$ at some point $x=a$
- **SIMPLE DIFFERENTIAL EQUATIONS** can be solved by separation of variables and integration.
  For example, the equation $f(x) = g(y)\frac{dy}{dx}$ can be written as $f(x)dx = g(y)dy$ and can be solved by integrating both sides: $\int f(x) = \int g(y)\frac{dy}{dx}$

FORMULAS FROM PRECALCULUS

LOGARITHMS AND EXPONENTIALS

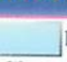
1. $y=\ln x$ if and only if $x=e^y=\exp(y)$
2. $\ln e^x = x$
3. $e^{\ln x} = x$
4. $e^x e^y = e^{x+y}$
5. $\frac{e^x}{e^y} = e^{x-y}$
6. $(e^x)^y = e^{xy}$
7. $e^0=1$
8. $\ln(xy)=\ln x+\ln y$
9. $\ln(x/y)=\ln x-\ln y$
10. $\ln(x^y)=y\ln x$
11. $\ln 1=0$
12. $\ln e=1$

ALGEBRAIC FORMULAS

1. If $a \neq 0$, the solutions to $ax^2 + bx + c = 0$ are given by
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
2. Point-slope equation of a line: $y - y_0 = m(x - x_0)$

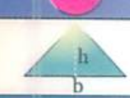
GEOMETRIC FORMULAS

1. Rectangle, width w. height h.
   - Area: A = wh Perimeter: p=2w + 2h



2. Circle, radius r. - Area: $A = \pi r^2$.
   - Circumference: $c = 2\pi r$

3. Triangle, base b. height h.
   - Area: A = 1/2 bh

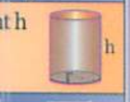


4. Rectangular Prism, length l, width w, height h.
   - Volume: V = lwh
   - Surface Area: A = 2wl + 2wh + 2hl

5. Sphere, radius r
   - Volume: $V=4/3\pi r^3$
   - Surface Area: $A = 4\pi r^2$

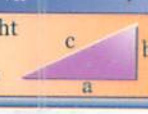
6. Trapezoid, parallel sides of length h_1, h_2, distance between parallel sides w.
   - Area: $A = 1/2\ (h_1 + h_2)\ w$

7. Right Circular Cylinder, radius r, height h
   - Volume: $V=\pi r^2 h$
   - Lateral surface Area: $A=2\pi rh$



8. Right Circular Cone, radius r, height h
   - Volume: $V=1/3\pi r^2 h$
   - Area of curved surface: $A=\pi r\sqrt{r^2+h^2}$

9. Pythagorean Theorem. If a right triangle has hypotenuse c and sides a and b, then $c^2 = a^2 + b^2$



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